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ORIGINAL RESEARCH PAPER

Distributed event-triggered output feedback H_∞ control for multi-agent systems with transmission delays

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Abstract
The output feedback H_∞ consensus control problem of multi-agent systems is studied using an event-triggered control strategy. Two types of transmission delays, one from the system output to the output feedback controller (OFC) and the other from the OFC to the zero-order holder, are considered. This causes the OFC and the system not to be updated in the same time intervals. An interval dividing approach is applied to such that the whole system can be updated in the same time intervals. An event-triggered OFC with H_∞ performance is proposed for multi-agent systems to achieve consensus. By constructing an appropriate Lyapunov–Krasovskii functional, sufficient conditions based on linear matrix inequality are derived to guarantee the consensus achievement. Finally, the theoretical results are validated using computer simulation.

1 | Introduction

Multi-agent systems have aroused extensive attention due to their autonomy, fault tolerance, flexibility, extensibility and collaboration. In recent decades, coordination of MASs has been extensively applied in different fields such as formation control, flocking, software development, multi-robot coordination and smart grids [1–4]. Consensus means that all the agents can reach a common value by only local information exchange. Many scholars have carried out a series of researches on the related issues of consensus from different aspects, such as the problem of finite-time consensus [5–8], consensus with time-varying delays [9–11], and consensus with different topologies [12–14], to name just a few.

The main idea of ETC strategy is to use the opportunistic aperiodic sampling instead of the classic periodic sampling to improve the efficiency. The ETC method uses a trigger function to replace the time constant in classic periodic sampling. When system is still running under the ideal state, the event will not be triggered. Otherwise, it will be triggered. As a result, ETC method can reduce the frequency of information transmission between agents to save energy. Therefore, how to accurately determine the updating time instants of control signals is the key to study this kind of problems. In 1999, [15] and [16] first proposed the ETC method. In 2012, [17] adopted centralized and distributed ETC method to analyze the consensus problem of MASs, respectively. Since then, more and more scholars have applied event-triggered strategies to MASs with different topologies [18–20], such as output feedback control [20–24], H_∞ control [25–30] etc, and have achieved fruitful research results in this field.

Event-triggered H_∞ consensus control is an important aspect for MASs, which has been deeply studied by a large number of literatures so far. In [25], the consensus control of MASs with switched topologies is investigated. Considering the uncertainty of communication networks in practical application, an event-triggered H_∞ consensus controller is proposed in switching networks subject to Markov chains using local information exchange via state-feedback. A sufficient condition based LMI for H_∞ consensus is given. In [26], aperiodic and periodic ETC methods are proposed for MASs to achieve H_∞ consensus. The event-triggered method is combined with the time-triggered method, and a fixed lower limit of sampling time interval is given to guarantee the avoidance of the Zeno behaviour. In [27], H_∞ control of MASs is investigated in directed networks via ETC method. In the case with external disturbances, a new distributed sampling method is proposed, and the Zeno-behaviour is completely excluded. In [28], the H_∞ consensus problem of MASs with missing measurements and external disturbance is considered, in which the considered system is in discrete-time and time-varying. Redundant channels are introduced to enhance the reliability of information transmission. An observer-based ETC method is proposed to reach consensus with H_∞ performance in a limited range. In [29], the H_∞ consensus control for discrete-time MASs with Markov switching topology is studied. An ETC strategy is proposed, which takes into account the influence of information exchange between neighbors and the channel noise due to environmental uncertainty. In [30], the consensus problem for MASs with external disturbance is investigated based on event-triggered scheme. A control algorithm is presented to achieve the control object by defining a control output to turn the consensus problem into H_∞ one. Time-delay is also a key factor in information transmission in practical applications. In the literatures mentioned above, only part of them consider the information transmission delay and the others do not. The above literature analysis inspires us to do the work in this paper, in which two kind of transmission delay are considered.

The output feedback H_∞ consensus problem of MASs is considered in this paper based on ETC strategy. Using the ETC method, the output signal is sampled and transmitted to the OFC side, and then sampled and transmitted to the ZOH. There are two kinds of transmission delays in this process, one from the output of system to the OFC and the other from the OFC to the ZOH. This causes the output feedback controller and system to be updated in different time intervals. By using interval decomposition method, the output feedback controller and system are unified into identical time intervals, and then the closed-loop system (CLS) of whole system is obtained. Since the system states are not measurable, an observer-based event-triggered OFC is presented for the followers to follow the leader. By constructing a Lyapunov–Krasovsky functional, sufficient conditions for consensus convergence and H_∞ performance are obtained in the form of LMI. The contributions of this work are summarized as follows. First, a novel event-triggered distributed output feedback controller is proposed for MAS to achieve leader-following consensus. In the proposed algorithm, both the controller and the trigger function are distributed only depending on the local information of the neighboring agents. Second, sufficient conditions based on LMI are derived to guarantee asymptotic stability and H_∞ performance of the considered system. The algorithms based on LMI to solve ETC problem were also proposed in [31–34], however, only one kind of transmission delay is considered in these literatures. Third, compared with [25–31, 35, 36], two kinds of transmission delay are considered in this paper. As far as we know, the work in this paper has rarely appeared in the literature except for [32]. In [32], the ETC problem via output feedback is applied to network control systems to achieve H_∞ performance. Two kinds of transmission delay are also considered and then a kind of interval decomposition method is applied to acquire a unified closed-loop system. However, due to the distributed requirement of MASs for controller and trigger function, the method proposed in [32] cannot be applied directly and the interval decomposition for MASs is more challenging.

The structure of this work is given below. In Section 2, we introduce some needed lemmas and concepts on algebraic graphic theory. The system model and problem are specified in Section 3. In Section 4, we propose the output feedback controller and analyze its stability. In Section 5, two instances of simulations are given to verify the feasibility of the results. We conclude this article in Section 6.

2 | PRELIMINARIES

In multi-agent systems, a directed graph denoted by $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{A})$ is used to represent the communication relationship between agents, where vertex set $\mathcal{V} = \{v_1, v_2, \dots, v_N\}$ represents N agents, and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the edge set. A directed edge $(v_j, v_i) \in \mathcal{E}$ means that agent i can sense information from agent j, in other words, agent i can receive information from agent j. For the weighted adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathcal{R}^{N\times N}$, if $(v_j, v_i) \in \mathcal{E}$, then $a_{ij} > 0$, otherwise $a_{ij} = 0$. The set of all adjacent agents of agent i is denoted by $\mathcal{N}_i = \{v_j \in \mathcal{V} : (v_j, v_i) \in \mathcal{E}, v_j \neq v_i\}$. The in-degree matrix $\mathcal{D} = [d_{ij}]_{N\times N}$ is a diagonal matrix with $d_{ii} = \sum_{j\in\mathcal{N}_i} a_{ij}$. The Laplacian matrix of graph $\mathcal{G}$ is defined as $\mathcal{L} = \mathcal{D} - \mathcal{A} = [l_{ij}]_{N\times N}$ where

$$\begin{cases} l_{ii} = \Sigma_{j=1}^{N} a_{ij} & i = j, \\ l_{ij} = -a_{ij} & i \neq j. \end{cases}$$

Let $\mathcal{B} = \mathrm{diag}(b_1, b_2, \dots, b_n)$ to be a diagonal matrix, and $b_i > 0$, if agent i can sense the leader, otherwise $b_i = 0$.

The following lemmas are useful in our theoretical analysis.

Lemma 1 [32]. *For any positive definite matrix Q, if constant $\beta > 0$, then in the interval $[0, \beta]$, the following inequality holds for the integrable vector function $\varpi(s)$:*

$$\left[\int_0^\beta \varpi(s)ds\right]^T Q \left[\int_0^\beta \varpi(s)ds\right] \leqslant \beta \left[\int_0^\beta \varpi(s)^T Q \varpi(s)ds\right].$$

Lemma 2 [33].

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{12}^T & T_{22} \end{pmatrix} < 0,$$

if and only if

$$T_{11} < 0,\ T_{22} - T_{12}^T T_{11}^{-1} T_{12} < 0,$$

or equivalently

$$T_{22} < 0,\ T_{11} - T_{12} T_{22}^{-1} T_{12}^T < 0.$$

where T_{11}, T_{12}, and T_{22} are matrices with appropriate dimensions.

3 | PROBLEM STATEMENT

Consider a class of MASs with N followers and a leader. The ith, $i = 1, 2, \dots, N$, follower's dynamic is

$$\begin{cases} \dot{x}^i = Ax^i + Bu^i + B_\omega \omega^i, \\ y_1^i = C_1 x^i, \\ y_2^i = C_2 x^i, \end{cases} \tag{1}$$

where $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times p}$, $B_\omega \in \mathcal{R}^{n\times q}$, $C_1 \in \mathcal{R}^{r\times n}$, and $C_2 \in \mathcal{R}^{n\times n}$ are matrices, $x^i \in \mathcal{R}^n$, $u^i \in \mathcal{R}^p$, $y_2^i \in \mathcal{R}^n$, $y_1^i \in \mathcal{R}^r$, $\omega^i \subset \mathcal{R}^q$ are the state vector, controller, measured output, controlled output, and disturbance input, respectively. The dynamic of the leader labelled by 0 is

$$\begin{cases} \dot{x}^0 = Ax^0, \\ y_1^0 = C_1 x^0, \\ y_2^0 = C_2 x^0. \end{cases} \tag{2}$$

We denote the release times of agent i by $t_0^i h, t_1^i h, t_2^i h, \dots$, where $t_0^i = 0$ is the initial time of the ith agent and $\{t_0^i, t_1^i, t_2^i, \dots\} \subset \{0, 1, 2, \dots\}$. The output $y_2^i(t)$ takes h as the sampling period and samples at time instant kh, where $h > 0$. Two types of transmission delays are considered. One is the transmission delay from system output to output feedback controller, denoted by ι_k^i. The other is from the OFC to the ZOH, denoted by ς_k^i. We make the hypothesis that $\iota_k^i \in [0, \bar{\iota})$ and $\varsigma_k^i \in [0, \bar{\varsigma})$, where $\bar{\iota}$ and $\bar{\varsigma}$ are upper bounds of ι_k^i and ς_k^i, respectively. Without loss of generality, let $\bar{\varsigma} = m_1 h$ and $\bar{\iota} = m_2 h$, where $m_1,\ m_2 > 0$. Motivated by the works in [37], a novel event-triggered condition requiring only local information:

$$\left[\varepsilon^i\left(t_k^i h + p^i h\right) - \varepsilon^i\left(t_k^i h\right)\right]^T C_2^T \Omega C_2 \left[\varepsilon^i\left(t_k^i h + p^i h\right) - \varepsilon^i\left(t_k^i h\right)\right]$$
$$\leqslant \sigma \varepsilon^i\left(t_k^i h + p^i h\right)^T C_2^T \Omega C_2 \varepsilon^i\left(t_k^i h + p^i h\right), \quad i = 1, 2, \dots, N, \tag{3}$$

is constructed to judge whether the output signal is being transferred to the OFC or not, where $\varepsilon^i = -\sum_{j\in\mathcal{N}_i} a_{ij}(x^i - x^j) - b_i(x^i - x^0)$, $\Omega > 0$, $p^i = 1, 2, \dots$ and $0 \leq \sigma < 1$.

Remark 1. When the inequality (3) holds, the sample output $y_2^i(t_k^i h + p^i h)$ of agent i will not be transferred to the OFC. Only when the inequality (3) fails to hold, it will be transmitted to output feedback controller. It can be seen from the information transmission mechanism that event-triggered design can save network bandwidth and energy. Obviously, when σ in (3) is equal to 0, it becomes time-triggered scheme as the special case of ETC scheme.

4 | OUTPUT FEEDBACK H_∞ CONTROL VIA ETC STRATEGY

From the event-triggered condition (3), the $(k+1)$th release time of agent i is $t_{k+1}^i h = t_k^i h + d_k^i h$, where $d_k^i = \min_j \{ j \,|\, [\varepsilon^i(t_k^i h + jh) - \varepsilon^i(t_k^i h)]^T C_2^T \Omega C_2 [\varepsilon^i(t_k^i h + jh) - \varepsilon^i(t_k^i h)] > \sigma \varepsilon^i(t_k^i h + jh)^T C_2^T \Omega C_2 \varepsilon^i(t_k^i h + jh)\}$. We assume that d_k^i is finite, that is, there exists a positive integer l such that $d_k^i \leqslant l$.

Let $\bar{x}^i = x^i - x^0$, $\bar{y}_2^i = y_2^i - y_2^0$, and $\bar{y}_1^i = y_1^i - y_1^0$, one has

$$\begin{cases} \dot{\bar{x}}^i = A\bar{x}^i + Bu^i + B_\omega \omega^i, \\ \bar{y}_1^i = C_1 \bar{x}^i, \\ \bar{y}_2^i = C_2 \bar{x}^i. \end{cases} \tag{4}$$

Let $\hat{x}^i$ and $\hat{x}^0$ be the estimates of x^i and x^0, respectively, and construct observers as

$$\dot{\hat{x}}^i = A\hat{x}^i + Bu^i + L\left(y_2^i\left(t_k^i h\right) - C_2 \hat{x}^i\left(t_k^i h\right)\right),$$
$$t \in \left[t_k^i h + \bar{\iota}, t_{k+1}^i h + \bar{\iota}\right), \tag{5}$$

and

$$\dot{\hat{x}}^0 = A\hat{x}^0 + L(y_2^0 - C_2 \hat{x}^0). \tag{6}$$

An observer-based dynamical OFC is presented as the following:

$$u^i(t) = K\tilde{\varepsilon}^i(t_k^i h), t \in [t_k^i h + \bar{\iota}, t_{k+1}^i h + \bar{\iota}), \tag{7}$$

where

$$\tilde{\varepsilon}^i = -\sum_{j\in\mathcal{N}_i} a_{ij}(\hat{x}^i - \hat{x}^j) - b_i(\hat{x}^i - \hat{x}^0),$$

and let

$$u^i(t) = K\varepsilon^i(t_0^i h), t \in [t_0^i h, t_0^i h + \bar{\iota} + \bar{\varsigma}), \tag{8}$$

where $x^i(t_0^i h)$ is the initial value of x^i.

Let $\tilde{x}^i = \hat{x}^i - \hat{x}^0$, one has

$$\dot{\tilde{x}}^i(t) = A\tilde{x}^i(t) + Bu^i(t) + L\left(\bar{y}_2^i\left(t_k^i h\right) - C_2 \tilde{x}^i\left(t_k^i h\right)\right),$$
$$t \in \left[t_k^i h + \bar{\iota}, t_{k+1}^i h + \bar{\iota}\right). \tag{9}$$

Remark 2. From the continuity of $\tilde{x}^i(t)$ on the interval $[t_k^i h + \bar{\iota}, t_{k+1}^i h + \bar{\iota})$ and $\tilde{x}^i(t_{k+1}^i h + \bar{\iota}) = \lim_{t\to(t_{k+1}^i h + \bar{\iota})^-} \tilde{x}^i(t)$, one has $\tilde{x}^i(t)$ on $[t_k^i h + \bar{\iota}, t_{k+1}^i h + \bar{\iota}]$, and then $\tilde{x}^i(t)$ on $[t_0^i, +\infty)$ are continuous. For the same reason, $\bar{x}^i(t)$ is continuous on $[t_0^i, +\infty)$ as well.

Remark 3. Note that the event-triggered condition (3) and controller (8) are distributed depending only on local information of neighboring agents. The event-triggered control method is applied in this paper, which can reduce unnecessary energy consumption.

Because there are two type of time-delays ι_k^i and ς_k^i, the dynamic output feedback controller (7) is updated based on $\bar{y}_2^i(t_k^i h)$ with a time-delay ι_k^i in $[t_k^i h+\bar{\iota}, t_{k+1}^i h+\bar{\iota}]$, while the system (9) is updated based on the sample control signal $u^i(t)$ with a delay ς_k^i in time interval $[t_k^i h+\bar{\iota}+\bar{\varsigma}, t_{k+1}^i h+\bar{\iota}+\bar{\varsigma})$. In other words, systems (7) and (9) are updated in different time intervals, so the CLS cannot be obtained from the two equations directly. In the following, the closed-loop system is derived by using an interval partition method. We divide the time interval of (9) using the updating time instants of (7).

Considering $[t_k^i h+\bar{\iota}+\bar{\varsigma}, t_{k+1}^i h+\bar{\iota}+\bar{\varsigma})$ and noting that $t_{k+1}^i h+\bar{\iota}+\bar{\varsigma} < t_k^i h+lh+m_1 h+m_2 h$, we can find two positive integers $t_{m_{k_1}}^i, t_{m_{k_2}}^i \in \{0,1,2,\dots\}$, satisfying $t_k^i \leqslant t_{m_{k_1}}^i < t_{m_{k_2}}^i$, $t_{m_{k_1}}^i < t_k^i+m_1+m_2$ and $t_{m_{k_2}}^i < t_k^i+l+m_1+m_2$ such that

$$t_k^i h+\bar{\iota}+\bar{\varsigma} \in \left[t_{m_{k_1}}^i h+\bar{\iota}, t_{m_{k_1}+1}^i h+\bar{\iota}\right),$$

and

$$t_{k+1}^i h+\bar{\iota}+\bar{\varsigma} \in \left[t_{m_{k_2}}^i h+\bar{\iota}, t_{m_{k_2}+1}^i h+\bar{\iota}\right).$$

Then we have the following interval decomposition:

$$\left[t_k^i h+\bar{\iota}+\bar{\varsigma}, t_{k+1}^i h+\bar{\iota}+\bar{\varsigma}\right) = \mathcal{I}^{0,\kappa^i} \bigcup_{s=1}^{m_{k_2}-m_{k_1}-1} \mathcal{I}_s^{\kappa^i} \bigcup \mathcal{I}^{1,\kappa^i}, \tag{10}$$

where

$$\kappa^i = \begin{cases} t_{m_{k_1}}^i, & t \in \left[t_k^i h+\bar{\iota}+\bar{\varsigma}, t_{m_{k_1}+1}^i h+\bar{\iota}\right), \\ t_{m_{k_1}+s}^i, & t \in \left[t_{m_{k_1}+s}^i h+\bar{\iota}, t_{m_{k_1}+s+1}^i h+\bar{\iota}\right), \\ & s = 1,\dots, m_{k_2}-m_{k_1}-1, \\ t_{m_{k_2}}^i, & t \in \left[t_{m_{k_2}}^i h+\bar{\iota}, t_{k+1}^i h+\bar{\iota}+\bar{\varsigma}\right), \end{cases} \tag{11}$$

and

$$\mathcal{I}^{\kappa^i} = \begin{cases} \mathcal{I}^{0,\kappa^i} = \left[t_k^i h+\bar{\iota}+\bar{\varsigma}, t_{m_{k_1}+1}^i h+\bar{\iota}\right), & \kappa^i = t_{m_{k_1}}^i, \\ \mathcal{I}_s^{\kappa^i} = \left[t_{m_{k_1}+s}^i h+\bar{\iota}, t_{m_{k_1}+s+1}^i h+\bar{\iota}\right), & \kappa^i = t_{m_{k_1}+s}^i, \\ \quad s = 1,\dots, m_{k_2}-m_{k_1}-1, & \\ \mathcal{I}^{1,\kappa^i} = \left[t_{m_{k_2}}^i h+\bar{\iota}, t_{k+1}^i h+\bar{\iota}+\bar{\varsigma}\right), & \kappa^i = t_{m_{k_2}}^i. \end{cases} \tag{12}$$

From (11) and (12), the system (4), (9) and the dynamic OFC (7) can be rewritten as

$$\begin{aligned} \dot{\tilde{x}}^i(t) &= A\tilde{x}^i(t) + BK\tilde{\varepsilon}^i(\kappa^i h) \\ &\quad + L\left[\bar{y}_2^i(\kappa^i h) - C_2\tilde{x}^i(\kappa^i h)\right], t \in \mathcal{I}^{\kappa^i}, \end{aligned} \tag{13}$$

$$\dot{\bar{x}}^i(t) = A\bar{x}^i(t) + BK\tilde{\varepsilon}^i(t_k^i h) + B_\omega \omega^i(t), t \in \mathcal{I}^{\kappa^i}, \tag{14}$$

and

$$u^i(t) = K\tilde{\varepsilon}^i(t_k^i h), t \in \mathcal{I}^{\kappa^i}, \tag{15}$$

respectively.

When $t \in [t_k^i h+\bar{\iota}+\bar{\varsigma}, t_{m_{k_1}+1}^i h+\bar{\iota})$, we have $t_k^i h+\bar{\iota}+\bar{\varsigma} - t_{m_{k_1}}^i h \le t-\kappa^i h < t_{m_{k_1}+1}^i h+\bar{\iota} - t_{m_{k_1}}^i h < \bar{\iota}+lh$. It follows from $t_k^i h+\bar{\iota}+\bar{\varsigma} \ge t_{m_{k_1}}^i h+\bar{\iota}$ that $\bar{\iota} \le t-\kappa^i h < \bar{\iota}+lh$.

When $t \in [t_{m_{k_1}+s}^i h+\bar{\iota}, t_{m_{k_1}+s+1}^i h+\bar{\iota})$, we have $t_{m_{k_1}+s}^i h+\bar{\iota} - t_{m_{k_1}+s}^i h \le t-\kappa^i h < t_{m_{k_1}+s+1}^i h+\bar{\iota} - t_{m_{k_1}+s}^i h$, that is, $\bar{\iota} \le t-\kappa^i h < \bar{\iota}+lh$.

When $t \in [t_{m_{k_2}}^i h+\bar{\iota}, t_{k+1}^i h+\bar{\iota}+\bar{\varsigma})$, we have $t_{m_{k_2}}^i h+\bar{\iota} - t_{m_{k_2}}^i h \le t-\kappa^i h < t_{k+1}^i h+\bar{\iota}+\bar{\varsigma} - t_{m_{k_2}}^i h$. It follows from $t_{k+1}^i h+\bar{\iota}+\bar{\varsigma} \le t_{m_{k_2}+1}^i h+\bar{\iota}$ that $\bar{\iota} \le t-\kappa^i h < \bar{\iota}+lh$.

Therefore, $\bar{\iota} \le t-\kappa^i h < \bar{\iota}+lh, t \in \mathcal{I}^{\kappa^i}$.

When $t \in [t_k^i h+\bar{\iota}+\bar{\varsigma}, t_{k+1}^i h+\bar{\iota}+\bar{\varsigma})$, it is easy to see that $t_k^i h+\bar{\iota}+\bar{\varsigma}-t_k^i h \le t-t_k^i h < t_{k+1}^i h+\bar{\iota}+\bar{\varsigma}-t_k^i h < \bar{\iota}+\bar{\varsigma}+lh$. That is, $\bar{\iota}+\bar{\varsigma} \le t-t_k^i h < \bar{\iota}+\bar{\varsigma}+lh, t \in \mathcal{I}^{\kappa^i}$.

Let

$$\vartheta_1(t) = t-\kappa^i h, \vartheta_2(t) = t-t_k^i h, t \in \mathcal{I}^{\kappa^i}, \tag{16}$$

where $\vartheta_1(t) \in [\bar{\iota}, \bar{\iota}+lh)$, $\vartheta_2(t) \in [\bar{\iota}+\bar{\varsigma}, \bar{\iota}+\bar{\varsigma}+lh)$, for all agents, satisfying $\dot{\vartheta}_1(t) = 1$ and $\dot{\vartheta}_2(t) = 1$.

We denoted the stack column vectors of x_i, $i = 1,2,\dots,N$, by $\mathrm{col}(x^i)$. Let $e^i = \bar{x}^i - \tilde{x}^i$, $e = \mathrm{col}(e^i)$. In the following lemma, CLS is derived according to (10).

Lemma 3. *Based on systems (13) and (14), the following CLS can be obtained:*

$$\begin{cases} \dot{\bar{x}}^i(t) = A\bar{x}^i(t) + BK\tilde{\varepsilon}^i(t) - BK\int_{t-\vartheta_2(t)}^t \dot{\tilde{\varepsilon}}^i(r)dr + B_\omega \omega^i(t), \\ \dot{e}^i(t) = LC_2\bar{x}^i(t) - LC_2\varepsilon^i(t) + (A-LC_2)e^i(t) \\ \quad - BK\int_{t-\vartheta_2(t)}^t \dot{\tilde{\varepsilon}}^i(r)dr + BK\int_{t-\vartheta_1(t)}^t \dot{\tilde{\varepsilon}}^i(r)dr \\ \quad - LC_2\int_{t-\vartheta_1(t)}^t \dot{\bar{x}}^i(r)dr + LC_2\int_{t-\iota(t)}^t \dot{\varepsilon}^i(r)dr \\ \quad - L\epsilon_{\kappa^i}(t) + B_\omega \omega^i(t), \quad t \in \mathcal{I}^{\kappa^i}, \quad i = 1,2,\dots,N, \end{cases} \tag{17}$$

where the functions $\iota(t)$ and $\epsilon_{\kappa^i}(t)$ will be determined later.

Proof. Similar to [37], we decompose the time interval $\mathcal{I}^{\kappa^i}$. For $\mathcal{I}^{0,\kappa^i}$, noticing that $t_k^i h + \bar{\iota} + \bar{\varsigma} \in [t_{m_{k_1}}^i h + \bar{\iota}, t_{m_{k_1}+1}^i h + \bar{\iota})$, there exists a positive number $n_0^i = \min_j \{jh | t_k^i h + \bar{\iota} + \bar{\varsigma} < t_{m_{k_1}}^i h + \bar{\iota} + jh\}$. Consider two cases:

Case 1. If $t_{m_{k_1}}^i h + \bar{\iota} + n_0^i h \geq t_{m_{k_1}+1}^i h + \bar{\iota}$, let

$$\iota^i(t) = t - t_{m_{k_1}}^i h - (n_0^i - 1)h, t \in [t_k^i h + \bar{\iota} + \bar{\varsigma}, t_{m_{k_1}+1}^i h + \bar{\iota}),$$

and

$$\begin{aligned} \epsilon_{\kappa^i}(t) &= C_2 \varepsilon^i (t_{m_{k_1}}^i h) - C_2 \varepsilon^i (t_{m_{k_1}}^i h + (n_0^i - 1)h), \\ &\quad t \in [t_k^i h + \bar{\iota} + \bar{\varsigma}, t_{m_{k_1}+1}^i h + \bar{\iota}). \end{aligned}$$

From the definition of ι^i, one has

$$\iota^i \geqslant t_k^i h + \bar{\iota} + \bar{\varsigma} - t_{m_{k_1}}^i h - (n_0^i - 1)h,$$

and

$$\iota^i < t_{m_{k_1}+1}^i h + \bar{\iota} - t_{m_{k_1}}^i h - (n_0^i - 1)h.$$

From the definition of n_0^i, one has $t_k^i h + \bar{\iota} + \bar{\varsigma} \geq t_{m_{k_1}}^i h + \bar{\iota} + n_0^i h - h$. Then

$$\iota^i \geqslant t_{m_{k_1}}^i h + \bar{\iota} + n_0^i h - h - t_{m_{k_1}}^i h - (n_0^i - 1)h = \bar{\iota}.$$

Since $t_{m_{k_1}}^i h + \bar{\iota} + n_0^i h \geq t_{m_{k_1}+1}^i h + \bar{\iota}$, one gets

$$\iota^i < t_{m_{k_1}}^i h + \bar{\iota} + n_0^i h - t_{m_{k_1}}^i h - (n_0^i - 1)h = \bar{\iota} + h.$$

Therefore,

$$\bar{\iota} \leqslant \iota^i(t) < \bar{\iota} + h.$$

Case 2. If $t_{m_{k_1}}^i h + \bar{\iota} + n_0^i h < t_{m_{k_1}+1}^i h + \bar{\iota}$, the following intervals:

$$[t_k^i h + \bar{\iota} + \bar{\varsigma}, t_{m_{k_1}}^i h + \bar{\iota} + n_0^i h), [t_{m_{k_1}}^i h + \bar{\iota} + d^i h, t_{m_{k_1}}^i h + \bar{\iota} + d^i h + h)$$

are considered. We can find some constant N^{0,κ^i} such that

$$t_{m_{k_1}}^i h + \bar{\iota} + N^{0,\kappa^i} h < t_{m_{k_1}+1}^i h + \bar{\iota} < t_{m_{k_1}}^i h + \bar{\iota} + N^{0,\kappa^i} h + h,$$

and $C_2 \varepsilon^i (t_{m_{k_1}}^i h + (n_0^i - 1)h)$ and $C_2 \varepsilon^i (t_{m_{k_1}}^i h + d^i h), d^i = n_0^i, \ldots, N^{0,\kappa^i}$ satisfy condition (3). Then, $\mathcal{I}^{0,\kappa^i}$ can be divided into

$$\mathcal{I}^{0,\kappa^i} = \mathcal{J}_{n_0^i - 1}^{0,\kappa^i} \bigcup_{d^i = n_0^i}^{N^{0,\kappa^i}} \mathcal{J}_{d^i}^{0,\kappa^i}, \kappa^i = t_{m_{k_1}}^i,$$

where

$$\begin{aligned} \mathcal{J}_{n_0^i - 1}^{0,\kappa^i} &= \left[t_k^i h + \bar{\iota} + \bar{\varsigma}, t_{m_{k_1}}^i h + \bar{\iota} + n_0^i h\right), \\ \mathcal{J}_{d^i}^{0,\kappa^i} &= \left[t_{m_{k_1}}^i h + \bar{\iota} + d^i h, t_{m_{k_1}}^i h + \bar{\iota} + d^i h + h\right), \\ d^i &= n_0^i, \ldots, N^{0,\kappa^i} - 1, \\ \mathcal{J}_{N^{0,\kappa^i}}^{0,\kappa^i} &= \left[t_{m_{k_1}}^i h + \bar{\iota} + N^{0,\kappa^i} h, t_{m_{k_1}+1}^i h + \bar{\iota}\right), \end{aligned}$$

and $\kappa^i = t_{m_{k_1}}^i$.

Let

$$\begin{aligned} \varpi_1\left((n_0^i - 1)h\right) &= C_2 \varepsilon^i \left(t_{m_{k_1}}^i h\right) - C_2 \varepsilon^i \left(t_{m_{k_1}}^i h + \left(n_0^i - 1\right)h\right), \\ \varpi_1(d^i h) &= C_2 \varepsilon^i \left(t_{m_{k_1}}^i h\right) - C_2 \varepsilon^i \left(t_{m_{k_1}}^i h + d^i h\right), \\ \varpi_1\left(N^{0,\kappa^i} h\right) &= C_2 \varepsilon^i \left(t_{m_{k_1}}^i h\right) - C_2 \varepsilon^i \left(t_{m_{k_1}}^i h + N^{0,\kappa^i} h\right). \end{aligned} \tag{18}$$

Define

$$\iota^i(t) = \begin{cases} t - t_{m_{k_1}}^i h - (n_0^i - 1)h, & t \in \mathcal{J}_{n_0^i - 1}^{0,\kappa^i}, \\ t - t_{m_{k_1}}^i h - d^i h, & t \in \mathcal{J}_{d^i}^{0,\kappa^i}, d^i = n_0^i, \\ \quad \ldots, N^{0,\kappa^i} - 1, & \\ t - t_{m_{k_1}}^i h - N^{0,\kappa^i} h, & t \in \mathcal{J}_{N^{0,\kappa^i}}^{0,\kappa^i}, \end{cases} \tag{19}$$

and

$$\epsilon_{\kappa^i}(t) = \begin{cases} \varpi_1((n_0^i - 1)h), & t \in \mathcal{J}_{n_0^i - 1}^{0,\kappa^i}, \\ \varpi_1(d^i h), & t \in \mathcal{J}_{d^i}^{0,\kappa^i}, d^i = n_0^i, \ldots, N^{0,\kappa^i} - 1, \\ \varpi_1(N^{0,\kappa^i} h), & t \in \mathcal{J}_{N^{0,\kappa^i}}^{0,\kappa^i}. \end{cases} \tag{20}$$

Just like in *Case* 1, we can also obtain that

$$\bar{\iota} \leqslant \iota^i(t) < \bar{\iota} + h.$$

Similarly, $\mathcal{I}_s^{\kappa^i}$ and $\mathcal{I}^{1,\kappa^i}$ can be divided into

$$\mathcal{I}_s^{\kappa^i} = \bigcup_{d^i = 0}^{N_s^{\kappa^i}} \mathcal{J}_{s,d^i}^{\kappa^i}, \kappa^i = t_{m_{k_1}+s}^i, s = 1, \ldots, m_{k_2} - m_{k_1} - 1,$$

and

$$\mathcal{I}^{1,\kappa^i} = \bigcup_{d^i = 0}^{N^{1,\kappa^i}} \mathcal{J}_{d^i}^{1,\kappa^i}, \kappa^i = t_{m_{k_2}}^i,$$

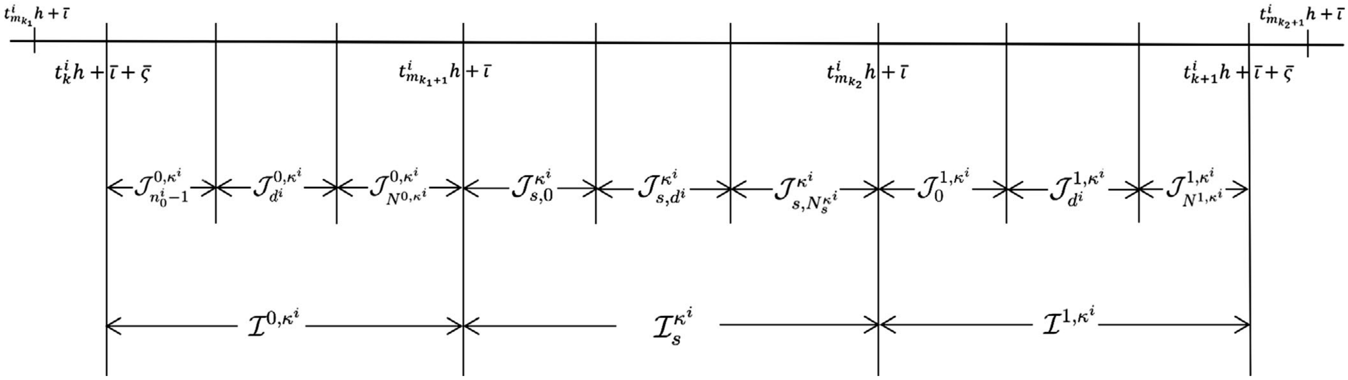

FIGURE 1 An example of interval decomposition.

respectively, where

$$
\begin{aligned}
\mathcal{J}^{\kappa^i}_{s,0} &= \left[t^i_{m_{k_1}+s}h+\bar{\iota},\, t^i_{m_{k_1}+s}h+\bar{\iota}+h\right),\\
\mathcal{J}^{\kappa^i}_{s,d^i} &= \left[t^i_{m_{k_1}+s}h+\bar{\iota}+d^ih,\, t^i_{m_{k_1}+s}h+\bar{\iota}+d^ih+h\right),\\
d^i &= 1,\dots,N^{\kappa^i}_s-1,\\
\mathcal{J}^{\kappa^i}_{s,N^{\kappa^i}_s} &= \left[t^i_{m_{k_1}+s}h+\bar{\iota}+N^{\kappa^i}_s h,\, t^i_{m_{k_1}+s+1}h+\bar{\iota}\right),\\
\kappa^i &= t^i_{m_{k_1}+s}, \quad s=1,\dots,m_{k_2}-m_{k_1}-1.\\
\mathcal{J}^{1,\kappa^i}_{0} &= \left[t^i_{m_{k_2}}h+\bar{\iota},\, t^i_{m_{k_2}}h+\bar{\iota}+h\right),\\
\mathcal{J}^{1,\kappa^i}_{d^i} &= \left[t^i_{m_{k_2}}h+\bar{\iota}+d^ih,\, t^i_{m_{k_2}}h+\bar{\iota}+d^ih+h\right),\\
d^i &= 1,\dots,N^{1,\kappa^i}-1,\\
\mathcal{J}^{1,\kappa^i}_{N^{1,\kappa^i}} &= \left[t^i_{m_{k_2}}h+\bar{\iota}+N^{1,\kappa^i}h,\, t^i_{k+1}h+\bar{\iota}+\bar{\varsigma}\right)
\end{aligned}
$$

and $\kappa^i = t^i_{m_{k_2}}$.

To facilitate the understanding of interval decomposition methods, an illustrative example is given in Figure 1.

Let

$$
\begin{aligned}
\varpi_2(0) &= C_2\varepsilon^i\left(t^i_{m_{k_1}+s}h\right) - C_2\varepsilon^i\left(t^i_{m_{k_1}+s}h\right),\\
\varpi_2(d^ih) &= C_2\varepsilon^i\left(t^i_{m_{k_1}+s}h\right) - C_2\varepsilon^i\left(t^i_{m_{k_1}+s}h+d^ih\right),\\
\varpi_2\left(N^{\kappa^i}_s h\right) &= C_2\varepsilon^i\left(t^i_{m_{k_1}+s}h\right) - C_2\varepsilon^i\left(t^i_{m_{k_1}+s}h+N^{\kappa^i}_s h\right),\\
\varpi_3(0) &= C_2\varepsilon^i\left(t^i_{m_{k_2}}h\right) - C_2\varepsilon^i\left(t^i_{m_{k_2}}h\right),\\
\varpi_3(d^ih) &= C_2\varepsilon^i\left(t^i_{m_{k_2}}h\right) - C_2\varepsilon^i\left(t^i_{m_{k_2}}h+d^ih\right),\\
\varpi_3\left(N^{1,\kappa^i}h\right) &= C_2\varepsilon^i\left(t^i_{m_{k_2}}h\right) - C_2\varepsilon^i\left(t^i_{m_{k_2}}h+N^{1,\kappa^i}h\right).
\end{aligned}
\tag{21}
$$

Define

$$
\iota^i(t) = \begin{cases}
t - t^i_{m_{k_1}+s}h, & t\in\mathcal{J}^{\kappa^i}_{s,0},\ s=1,\dots,\\
m_{k_2}-m_{k_1}-1, & \\
t - t^i_{m_{k_1}+s}h - d^ih, & t\in\mathcal{J}^{\kappa^i}_{s,d^i},\ d^i=1,\dots,\\
N^{\kappa^i}_s-1, & \\
t - t^i_{m_{k_1}+s}h - N^{\kappa^i}_s h, & t\in\mathcal{J}^{\kappa^i}_{s,N^{\kappa^i}_s},\\
t - t^i_{m_{k_2}}h, & t\in\mathcal{J}^{1,\kappa^i}_{0},\\
t - t^i_{m_{k_2}}h - d^ih, & t\in\mathcal{J}^{1,\kappa^i}_{d^i},\\
d^i=1,\dots,N^{1,\kappa^i}-1, & \\
t - t^i_{m_{k_2}}h - N^{1,\kappa^i}h, & t\in\mathcal{J}^{1,\kappa^i}_{N^{1,\kappa^i}},
\end{cases}
\tag{22}
$$

and

$$
\epsilon_{\kappa^i}(t) = \begin{cases}
\varpi_2(0), & t\in\mathcal{J}^{\kappa^i}_{s,0},\ s=1,\dots,m_{k_2}-m_{k_1}-1,\\
\varpi_2(d^ih), & t\in\mathcal{J}^{\kappa^i}_{s,d^i},\ d^i=1,\dots,N^{\kappa^i}_s-1,\\
\varpi_2(N^{\kappa^i}_s h), & t\in\mathcal{J}^{\kappa^i}_{s,N^{\kappa^i}_s},\\
\varpi_3(0), & t\in\mathcal{J}^{1,\kappa^i}_{0},\\
\varpi_3(d^ih), & t\in\mathcal{J}^{1,\kappa^i}_{d^i},\ d^i=1,\dots,N^{1,\kappa^i}-1,\\
\varpi_3(N^{1,\kappa^i}h), & t\in\mathcal{J}^{1,\kappa^i}_{N^{1,\kappa^i}}.
\end{cases}
\tag{23}
$$

By a similar analysis, we can obtain that

$$
\bar{\iota} \leqslant \iota^i(t) < \bar{\iota}+h.
$$

Let $\iota(t)\in[\bar{\iota},\bar{\iota}+h)$ with $\dot{\iota}(t)=1$, for all agents, $t\in\mathcal{I}^{\kappa^i}$.

From the definition of $\epsilon_{\kappa^i}(t)$ and (3), for $t\in\mathcal{I}^{\kappa^i}$, we have

$$
\begin{aligned}
\epsilon^T_{\kappa^i}\Omega\epsilon_{\kappa^i} &\leqslant \sigma\varepsilon^i(t-\iota)^T C_2^T\Omega C_2\varepsilon^i(t-\iota)\\
&= \sigma\left[C_2\varepsilon^i - C_2\int_{t-\iota}^{t}\dot{\varepsilon}^i(r)dr\right]^T\\
&\quad \Omega\left[C_2\varepsilon^i - C_2\int_{t-\iota}^{t}\dot{\varepsilon}^i(r)dr\right],
\end{aligned}
$$

that is,

$$-\epsilon_{\kappa^i}^T\Omega\epsilon_{\kappa^i}+\sigma\left[C_2\varepsilon^i-C_2\int_{t-\iota}^{t}\dot{\varepsilon}^i(r)dr\right]^T$$
$$\Omega[C_2\varepsilon^i-C_2\int_{t-\iota}^{t}\dot{\varepsilon}^i(r)dr]\geqslant 0,\ i=1,2,\dots,N. \tag{24}$$

On the basis of the above analysis, we can easily derive that

$$\begin{aligned}\bar{y}_2^i(\kappa^i h)&=C_2\varepsilon^i(t-\iota(t))+\epsilon_{\kappa^i}(t)\\&=C_2\varepsilon^i(t)-C_2\int_{t-\iota(t)}^{t}\dot{\varepsilon}^i(r)dr+\epsilon_{\kappa^i}(t).\end{aligned} \tag{25}$$

From (4) to (13), the error dynamics is given by

$$\begin{aligned}\dot{e}^i&=A\bar{x}^i+BK\tilde{\varepsilon}^i\left(t_k^i h\right)+B_\omega\omega^i-A\tilde{x}^i-BK\tilde{\varepsilon}^i(\kappa^i h)\\&\quad-L\bar{y}_2^i(\kappa^i h)+LC_2\tilde{x}^i(\kappa^i h).\end{aligned} \tag{26}$$

By (16) and (25), the CLS (17) can be obtained. □

Remark 4. The updating interval of (7) and (9) is different due to the transmission delays ι_k^i and ς_k^i. It is challenging for stability analysis. An interval decomposition method is used such that system (7) and (9) are updated in the same time interval.

Remark 5. In [31, 34 37], the interval decomposition method has also been used in the event-triggered control problem. The main difference is that we need to obtain a unified closed-loop system due to existing two kinds of transmission delay. In Lemma 2, a CLS is obtained.

Definition 1. For the CLS (17) and given $\gamma>0$, if:

1. $\lim_{t\to\infty}\|\bar{x}^i(t)\|=0$, asymptotically for all agents and any initial states as the disturbance vanishing;
2. $\|\bar{y}_1(t)^T\bar{y}_1(t)\|_2\leqslant\gamma\|\omega(t)^T\omega(t)\|_2$, holds, then, controller (7) is called H_∞ consensus OFC and the CLS is said to have an H_∞ performance with an index γ.

In the following, sufficient conditions based on LMI are given to ensure the existence of the H_∞ consensus OFC.

Lemma 4. *There exists an H_∞ consensus OFC (7) for system (1) and (2), if there exist matrices L, K, $\Omega>0$ and $W>0$, and constants $h>0$ and $\sigma>0$ such that*

$$\begin{pmatrix}\Sigma & \Gamma_1^T(I_N\otimes W) & \Gamma_2^T(I_N\otimes W) & \Gamma_3^T(I_N\otimes\Omega)\\ * & -\frac{1}{a_1}(I_N\otimes W) & 0 & 0\\ * & * & -\frac{1}{a_1}(I_N\otimes W) & 0\\ * & * & * & -\frac{1}{\sigma}(I_N\otimes\Omega)\end{pmatrix}<0, \tag{27}$$

where

$$\Sigma=\begin{pmatrix}\Sigma_{11} & \Sigma_{12} & 0\\ * & \Sigma_{22} & \Sigma_{23}\\ * & * & \Sigma_{33}\end{pmatrix},\Sigma_{11}=G_{11}+I_N\otimes C_1^TC_1,$$

$$\begin{aligned}G_{11}&=I_N\otimes(WA+A^TW)-\mathcal{H}\otimes WBK\\&\quad-\mathcal{H}^T\otimes K^TB^TW,\end{aligned}$$

$$\Sigma_{12}=\begin{pmatrix}\Sigma_{13} & \Sigma_{14}\end{pmatrix},$$

$$\Sigma_{13}=\begin{pmatrix}\begin{array}{c}\mathcal{H}\\ \otimes WBK+\mathcal{H}^T\otimes C_2^TL^TW\\ +I_N\otimes C_2^TL^TW\\ \otimes WBK\end{array} & \mathcal{H}\end{pmatrix},$$

$$\Sigma_{14}=\begin{pmatrix}-\mathcal{H}\otimes WBK & I_N\otimes WB_\omega\end{pmatrix},$$

$$\Sigma_{22}=\begin{pmatrix}G_{31} & \mathcal{H}\otimes WBK & -\mathcal{H}\otimes WBK & I_N\otimes WB_\omega\\ * & -\frac{1}{3a_1}(I_N\otimes W) & 0 & 0\\ * & * & -\frac{1}{2a_1}(I_N\otimes W) & 0\\ * & * & * & -\gamma^2I_{Nn}\end{pmatrix},$$

96pt]$G_{31}=I_N\otimes(WA+A^TW-WLC_2-C_2^TL^TW)$,

$$\Sigma_{23}=\begin{pmatrix}\Sigma_{24} & \Sigma_{25}\end{pmatrix},$$

$$\Sigma_{24}=\begin{pmatrix}-(I_N\otimes WLC_2+\mathcal{H}\otimes WBK) & \begin{array}{c}I_N\otimes WLC_2\\ +\mathcal{H}\otimes WBK\end{array}\\ 0_{9N\times 3N} & 0_{9N\times 3N}\end{pmatrix},$$

$$\Sigma_{25}=\begin{pmatrix}-\mathcal{H}\otimes WLC_2 & -I_N\otimes WL\\ 0_{9N\times 3N} & 0_{9N\times 3N}\end{pmatrix},$$

$$\Sigma_{33}=\begin{pmatrix}-a_1(I_N\otimes W) & 0 & \begin{array}{c}0\\0\end{array}\\ 0 & -\frac{1}{2a_1}(I_N\otimes W) & \begin{array}{c}0\\0\end{array}\\ 0 & 0 & \begin{array}{c}-\frac{1}{3a_1}(I_N\otimes W)\\0\end{array}\\ 0 & 0 & \begin{array}{c}0\\-I_N\otimes\Omega\end{array}\end{pmatrix},$$

$$\Gamma_1 = (\Gamma_{11}\Gamma_{12}),$$

$$\Gamma_{11} = \big(I_N \otimes A - \mathcal{H} \otimes BK\mathcal{H} \otimes BK + \mathcal{H}^T \otimes C_2^T L^T + I_N \otimes C_2^T L^T\big),$$

$$\Gamma_{12} = (\mathcal{H} \otimes BK - \mathcal{H} \otimes BKI_N \otimes B_\omega 0_{3N\times 12N}),$$

$$\Gamma_2 = (\Gamma_{21}\Gamma_{22}\Gamma_{23}),$$

$$\Gamma_{21} = \big(0_{3N\times 3N} I_N \otimes (A - LC_2)\mathcal{H} \otimes BK\big),$$

$$\Gamma_{22} = \big(-\mathcal{H} \otimes BKI_N \otimes B_\omega - (I_N \otimes LC_2 + \mathcal{H} \otimes BK)\big),$$

$$\Gamma_{23} = \big(I_N \otimes LC_2 + \mathcal{H} \otimes BK - \mathcal{H} \otimes LC_2 - I_N \otimes L\big),$$

$$\Gamma_3 = \big(\mathcal{H} \otimes C_2 0_{3N\times 18N} - \mathcal{H} \otimes C_2 0_{3N\times 3N}\big),$$

$$a_1 = (m_1 + m_2 + l)h.$$

Proof. Construct a Lyapunov–Krasovskii functional $U(t) = U_1(t) + U_2(t)$, where

$$U_1(t) = \bar{x}(t)^T (I_N \otimes W)\bar{x}(t) + \sum_{i=1}^{N} \int_{t-a_1}^{t} \int_{\beta}^{t} \dot{\bar{x}}^i(r)^T W \dot{\bar{x}}^i(r)\,dr\,d\beta, \tag{28}$$

$$U_2(t) = e(t)^T (I_N \otimes W)e(t) + \sum_{i=1}^{N} \int_{t-a_1}^{t} \int_{\beta}^{t} \dot{e}^i(r)^T W \dot{e}^i(r)\,dr\,d\beta. \tag{29}$$

The time derivatives of $U_1(t)$ and $U_2(t)$ along trajectories of CLS (17) are

$$\dot{U}_1 = \dot{\bar{x}}^T (I_N \otimes W)\bar{x} + \bar{x}^T (I_N \otimes W)\dot{\bar{x}} + \sum_{i=1}^{N} a_1 (\dot{\bar{x}}^i)^T W \dot{\bar{x}}^i - \sum_{i=1}^{N} \int_{t-a_1}^{t} \dot{\bar{x}}^i(r)^T W \dot{\bar{x}}^i(r)\,dr, \tag{30}$$

and

$$\dot{U}_2 = \dot{e}^T (I_N \otimes W)e + e^T (I_N \otimes W)\dot{e} + \sum_{i=1}^{N} a_1 (\dot{e}^i)^T W \dot{e}^i - \sum_{i=1}^{N} \int_{t-a_1}^{t} \dot{e}^i(r)^T W \dot{e}^i(r)\,dr. \tag{31}$$

From Lemma 1, we obtain that

$$\begin{aligned}
&-\int_{t-a_1}^{t} \dot{\bar{x}}^i(r)^T W \dot{\bar{x}}^i(r)\,dr \\
&\leqslant -\frac{1}{3}\int_{t-\vartheta_2(t)}^{t} \dot{\bar{x}}^i(r)^T W \dot{\bar{x}}^i(r)\,dr - \frac{1}{3}\int_{t-\vartheta_1(t)}^{t} \dot{\bar{x}}^i(r)^T W \dot{\bar{x}}^i(r)\,dr - \frac{1}{3}\int_{t-\iota(t)}^{t} \dot{\bar{x}}^i(r)^T W \dot{\bar{x}}^i(r)\,dr \\
&\leqslant -\frac{1}{3a_1}\Big[\int_{t-\vartheta_2(t)}^{t} \dot{\bar{x}}^i(r)^T dr W \int_{t-\vartheta_2(t)}^{t} \dot{\bar{x}}^i(r)\,dr + \int_{t-\vartheta_1(t)}^{t} \dot{\bar{x}}^i(r)^T dr W \int_{t-\vartheta_1(t)}^{t} \dot{\bar{x}}^i(r)\,dr \\
&\quad + \int_{t-\iota(t)}^{t} \dot{\bar{x}}^i(r)^T dr W \int_{t-\iota(t)}^{t} \dot{\bar{x}}^i(r)\,dr\Big],
\end{aligned}$$

and

$$\begin{aligned}
&-\int_{t-a_1}^{t} \dot{e}^i(r)^T W \dot{e}^i(r)\,dr \\
&\leqslant -\frac{1}{2}\int_{t-\vartheta_2(t)}^{t} \dot{e}^i(r)^T W \dot{e}^i(r)\,dr - \frac{1}{2}\int_{t-\vartheta_1(t)}^{t} \dot{e}^i(r)^T W \dot{e}^i(r)\,dr \\
&\leqslant -\frac{1}{2a_1}\Big[\int_{t-\vartheta_2(t)}^{t} \dot{e}^i(r)^T dr W \int_{t-\vartheta_2(t)}^{t} \dot{e}^i(r)\,dr + \int_{t-\vartheta_1(t)}^{t} \dot{e}^i(r)^T ds W \int_{t-\vartheta_1(t)}^{t} \dot{e}^i(r)\,dr\Big].
\end{aligned}$$

Let

$$\xi_1(t) = \mathrm{col}\left(\int_{t-\vartheta_2(t)}^{t} \dot{\bar{x}}^i(r)\,dr\right),$$

$$\xi_2(t) = \mathrm{col}\left(\int_{t-\vartheta_2(t)}^{t} \dot{e}^i(r)\,dr\right),$$

$$\xi_3(t) = \mathrm{col}\left(\int_{t-\vartheta_1(t)}^{t} \dot{\bar{x}}^i(r)\,dr\right),$$

$$\xi_4(t) = \mathrm{col}\left(\int_{t-\vartheta_1(t)}^{t} \dot{e}^i(r)\,dr\right),$$

$$\xi_5(t) = \mathrm{col}\left(\int_{t-\iota(t)}^{t} \dot{\bar{x}}^i(r)\,dr\right),$$

we have

$$\dot{U}_1 \leqslant -\frac{1}{3a_1}\mathrm{col}\left(\int_{t-\vartheta_2}^{t}\dot{\bar{x}}^i(r)dr\right)^T(I_N\otimes W)\mathrm{col}\left(\int_{t-\vartheta_2}^{t}\dot{\bar{x}}^i(r)dr\right)$$
$$-\frac{1}{3a_1}\mathrm{col}\left(\int_{t-\vartheta_1}^{t}\dot{\bar{x}}^i(r)dr\right)^T(I_N\otimes W)\mathrm{col}\left(\int_{t-\vartheta_1}^{t}\dot{\bar{x}}^i(r)dr\right)$$
$$-\frac{1}{3a_1}\mathrm{col}\left(\int_{t-\iota}^{t}\dot{\bar{x}}^i(r)dr\right)^T(I_N\otimes W)\mathrm{col}\left(\int_{t-\iota}^{t}\dot{\bar{x}}^i(r)dr\right)$$
$$+\dot{\bar{x}}^T(I_N\otimes W)\bar{x}+\bar{x}^T(I_N\otimes W)\dot{\bar{x}}+\sum_{i=1}^{N}a_1\dot{\bar{x}}^{i^T}W\dot{\bar{x}}^i$$
$$\leqslant -\frac{1}{3a_1}\xi_1^T(I_N\otimes W)\xi_1-\frac{1}{3a_1}\xi_3^T(I_N\otimes W)\xi_3$$
$$-\frac{1}{3a_1}\xi_5^T(I_N\otimes W)\xi_5+[(I_N\otimes A)\bar{x}+(I_N\otimes BK)\tilde{\varepsilon}$$
$$-(I_N\otimes BK)\mathrm{col}\left(\int_{t-\vartheta_2}^{t}\dot{\tilde{\varepsilon}}^i(r)dr\right)+(I_N\otimes B_\omega)\omega]^T(I_N\otimes W)\bar{x}$$
$$+\bar{x}^T(I_N\otimes W)[(I_N\otimes A)\bar{x}+(I_N\otimes BK)\tilde{\varepsilon}$$
$$-(I_N\otimes BK)\mathrm{col}\left(\int_{t-\vartheta_2}^{t}\dot{\tilde{\varepsilon}}^i(r)dr\right)+(I_N\otimes B_\omega)\omega]$$
$$+\sum_{i=1}^{N}a_1[A\bar{x}^i+BK\tilde{\varepsilon}^i-BK\int_{t-\vartheta_2}^{t}\dot{\tilde{\varepsilon}}^i(r)dr+B_\omega\omega^i]^T$$
$$\times W[A\bar{x}^i+BK\tilde{\varepsilon}^i-BK\int_{t-\vartheta_2}^{t}\dot{\tilde{\varepsilon}}^i(r)dr+B_\omega\omega^i], \tag{32}$$

and

$$\dot{U}_2 \leqslant -\frac{1}{2a_1}\mathrm{col}\left(\int_{t-\vartheta_2}^{t}\dot{e}^i(r)dr\right)^T(I_N\otimes W)\mathrm{col}\left(\int_{t-\vartheta_2}^{t}\dot{e}^i(r)dr\right)$$
$$-\frac{1}{2a_1}\mathrm{col}\left(\int_{t-\vartheta_1}^{t}\dot{e}^i(r)dr\right)^T(I_N\otimes W)\mathrm{col}\left(\int_{t-\vartheta_1}^{t}\dot{e}^i(r)dr\right)$$
$$+\dot{e}^T(I_N\otimes W)e+e^T(I_N\otimes W)\dot{e}+\sum_{i=1}^{N}a_1\dot{e}^i(t)^TW\dot{e}^i(t)$$
$$\leqslant -\frac{1}{2a_1}\xi_2^T(I_N\otimes W)\xi_2-\frac{1}{2a_1}\xi_4^T(I_N\otimes W)\xi_4$$
$$+\Bigg[(I_N\otimes LC_2)\bar{x}-(I_N\otimes LC_2)\varepsilon+I_N\otimes(A-LC_2)e$$
$$-(I_N\otimes BK)\mathrm{col}\left(\int_{t-\vartheta_2}^{t}\dot{\tilde{\varepsilon}}^i(r)dr\right)-(I_N\otimes LC_2)\,\mathrm{col}$$
$$\times\left(\int_{t-\vartheta_1}^{t}\dot{\bar{x}}^i(r)dr\right)+(I_N\otimes LC_2)\mathrm{col}$$
$$\times\left(\int_{t-\iota}^{t}\dot{\varepsilon}^i(r)dr\right)-(I_N\otimes L)\epsilon_k+(I_N\otimes B_\omega)\omega\Bigg]^T$$
$$+(I_N\otimes W)e+e^T(I_N\otimes W)$$
$$\times\Bigg[(I_N\otimes LC_2)\bar{x}-(I_N\otimes LC_2)\varepsilon$$
$$+I_N\otimes(A-LC_2)e-(I_N\otimes BK)\mathrm{col}\left(\int_{t-\vartheta_2}^{t}\dot{\tilde{\varepsilon}}^i(r)dr\right)$$
$$-(I_N\otimes LC_2)\mathrm{col}\left(\int_{t-\vartheta_1}^{t}\dot{\bar{x}}^i(r)dr\right)$$
$$+(I_N\otimes LC_2)\mathrm{col}\left(\int_{t-\iota}^{t}\dot{\varepsilon}^i(r)dr\right)$$
$$-(I_N\otimes L)\epsilon_k+(I_N\otimes B_\omega)\omega\Bigg]+\sum_{i=1}^{N}a_1\Bigg[LC_2\bar{x}^i-LC_2\varepsilon^i$$
$$+(A-LC_2)e^i-BK\int_{t-\vartheta_2}^{t}\dot{\tilde{\varepsilon}}^i(r)dr+BK\int_{t-\vartheta_1}^{t}\dot{\tilde{\varepsilon}}^i(r)dr$$
$$-LC_2\int_{t-\vartheta_1}^{t}\dot{\bar{x}}^i(r)dr+LC_2\int_{t-\iota}^{t}\dot{\varepsilon}^i(r)dr$$
$$-L\epsilon_{\kappa^i}+B_\omega\omega^i\Bigg]^TW$$
$$\times\Bigg[LC_2\bar{x}^i-LC_2\varepsilon^i(A-LC_2)e^i-BK\int_{t-\vartheta_2}^{t}\dot{\tilde{\varepsilon}}^i(r)dr$$
$$+BK\int_{t-\vartheta_1}^{t}\dot{\tilde{\varepsilon}}^i(r)dr-LC_2\int_{t-\vartheta_1}^{t}\dot{\bar{x}}^i(r)dr$$
$$+LC_2\int_{t-\iota}^{t}\dot{\varepsilon}^i(r)dr-L\epsilon_{\kappa^i}+B_\omega\omega^i\Bigg]. \tag{33}$$

Note that

$$\mathrm{col}\left(\int_{t-\vartheta_2}^{t}\dot{\tilde{\varepsilon}}^i(r)dr\right)=(\mathcal{H}\otimes I_n)\mathrm{col}\left(\int_{t-\vartheta_2(t)}^{t}\dot{\tilde{x}}^i(r)dr\right)$$
$$=(\mathcal{H}\otimes I_n)\mathrm{col}\left(\int_{t-\vartheta_2}^{t}(\dot{\bar{x}}^i(r)-\dot{e}^i(r))dr\right)$$
$$=(\mathcal{H}\otimes I_n)(\xi_1-\xi_2), \tag{34}$$

$$\mathrm{col}\left(\int_{t-\vartheta_1}^{t}\dot{\tilde{x}}^i(r)dr\right)=(\mathcal{H}\otimes I_n)\mathrm{col}\left(\int_{t-\vartheta_1}^{t}(\dot{\bar{x}}^i(r)-\dot{e}^i(r))dr\right)$$
$$=\xi_3-\xi_4, \tag{35}$$

$$\operatorname{col}\left(\int_{t-\vartheta_1}^{t} \dot{\bar{\varepsilon}}^i(r)dr\right) = (\mathcal{H} \otimes I_n)\operatorname{col}\left(\int_{t-\vartheta_1(t)}^{t} \dot{\tilde{x}}^j(r)dr\right)$$
$$= (\mathcal{H} \otimes I_n)(\xi_3 - \xi_4), \tag{36}$$

and

$$\operatorname{col}\left(\int_{t-\iota}^{t} \dot{\varepsilon}^i(r)dr\right) = (\mathcal{H} \otimes I_n)\operatorname{col}\left(\int_{t-\iota(t)}^{t} \dot{\tilde{x}}^j(r)dr\right)$$
$$= (\mathcal{H} \otimes I_n)\xi_5(t). \tag{37}$$

Then, (32) and (33) can be rewritten as

$$\dot{U}_1 \leqslant -\frac{1}{3a_1}\xi_1^T(I_N \otimes W)\xi_1 - \frac{1}{3a_1}\xi_3^T(I_N \otimes W)\xi_3$$
$$- \frac{1}{3a_1}\xi_5^T(I_N \otimes W)\xi_5 + \zeta_1^T(G_1 + G_2)\zeta_1, \tag{38}$$

and

$$\dot{U}_2 \leqslant -\frac{1}{2a_1}\xi_2^T(I_N \otimes W)\xi_2 - \frac{1}{2a_1}\xi_4^T(I_N \otimes W)\xi_4$$
$$+ \zeta_2^T(G_3 + G_4)\zeta_2, \tag{39}$$

where

$$\zeta_1 = \left(\bar{x}^T e^T \ \xi_1^T \ \xi_2^T \ \omega^T\right)^T,$$
$$\zeta_2 = \left(e^T \ \bar{x}^T \ \xi_1^T \ \xi_2^T \ \xi_3^T \ \xi_4^T \ \xi_5^T \ \epsilon_\kappa^T \ \omega^T\right)^T,$$
$$G_1 = (G_{11} \ G_{12} \ G_{13} \ 0),$$
$$G_{11} = I_N \otimes (WA + A^T W) - \mathcal{H} \otimes WBK - \mathcal{H}^T \otimes K^T B^T W,$$
$$G_{12} = (\mathcal{H} \otimes WBK \ \mathcal{H} \otimes WBK \ -\mathcal{H} \otimes WBK \ I_N \otimes WB_\omega),$$
$$G_{13} = (\mathcal{H} \otimes WBK \ \mathcal{H} \otimes WBK \ -\mathcal{H} \otimes WBK \ I_N \otimes WB_\omega)^T,$$
$$G_2 = G_{21} a_1 (I_N \otimes W) G_{21}^T,$$
$$G_{21} = \begin{pmatrix} (I_N \otimes A - \mathcal{H} \otimes BK)^T \\ (\mathcal{H} \otimes BK)^T \\ (\mathcal{H} \otimes BK)^T \\ (-\mathcal{H} \otimes BK)^T \\ (I_N \otimes B_\omega)^T \end{pmatrix},$$
$$G_3 = \begin{pmatrix} G_{31} & G_{32} & G_{33} & G_{34} \\ G_{35} & 0 & 0 & 0 \\ G_{36} & 0 & 0 & 0 \\ G_{37} & 0 & 0 & 0 \end{pmatrix},$$
$$G_{31} = I_N \otimes (WA + A^T W - WLC_2 - C_2^T L^T W),$$
$$G_{32} = \begin{pmatrix} \mathcal{H}^T \otimes C_2^T L^T W + I_N \otimes C_2^T L^T W & \mathcal{H} \\ \otimes WBK - \mathcal{H} \otimes WBK & \end{pmatrix},$$
$$G_{33} = \begin{pmatrix} -(I_N \otimes WLC_2 + \mathcal{H} \otimes WBK) & I_N \otimes WLC_2 \\ & +\mathcal{H} \otimes WBK \end{pmatrix},$$
$$G_{34} = \left(-\mathcal{H} \otimes WLC_2 \quad -I_N \otimes WL \quad I_N \otimes WB_\omega\right),$$
$$G_{35} = \begin{pmatrix} \mathcal{H} \otimes WLC_2 + I_N \otimes WLC_2 \\ (\mathcal{H} \otimes WBK)^T \\ -(\mathcal{H} \otimes WBK)^T \end{pmatrix},$$
$$G_{36} = \begin{pmatrix} -(I_N \otimes WLC_2 + \mathcal{H} \otimes WBK)^T \\ (I_N \otimes WLC_2 + \mathcal{H} \otimes WBK)^T \end{pmatrix},$$
$$G_{37} = \begin{pmatrix} -(\mathcal{H} \otimes WLC_2)^T \\ -(I_N \otimes WL)^T \\ (I_N \otimes WB_\omega)^T \end{pmatrix}, \quad G_4 = G_{41} a_1 (I_N \otimes W) G_{41}^T,$$

$$G_{41} = \begin{pmatrix} I_N \otimes (A - LC_2)^T \\ \mathcal{H} \otimes LC_2 + I_N \otimes LC_2 \\ (\mathcal{H} \otimes BK)^T \\ -(\mathcal{H} \otimes BK)^T \\ -(I_N \otimes LC_2 + \mathcal{H} \otimes BK)^T \\ (I_N \otimes LC_2 + \mathcal{H} \otimes BK)^T \\ -(\mathcal{H} \otimes LC_2)^T \\ -(I_N \otimes L)^T \\ (I_N \otimes B_\omega)^T \end{pmatrix}.$$

Moreover, by (38) and (39), we have

$$\dot{U}(t) \leqslant \zeta_1^T(G_1 + G_2)\zeta_1 + \zeta_2^T(G_3 + G_4)\zeta_2$$
$$- \frac{1}{3a_1}\xi_1^T(t)(I_N \otimes W)\xi_1(t)$$
$$- \frac{1}{3a_1}\xi_3^T(t)(I_N \otimes W)\xi_3(t) - \frac{1}{2a_1}\xi_2^T(t)(I_N \otimes W)\xi_2(t)$$
$$- \frac{1}{2a_1}\xi_4^T(t)(I_N \otimes W)\xi_4(t) - \frac{1}{3a_1}\xi_5^T(t)(I_N \otimes W)\xi_5(t)$$
$$- \epsilon_\kappa(t)^T(I_N \otimes \Omega)\epsilon_\kappa(t)$$
$$+ \sigma[(\mathcal{H} \otimes C_2)\bar{x}(t) - (\mathcal{H} \otimes C_2)\xi_5(t)]^T(I_N \otimes \Omega)$$
$$\times [(\mathcal{H} \otimes C_2)\bar{x}(t) - (\mathcal{H} \otimes C_2)\xi_5(t)].$$

Define $U_3(t)$ as:

$$U_3(t) = \bar{y}_1^i(t)^T \bar{y}_1^i(t) - \gamma^2 \omega^i(t)^T \omega^i(t), \tag{40}$$

that is,

$$U_3(t) = \bar{x}^T(t)(I_N \otimes C_1^T C_1)\bar{x}(t) - \gamma^2 \omega^T(t)\omega(t). \tag{41}$$

From (38) to (41), we have

$$\dot{U}(t) + U_3(t) \leqslant \zeta^T \left[\Sigma + \begin{pmatrix} \Gamma_1^T & \Gamma_2^T & \Gamma_3^T \end{pmatrix} \Sigma_0 \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \end{pmatrix} \right] \zeta, \tag{42}$$

where

$$\zeta = \begin{pmatrix} \bar{x}^T & e^T & \xi_1^T & \xi_2^T & \omega^T & \xi_3^T & \xi_4^T & \xi_5^T & \epsilon_\kappa^T \end{pmatrix}^T,$$

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} & 0 \\ * & \Sigma_{22} & \Sigma_{23} \\ * & * & \Sigma_{33} \end{pmatrix}, \Sigma_{11} = G_{11} + I_N \otimes C_1^T C_1,$$

$$G_{11} = I_N \otimes (WA + A^T W) - \mathcal{H} \otimes WBK - \mathcal{H}^T \otimes K^T B^T W,$$

$$\Sigma_{12} = \begin{pmatrix} \Sigma_{13} & \Sigma_{14} \end{pmatrix},$$

$$\Sigma_{13} = \begin{pmatrix} \mathcal{H} \otimes WBK + \mathcal{H}^T \otimes C_2^T L^T W + I_N \otimes C_2^T L^T W \\ \mathcal{H} \otimes WBK \end{pmatrix},$$

$$\Sigma_{14} = \begin{pmatrix} -\mathcal{H} \otimes WBK & I_N \otimes WB_\omega \end{pmatrix},$$

$$\Sigma_{22} = \begin{pmatrix} G_{31} & \mathcal{H} \otimes WBK & -\mathcal{H} \otimes WBK & I_N \otimes WB_\omega \\ * & -\frac{1}{3a_1}(I_N \otimes W) & 0 & 0 \\ * & * & -\frac{1}{2a_1}(I_N \otimes W) & 0 \\ * & * & * & -\gamma^2 I_{Nn} \end{pmatrix},$$

$$G_{31} = I_N \otimes (WA + A^T W - WLC_2 - C_2^T L^T W),$$

$$\Sigma_{23} = \begin{pmatrix} \Sigma_{24} & \Sigma_{25} \end{pmatrix},$$

$$\Sigma_{24} = \begin{pmatrix} -(I_N \otimes WLC_2 + \mathcal{H} \otimes WBK) & I_N \otimes WLC_2 \\ +\mathcal{H} \otimes WBK & \\ 0_{9N\times 3N} & 0_{9N\times 3N} \end{pmatrix},$$

$$\Sigma_{25} = \begin{pmatrix} -\mathcal{H} \otimes WLC_2 & -I_N \otimes WL \\ 0_{9N\times 3N} & 0_{9N\times 3N} \end{pmatrix},$$

$$\Sigma_{33} = \begin{pmatrix} -a_1(I_N \otimes W) & 0 & 0 & 0 \\ 0 & -\frac{1}{2a_1}(I_N \otimes W) & 0 & 0 \\ 0 & 0 & -\frac{1}{3a_1}(I_N \otimes W) & 0 \\ 0 & 0 & 0 & -I_N \otimes \Omega \end{pmatrix},$$

$$\Sigma_0 = \begin{pmatrix} a_1 I_N \otimes W & 0 & 0 \\ 0 & a_1 I_N \otimes W & 0 \\ 0 & 0 & \sigma I_N \otimes \Omega \end{pmatrix},$$

$$\Gamma_1 = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \end{pmatrix},$$

$$\Gamma_{11} = \big(I_N \otimes A - \mathcal{H} \otimes BK\ \mathcal{H} \otimes BK + \mathcal{H}^T \otimes C_2^T L^T + I_N \otimes C_2^T L^T\big),$$

$$\Gamma_{12} = \begin{pmatrix} \mathcal{H} \otimes BK & -\mathcal{H} \otimes BK & I_N \otimes B_\omega & 0_{3N\times 12N} \end{pmatrix},$$

$$\Gamma_2 = \begin{pmatrix} \Gamma_{21} & \Gamma_{22} & \Gamma_{23} \end{pmatrix},$$

$$\Gamma_{21} = \begin{pmatrix} 0_{3N\times 3N} & I_N \otimes (A - LC_2) & \mathcal{H} \otimes BK \end{pmatrix},$$

$$\Gamma_{22} = \begin{pmatrix} -\mathcal{H} \otimes BK & I_N \otimes B_\omega & -(I_N \otimes LC_2 + \mathcal{H} \otimes BK) \end{pmatrix},$$

$$\Gamma_{23} = \begin{pmatrix} I_N \otimes LC_2 + \mathcal{H} \otimes BK & -\mathcal{H} \otimes LC_2 & -I_N \otimes L \end{pmatrix},$$

$$\Gamma_3 = \begin{pmatrix} \mathcal{H} \otimes C_2 & 0_{3N\times 18N} & -\mathcal{H} \otimes C_2 & 0_{3N\times 3N} \end{pmatrix},$$

$$a_1 = (m_1 + m_2 + l)h.$$

From Lemma 2, the condition (27) is equivalent to

$$\Sigma + \begin{pmatrix} \Gamma_1^T & \Gamma_2^T & \Gamma_3^T \end{pmatrix} \Sigma_0 \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \end{pmatrix} < 0,$$

which implies that

$$\dot{U} + \bar{y}_1^T \bar{y}_1 - \gamma^2 \omega^T \omega < 0, t \in \mathcal{I}^{\kappa^i}. \tag{43}$$

In the case of $t \in [t_0^i h, t_0^i h + \bar{\iota} + \bar{\varsigma})$, the derivative of $U_1(t)$ is given by

$$\begin{aligned} \dot{U}_1 &= \dot{\bar{x}}^T(I_N \otimes W)\bar{x} + \bar{x}^T(I_N \otimes W)\dot{\bar{x}} \\ &+ \sum_{i=1}^{N} a_1 (\dot{\bar{x}}^i)^T W \dot{\bar{x}}^i - \sum_{i=1}^{N} \int_{t-a_1}^{t} \dot{\bar{x}}^i(r)^T W \dot{\bar{x}}^i(r) dr \\ &\leqslant \zeta_0^T (G_{0_1} + G_{0_2})\zeta_0 - \left(\mathrm{col}\left(\int_{t_0^i h}^{t} \dot{\varepsilon}^i(s) ds \right) \right)^T \\ &\times a_1 (I_N \otimes W) \mathrm{col}\left(\int_{t_0^i h}^{t} \dot{\varepsilon}^i(r) dr \right). \end{aligned} \tag{44}$$

From (38) and (44), we have

$$\dot{U}_1(t) + U_3(t) \leqslant \zeta_0^T (\Sigma_1 + G_{0_2})\zeta_0, \tag{45}$$

where

$$\zeta_0 = \left(\bar{x}^T \quad \mathrm{col}\left(\int_{t_0^i h}^{t} \dot{\bar{x}}^i(s) ds \right)^T \quad \omega^T \right)^T,$$

$$G_{0_1} = \begin{pmatrix} G_{11} & \mathcal{H} \otimes WBK & I_N \otimes WB_\omega \\ (\mathcal{H} \otimes WBK)^T & 0 & 0 \\ (I_N \otimes WB_\omega)^T & 0 & 0 \end{pmatrix},$$

$$G_{11}=I_N\otimes(WA+A^TW)-\mathcal{H}\otimes WBK-\mathcal{H}^T\otimes K^TB^TW,$$

$$G_{0_2}=\begin{pmatrix}(I_N\otimes A-\mathcal{H}\otimes BK)^T\\(\mathcal{H}\otimes BK)^T\\(I_N\otimes B_\omega)^T\end{pmatrix}a_1(I_N\otimes W)\begin{pmatrix}(I_N\otimes A-\mathcal{H}\otimes BK)^T\\(\mathcal{H}\otimes BK)^T\\(I_N\otimes B_\omega)^T\end{pmatrix}^T,$$

$$\Sigma_1=\begin{pmatrix}G_{11} & \mathcal{H}\otimes WBK & I_N\otimes WB_\omega\\(\mathcal{H}\otimes WBK)^T & -\frac{1}{a_1}(I_N\otimes W) & 0\\(I_N\otimes WB_\omega)^T & 0 & -\gamma^2I_{Nn}\end{pmatrix}.$$

The condition (27) implies that the matrix

$$\begin{pmatrix}G_{11}+I_N\otimes C_1^TC_1 & \mathcal{H}\otimes WBK & I_N\otimes WB_\omega & (I_N\otimes WA-\mathcal{H}\otimes WBK)^T\\ * & -\frac{1}{a_1}(I_N\otimes W) & 0 & (\mathcal{H}\otimes WBK)^T\\ * & * & -\gamma^2I_{Nn} & I_N\otimes B_\omega^TW\\ * & * & * & -\frac{1}{a_1}(I_N\otimes W)\end{pmatrix}$$

is negative definite.

Therefore, from (45) and the above matrix, one has

$$\dot{U}_1+U_3<0, t\in[t_0^ih, t_0^ih+\bar{\iota}+\bar{\varsigma}).$$

Since $\bigcup_{k=0}^{\infty}[t_k^ih+\bar{\iota}+\bar{\varsigma}, t_{k+1}^ih+\bar{\iota}+\bar{\varsigma})\bigcup[t_0^ih, t_0^ih+\bar{\iota}+\bar{\varsigma})=[t_0,+\infty)$, and $\bar{x}(t), e(t)$ are continuous on $[t_0,+\infty)$, thus, $U(t)$ is continuous on $[t_0,+\infty)$. If $\omega(t)=0$, we get

$$\dot{U}(t)+\bar{y}_1(t)^T\bar{y}_1(t)<0,$$

Therefore, when the disturbance vanishes, the CLS is asymptotically stable. Furthermore, $\lim_{t\to\infty}\bar{x}^i(t)=0$.

Since $U(t)$ is continuous on $[t_0,+\infty)$, integrating the inequality (43) from t_0 to t yields

$$U(t)-U(t_0)<-\int_{t_0}^t\bar{y}_1(r)^T\bar{y}_1(r)dr+\gamma^2\int_{t_0}^t\omega(r)^T\omega(r)dr.$$

Using the 0 initial condition and when $t\to\infty$, one has

$$\int_0^\infty\bar{y}_1(r)^T\bar{y}_1(r)dr<\gamma^2\int_0^\infty\omega(r)^T\omega(r)dr.$$

Thus, $\|\bar{y}_1(t)^T\bar{y}_1(t)\|_2\leqslant\gamma\|\omega(t)^T\omega(t)\|_2$. □

The matrix inequality (27) with respect to W, L and K is not solvable. In the following theorem, we transform the matrix inequality (27) into an LMI-based feasible problem.

Theorem 1. *There exists an H_∞ consensus OFC (7) for system (1) and (2), if there exist matrices $\bar{L}$, $\bar{K}$, $W>0$ and $\Omega>0$, and given constants $h>0$ and $\sigma>0$ such that*

$$\begin{pmatrix}\bar{\Sigma} & \bar{\Gamma}_1^T & \bar{\Gamma}_2^T & \Gamma_3^T(I_N\otimes\Omega)\\ * & -\frac{1}{a_1}(I_N\otimes W) & 0 & 0\\ * & * & -\frac{1}{a_1}(I_N\otimes W) & 0\\ * & * & * & -\frac{1}{\sigma}(I_N\otimes\Omega)\end{pmatrix}<0, \tag{46}$$

where

$$\bar{\Sigma}=\begin{pmatrix}\bar{\Sigma}_{11} & \bar{\Sigma}_{12} & 0\\ * & \bar{\Sigma}_{22} & \bar{\Sigma}_{23}\\ * & * & \Sigma_{33}\end{pmatrix}, \bar{\Sigma}_{11}=\bar{G}_{11}+I_N\otimes C_1^TC_1,$$

$$\bar{G}_{11}=I_N\otimes(WA+A^TW)-\mathcal{H}\otimes\bar{K}-\mathcal{H}^T\otimes\bar{K}^T,$$

$$\bar{\Sigma}_{12}=\begin{pmatrix}\mathcal{H}\otimes\bar{K}+\mathcal{H}^T\otimes C_2^T\bar{L}^T+I_N\otimes C_2^T\bar{L}^T & \mathcal{H}\otimes\bar{K}\\ -\mathcal{H}\otimes\bar{K} & I_N\otimes WB_\omega\end{pmatrix},$$

$$\bar{\Sigma}_{22}=\begin{pmatrix}\bar{G}_{31} & \mathcal{H}\otimes\bar{K} & -\mathcal{H}\otimes\bar{K} & I_N\otimes WB_\omega\\ * & -\frac{1}{3a_1}(I_N\otimes W) & 0 & 0\\ * & * & -\frac{1}{2a_1}(I_N\otimes W) & 0\\ * & * & * & -\gamma^2I_{Nn}\end{pmatrix},$$

$$G_{31}=I_N\otimes(WA+A^TW-\bar{L}C_2-C_2^T\bar{L}^T),$$

$$\bar{\Sigma}_{23}=\begin{pmatrix}\bar{\Sigma}_{24} & \bar{\Sigma}_{25}\end{pmatrix},$$

$$\bar{\Sigma}_{24}=\begin{pmatrix}-(I_N\otimes\bar{L}C_2+\mathcal{H}\otimes\bar{K}) & I_N\otimes\bar{L}C_2+\mathcal{H}\otimes\bar{K}\\ 0_{9N\times3N} & 0_{9N\times3N}\end{pmatrix},$$

$$\bar{\Sigma}_{25}=\begin{pmatrix}-\mathcal{H}\otimes\bar{L}C_2 & -I_N\otimes\bar{L}\\ 0_{9N\times3N} & 0_{9N\times3N}\end{pmatrix},$$

$$\Sigma_{33}=\begin{pmatrix}-\frac{1}{a_1}(I_N\otimes W) & 0 & 0 & 0\\ 0 & -\frac{1}{2a_1}(I_N\otimes W) & 0 & 0\\ 0 & 0 & -\frac{1}{3a_1}(I_N\otimes W) & 0\\ 0 & 0 & 0 & -I_N\otimes\Omega\end{pmatrix},$$

and

$$\bar{\Gamma}_1=(\bar{\Gamma}_{11}\ \bar{\Gamma}_{12}),$$

$$\bar{\Gamma}_{11}=\big(I_N\otimes WA-\mathcal{H}\otimes\bar{K}\ \mathcal{H}\otimes\bar{K}+\mathcal{H}^T\otimes C_2^T\bar{L}^T$$
$$+I_N\otimes C_2^T\bar{L}^T\big),$$

$$\Gamma_{12}=\big(\mathcal{H}\otimes\bar{K}\quad -\mathcal{H}\otimes\bar{K}\quad I_N\otimes WB_\omega\quad 0_{3N\times 12N}\big),$$

$$\bar{\Gamma}_2=\big(\bar{\Gamma}_{21}\quad\bar{\Gamma}_{22}\quad\bar{\Gamma}_{23}\big),$$

$$\bar{\Gamma}_{21}=\big(0_{3N\times 3N}\quad I_N\otimes(WA-\bar{L}C_2)\quad \mathcal{H}\otimes\bar{K}\big),$$

$$\bar{\Gamma}_{22}=\big(-\mathcal{H}\otimes\bar{K}\quad I_N\otimes WB_\omega\quad -(I_N\otimes\bar{L}C_2+\mathcal{H}\otimes\bar{K})\big),$$

$$\bar{\Gamma}_{23}=\big(I_N\otimes\bar{L}C_2+\mathcal{H}\otimes\bar{K}\quad -\mathcal{H}\otimes\bar{L}C_2\quad -I_N\otimes\bar{L}\big),$$

$$\Gamma_3=\big(\mathcal{H}\otimes C_2\quad 0_{3N\times 18N}\quad -\mathcal{H}\otimes C_2\quad 0_{3N\times 3N}\quad\big).$$

Under this setting, the control gain and the observer gain are $K=(B^TB)^{-1}B^TW^{-1}\bar{K}$ *and* $L=W^{-1}\bar{L}$, *respectively.*

Proof. Let $\bar{K}=WBK$ and $\bar{L}=WL$. Then, we can obtain (46) from (27). □

Remark 6. The sufficient condition proposed in Theorem 1 for H_∞ consensus achieving is based on LMI. For LMI based algorithm, how to reduce its conservatism is an interesting topic worthy of further investigation in the future.

5 | NUMERICAL SIMULATIONS

In this section, we give two examples to show the validity of results. Consider a MAS consisting of one leader and four followers shown in Figure 2. Choose the parametric matrices of the MAS as:

$$A=\begin{pmatrix}-2&0&0.3\\2&-3&0\\1&0&-2\end{pmatrix},\quad B=\begin{pmatrix}1\\2\\1\end{pmatrix},\quad B_\omega=\begin{pmatrix}1\\1\\1\end{pmatrix},$$

$$C_1=\begin{pmatrix}0.1&0&0.2\\0.1&0.2&0\\0&0.3&0.1\end{pmatrix},\quad C_2=\begin{pmatrix}1&0&1\\1&2&0\\0&1&3\end{pmatrix},$$

$$\omega(t)=\begin{cases}\sin t, & t\in[0,20],\\ 0, & \textit{otherwise}.\end{cases}$$

Case 1 : Select the parameters: $h=0.02,\sigma=0.2,m_1=2,m_2=1$, and $l=6$. By calculating, we get $\gamma=12.6356$, matrices K,L and Ω are

$$K=\big(0.0232\quad 0.0064\quad 0.0109\big),$$

$$L=\begin{pmatrix}-0.0770&-0.0244&0.0112\\0.0384&-0.0157&-0.0139\\-0.0183&-0.0008&-0.0131\end{pmatrix},$$

and

$$\Omega=\begin{pmatrix}15.6891&-1.5107&-4.4787\\-1.5107&4.2762&-0.7225\\-4.4787&-0.7225&3.4434\end{pmatrix}.$$

Case 2 : Select the parameters: $h=0.02,\sigma=0.3,m_1=2,m_2=1$, and $l=6$. By calculating, we get $\gamma=4.6603$, matrices K,L,Ω are

$$K=\big(0.0236\quad 0.0079\quad 0.0125\big),\tag{47}$$

$$L=\begin{pmatrix}-0.0713&-0.0229&0.0099\\0.0256&-0.0155&-0.0106\\-0.0229&-0.0008&-0.0107\end{pmatrix},\tag{48}$$

and

$$\Omega=\begin{pmatrix}1.5655&-0.1485&-0.4492\\-0.1485&0.3826&-0.0593\\-0.4492&-0.0593&0.3232\end{pmatrix}.\tag{49}$$

We choose the initial values of $x(0)$ and $\hat{x}(0)$ as

$$x(0)=(0.7,0.6,0.1,-0.4,0.2,0.3,0.5,-0.1,0.5,$$
$$-0.1,0.4,0.1,0.5,0.8,-0.1)^T,$$

and

$$\hat{x}(0)=(-0.2,0.7,0.4,0.2,-0.2,0.3,0.5,0.1,$$
$$-0.1,0.2,0.5,0.3,0.4,0.7,0.2)^T,$$

and the transmission delays ι_k^i and ς_k^i are randomly generated in the interval $[0,2h]$ and $[0,h]$ respectively. Fig-

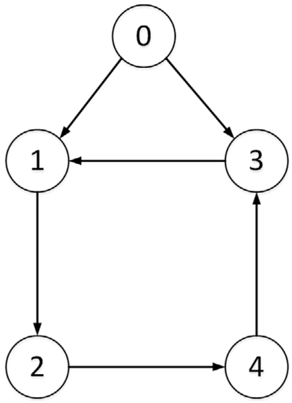

FIGURE 2 Connected graph

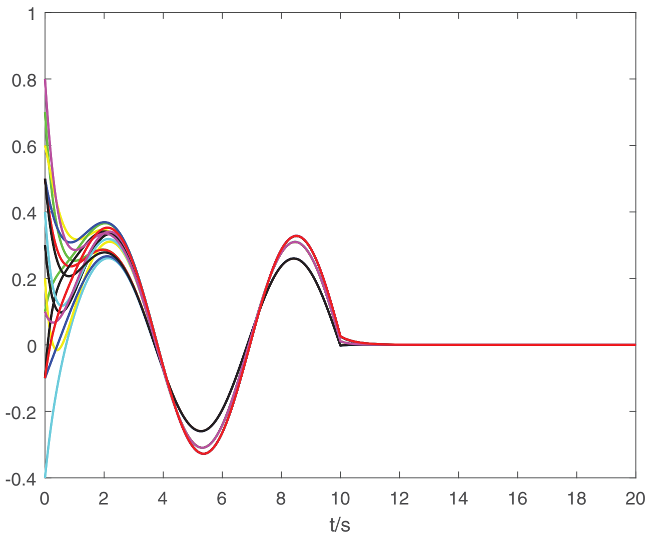

FIGURE 3 Trajectories of error dynamic

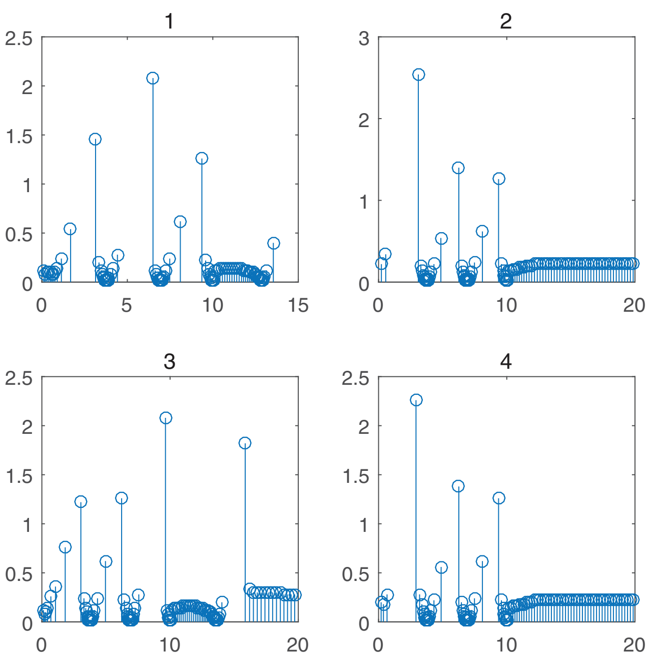

FIGURE 4 The corresponding release instants

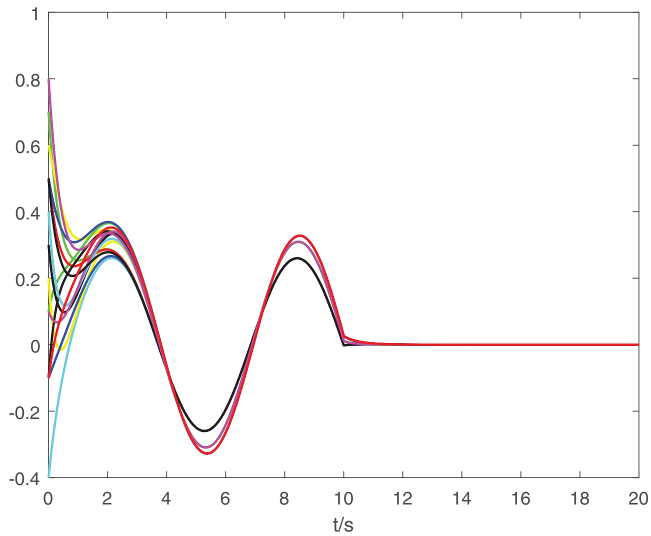

FIGURE 5 Trajectories of states

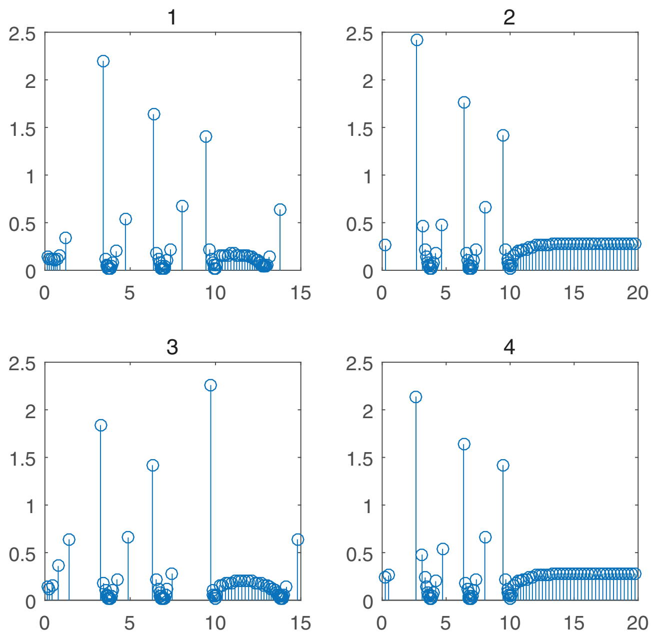

FIGURE 6 The corresponding release instants

ures 3 and 5 show the state trajectories of Case 1 and Case 2, respectively, while Figures 4 and 6 show the release time instants. The simulation results show that the Zeno-behaviour can be avoided and consensus can be reached asymptotically.

Remark 7. For multi-agent systems, computational complexity is an important problem we face when the number of agents is large. However, the LMI (46) in theorem 1 can be solved offline, and the event-triggered condition (3) and controller (8) are distributed only depending on local information exchange, which greatly reduce the computational burden when the number of agents is large.

6 | CONCLUSION

The consensus control of leader-following MASs is studied in this paper via event-triggered H_∞ consensus OFC. Due to taking two class of time-delay into account, the system and the output feedback controller have different update time intervals. By interval dividing, we obtain the CLS updated in the same time intervals. The event-triggered condition is adopted to reduce times of sampling and improve efficiency. Output feedback H_∞ control method is applied such that leader-following consensus is reached. In the future, it is important to reduce the conservatism of the sufficient conditions.

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