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Brief paper

Continuous observer design for a class of multi-output nonlinear systems with multi-rate sampled and delayed output measurements[☆]

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ABSTRACT

In this paper, continuous observer is designed for a class of multi-output nonlinear systems with multi-rate sampled and delayed output measurements. The time delay may be larger or less than the sampling intervals. The sampled and delayed measurements are used to update the observer whenever they are available. Sufficient conditions are presented to ensure global exponential stability of the observation errors by constructing a Lyapunov–Krasovskii function. A numerical example is given to illustrate the effectiveness of the proposed methods.

1. Introduction

Recently, the problem of design global convergent observers for nonlinear systems has made great progress. For the observation of nonlinear systems, one can use extended Luenberger observers (Zeitz, 1987), normal form observers (Bestle & Zeitz, 1983; Krener & Isidori, 1983; Xia & Gao, 1988, 1989), Lyapunov based observers (Raghavan & Hedrick, 1994; Thau, 1973), high-gain observers (Gauthier, Hammouri, & Othman, 1992; Gauthier & Kupka, 1994), sliding mode observers (Haskara, Özgüner, & Utkin, 1998) and moving horizon/optimization based observers (Michalska & Mayne, 1995). Among these methods, high-gain observers play an important role and can be used to a large class of nonlinear systems with a triangular structure after a coordinate change. New developments of high gain observers have been carried out in various directions (Andrieu, Praly, & Astolfi, 2009; Deza, 1991; Deza, Bossanne, Busvelle, Gauthier, & Rakotopara, 1993; Gauthier et al., 1992; Praly, 2003). For example, the result of Gauthier et al. (1992) is extended to a class of nonlinear systems where the nonlinear terms admit an incremental rate depending on the measured output (Praly, 2003). In Deza (1991), the authors considered observer design for multi-input and multi-output (MIMO) nonlinear systems. The result has been extended to a class of MIMO nonlinear systems, in which interconnection between the blocks are not allowed (Deza et al., 1993). Based on the observer normal form, another extension for the multi-output systems has been studied in Rudolph and Zeitz (1994). However, the nonlinearity of each block does not allow the unmeasurable states of its own block. Under the conditions of observability and triangular structure, a nonlinear system can be transformed into the block low triangular form considered in Shim, Son, and Seo (2001) by a coordinate transformation. Then, semi-global observer has been designed for nonlinear systems with interconnections between the subsystems (Shim et al., 2001). The nonlinear system with block lower triangular form is rather general when nonlinear changes of coordinates are allowed. It includes the control-affine multi-input and single-output (MISO) nonlinear systems which are strongly observable for any input (Gauthier et al., 1992) and the control-affine MIMO nonlinear systems which are strongly observable for any input for each output taken separately (Deza, 1991). Moreover, it can be used to express some physical systems. For example, the dynamical equations of a permanent magnet stepper motor can be transformed into the block lower triangular form Mahmoud and Khalil (2002). The estimation errors can converge to the origin in finite-time by using high gain observers

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in conjunction with applications of geometric homogeneity and Lyapunov theories (Li, Xia, & Shen, 2013; Shen & Huang, 2009; Shen & Xia, 2008).

It should be noted that the above results on observer design are based on continuous-time analysis. However, for a networked control system, the output is only available at discrete-time instants since it is usually transmitted through a shared band-limited digital communication network. Therefore, observer design for continuous systems with sampled and delayed output measurements has attracted the control community wide attention. There exist three main approaches to design observer for continuous systems with sampled and delayed measurements, for example, discrete time analysis based on a discretized model (Arcak & Nešić, 2004; Barbot, Monaco, & Normand-Cyrot, 1999; Nešić, Teel, & Kokotović, 1999), continuous time analysis followed by discretization (Khalil, 2004; Nešić & Teel, 2004; Postoyan & Nešić, 2012; Wang, Nešić, & Postoyan, 2015), and a mixed continuous and discrete time analysis without discretization (Ahmed-Ali & Lamnabhi-Lagarrigue, 2012; Ahmed-Ali, Van Assche, Massieu, & Dorléans, 2013; Deza, Busvelle, Gauthier, & Rakotopora, 1992; Karafyllis & Kravaris, 2009; Nadri, Hammouri, & Grajales, 2013; Raff et al. Raff, Kögel, & Allgöwer, 2008; Van Assche, Ahmed-Ali, Ham, & Lamnabhi-Lagarrigue, 2011; Zhang, Shen, & Xia, 2014). More specifically, two classes of global exponential observers have been presented for a class of continuous systems with sampled and delayed measurements in Ahmed-Ali, Van Assche et al. (2013). By using the same methods, exponential convergent observers were proposed for nonlinear systems with sampled and delayed measurements in Ahmed-Ali, Karafyllis, and Lamnabhi-Lagarrigue (2013). The observers designed in Ahmed-Ali, Karafyllis et al. (2013) and Ahmed-Ali, Van Assche et al. (2013) are in essence discontinuous. The authors in Zhang et al. (2014) proposed a continuous observer for a class of nonlinear systems with sampled and delayed measurements based on an auxiliary integral technique. But there is a constraint condition on time delay, that is the maximum delay must be less than the minimum sampling interval as in Ahmed-Ali, Karafyllis et al. (2013) and Ahmed-Ali, Van Assche et al. (2013).

In this paper, we address continuous observer design for a class of multi-output nonlinear systems with multi-rate sampled and delayed output measurements. The considered nonlinear systems are in continuous time while the outputs are in discrete time. In order to overcome the difficulties in analysis, we represent the sampled-data system as a continuous time system with successive delay components by some transformations. The time delays are more general than those in Ahmed-Ali, Karafyllis et al. (2013), Ahmed-Ali, Van Assche et al. (2013) and Zhang et al. (2014) since they may be larger or smaller than the sampling periods. Our main contributions include the following: (a) Continuous observer is designed for a class of multi-output nonlinear systems whenever the sampled and delayed measurements are available. (b) The observer is transformed into a continuous nonlinear system with time-varying delay by time delay method. Then, by constructing a Lyapunov–Krasovskii function, sufficient conditions are presented to ensure that the observation errors are globally exponentially stable. (c) Different high gains are used to dominate the nonlinear terms in each block. Then, upper bounds on each sampling period and time delay are also achieved.

This paper is organized as follows. In Section 2, continuous observers are presented for a class of multi-output nonlinear systems with multi-rate sampled and time delayed measurements. In Section 3, an example is used to illustrate the validity of the proposed design methods. Finally, Section 4 concludes the paper.

Throughout this paper, let $\mathbb{R}^n$ denote n-dimension real space, I denote an identity matrix, diag{} denote a diagonal matrix, and the superscript "$\top$" stand for matrix transposition. For any $x \in \mathbb{R}^n$, let $\|x\| = (x^\top x)^{1/2}$. For a continuous function $f : \mathbb{R} \to \mathbb{R}$ and $t \in \mathbb{R}$, let $\lim_{s\to t^-} f(s) = \lim_{s\to t, s<t} f(s)$. $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote the largest and the smallest eigenvalues of $P \in \mathbb{R}^{n\times n}$, respectively.

2. Main results

In this section, we consider the following multi-output nonlinear systems:

$$\begin{cases} \dot{x}(t) = Ax(t) + B(x(t), u(t)), \\ y(t) = Cx(t) = [C_1x^1(t), \ \dots \ C_mx^m(t)]^\top, \end{cases} \tag{1}$$

where the state $x(t) \in \mathbb{R}^n$, the input $u(t) \in \mathbb{R}^p$, the output $y(t) \in \mathbb{R}^m$, $x(t) = [x^1(t)^\top, \dots, x^m(t)^\top]^\top$, $x^i(t) \in \mathbb{R}^{\lambda_i}$ $(1 \le i \le m)$ is the ith partition of the state $x(t)$; $A = \mathrm{diag}\{A_1, \dots, A_m\}$, A_i is $\lambda_i \times \lambda_i$ matrix of Brunovsky form, that is $A_i = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}$, $C = \mathrm{diag}\{C_1, \dots, C_m\}$, $C_i = [1, 0, \dots, 0]_{1\times\lambda_i}$, and $B(x(t), u(t)) = [b^1(x(t), u(t))^\top, \dots, b^m(x(t), u(t))^\top]^\top$ in which the jth element of $b^i(\cdot)$, $b^i_j(\cdot)$ has the following structural dependence on the states:

$$b^i_j(t) = b^i_j(x^1(t), \dots, x^{i-1}(t); x^i_1(t), \dots, x^i_j(t); u(t)),$$

for all $1 \le i \le m$ and $1 \le j \le \lambda_i$. Thus, b^i_j is independent of the lower states $(x^i_{j+1}, \dots, x^i_{\lambda_i})$ of the ith block and the states of the lower blocks $(x^{i+1}, \dots, x^m)$. The ith block of the above system can be expressed as follows:

$$\begin{cases} \dot{x}^i_1(t) = x^i_2(t) + b^i_1(x(t)^{[1,i-1]}; x^i_1(t); u(t)), \\ \quad\vdots \\ \dot{x}^i_{\lambda_i-1}(t) = x^i_{\lambda_i}(t) + b^i_{\lambda_i-1}(x(t)^{[1,i-1]}; x(t)^i_{[1,\lambda_i-1]}; u(t)), \\ \dot{x}^i_{\lambda_i}(t) = b^i_{\lambda_i}(x(t)^{[1,i-1]}; x(t)^i_{[1,\lambda_i]}; u(t)), \end{cases} \tag{2}$$

where $x^i_j(t)$ is the jth element of the ith block $x^i(t)$. The abbreviation $x(t)^{[1,k]} := [x^1(t)^\top, \dots, x^k(t)^\top]^\top$ and $x(t)^i_{[1,j]} := [x^i_1(t), \dots, x^i_j(t)]^\top$ can be used to simplify the notation. We assume that there are m sensors in m channels to sample the output y at sampling instants t^i_k, and $t^i_k < t^i_{k+1}$ ($i = 1, \dots, m$ and $k = 0, 1, 2, \dots, \infty$), where $\{t^i_k\}$ ($i = 1, \dots, m$) are strictly increasing sequences and satisfy that $\lim_{k\to\infty} t^i_k = \infty$. The sampled measures are available at instants $t^i_k + \tau^i_k$ ($i = 1, \dots, m$), where $\tau^i_k > 0$ ($i = 1, \dots, m$) denote the transmission delay, which are unknown but have an upper bound $\bar{\tau}_i$. The nonlinear terms $b^i_j(\cdot)$ are assumed to satisfy the following global Lipschitz conditions with Lipschitz constant $l_1 > 0$,

$$\begin{aligned} &\big|b^i_j(x^1, \dots, x^{i-1}; x^i_1, \dots, x^i_j; u) - b^i_j(\hat{x}^1, \dots, \hat{x}^{i-1}; \hat{x}^i_1, \\ &\quad \dots, \hat{x}^i_j; u)\big| \le l_1\big(|x^1_1 - \hat{x}^1_1| + |x^1_2 - \hat{x}^1_2| + \cdots |x^i_j - \hat{x}^i_j|\big), \\ &\quad 1 \le i \le m, \ 1 \le j \le \lambda_i. \end{aligned} \tag{3}$$

Now, the explicit form of the ith block of the observer is given as follows:

$$\begin{cases} \dot{\hat{x}}^i_1(t) = \hat{x}^i_2(t) + L_i a^i_1 e^i_1(t^i_k) + b^i_1(\hat{x}(t)^{[1,i-1]}; \hat{x}^i_1(t); u(t)), \\ \quad\vdots \\ \dot{\hat{x}}^i_{\lambda_i-1}(t) = \hat{x}^i_{\lambda_i}(t) + L_i^{\lambda_i-1} a^i_{\lambda_i-1} e^i_1(t^i_k) \\ \quad + b^i_{\lambda_i-1}(\hat{x}(t)^{[1,i-1]}; \hat{x}(t)^i_{[1,\lambda_i-1]}; u(t)), \\ \dot{\hat{x}}^i_{\lambda_i}(t) = L_i^{\lambda_i} a^i_{\lambda_i} e^i_1(t^i_k) + b^i_{\lambda_i}(\hat{x}(t)^{[1,i-1]}; \hat{x}(t)^i_{[1,\lambda_i]}; u(t)), \\ \hat{x}^i_j(t^i_{k+1} + \tau^i_{k+1}) = \lim\limits_{t\to t^i_{k+1}+\tau^i_{k+1}{}^-} \hat{x}^i_j(t), \\ \quad j = 1, 2, \dots, \lambda_i, \ t \in [t^i_k + \tau^i_k, t^i_{k+1} + \tau^i_{k+1}), \ k \ge 0, \end{cases} \tag{4}$$

where $\hat{x}_j^i(t) = \hat{x}_{j0}^i$ for $t \in [t_0, t_0 + \tau_0^i]$ $(t_0 = t_0^i)$, $i = 1, \ldots, m$ and $j = 1, \ldots, \lambda_i$, $e_1^i(t_k^i) = x_1^i(t_k^i) - \hat{x}_1^i(t_k^i)$, $L_i \geq 1$ and a_j^i $(1 \leq i \leq m, 1 \leq j \leq \lambda_i)$ are positive real numbers, and will be given later. The definition of global exponential stable observer for the system (2) is given as follows.

Definition 1. We say that the system (4) is a *global exponential stable observer* for the system (2), if there exist a non-decreasing function $N : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ and a positive constant κ such that $\|\hat{x}(t) - x(t)\| \leq \exp(-\kappa(t - t_0))N(\|x_0\|, \|\hat{x}_0\|)$ for any $x_0 \in \mathbb{R}^n$, $\hat{x}_0 \in \mathbb{R}^n$.

Remark 1. The outputs y_i $(i = 1, \ldots, m)$ are transmitted through m channels, respectively. We can use m sensors to detect them. Therefore, although τ_k^i are unknown, we can obtain the instant that the sampled data at instants t_k^i is available. In other word, $e_1^i(t_k^i)$ is updated automatically whenever the sampled and delayed measurement $y_i(t_k^i)$ arrives.

From (2) and (4), the dynamics of the state error can be obtained:

$$\begin{cases} \dot{e}_1^i(t) = e_2^i(t) - L_i a_1^i e_1^i(t) + L_i a_1^i \displaystyle\int_{t_k^i}^{t} \dot{e}_1^i(s)ds + \tilde{b}_1^i, \\ \quad \vdots \\ \dot{e}_{\lambda_i-1}^i(t) = e_{\lambda_i}^i(t) - L_i^{\lambda_i-1} a_{\lambda_i-1}^i e_1^i(t) \\ \quad + L_i^{\lambda_i-1} a_{\lambda_i-1}^i \displaystyle\int_{t_k^i}^{t} \dot{e}_1^i(s)ds + \tilde{b}_{\lambda_i-1}^i, \\ \dot{e}_{\lambda_i}^i(t) = -L_i^{\lambda_i} a_{\lambda_i}^i e_1^i(t) + L_i^{\lambda_i} a_{\lambda_i}^i \displaystyle\int_{t_k^i}^{t} \dot{e}_1^i(s)ds + \tilde{b}_{\lambda_i}^i, \\ e_j^i(t_{k+1}^i + \tau_{k+1}^i) = \lim\limits_{t \to {t_{k+1}^i + \tau_{k+1}^i}^-} e_j^i(t), \\ \quad j = 1, \ldots, \lambda_i, \ t \in [t_k^i + \tau_k^i, t_{k+1}^i + \tau_{k+1}^i), \ k \geq 0, \end{cases} \tag{5}$$

where $e = [e^1(t)^\top, \ldots, e^m(t)^\top]^\top$, $e^i(t) = [e_1^i(t), \ldots, e_{\lambda_i}^i(t)]^\top$, $e_j^i(t) = x_j^i(t) - \hat{x}_j^i(t)$, $\tilde{b}_j^i = b_j^i(x(t)^{[1,i-1]}; x(t)_{[1,j]}^i; u(t)) - b_j^i(\hat{x}(t)^{[1,i-1]}; \hat{x}(t)_{[1,j]}^i; u(t))$, $(1 \leq i \leq m, 1 \leq j \leq \lambda_i)$.

Remark 2. Note that $\lim_{t \to t_{k+1}^i + \tau_{k+1}^i} e_j^i(t) = e_j^i(t_{k+1}^i + \tau_{k+1}^i)$, then $e^i(t)$ is continuous on $[t_k^i + \tau_k^i, t_{k+1}^i + \tau_{k+1}^i]$. On the other hand, the evolution process $e_1^i(t_k^i) = x_1^i(t_k^i) - \hat{x}_1^i(t_k^i)$ is updated at instants $t_k^i + \tau_k^i$, whereas the sampled measurement $y_i(t)$ is sampled at instants t_k^i. Therefore, the system (5) is continuous, delayed and hybrid in nature. Similar systems have been investigated in Ahmed-Ali, Van Assche et al. (2013), Karafyllis (2007a,b) and Karafyllis and Jiang (2007).

Let $\eta_i(t) = t - t_k^i$. Then, t_k^i in (5) can been expressed by

$$t_k^i = t - \eta_i(t). \tag{6}$$

Therefore, $0 < \eta_i(t) = t - t_k^i \leq t_{k+1}^i + \tau_{k+1}^i - t_k^i < h_i$, where $h_i > 0$. Our aim is to find the bounds of h_i such that the error system (5) is globally exponentially stable.

Consider the following change of coordinates $\varepsilon_j^i = \frac{e_j^i}{L_i^{\lambda_j^i - 1}}$, $1 \leq i \leq m$, $1 \leq j \leq \lambda_i$, where $\lambda_j^i = \Sigma_{k=1}^{i-1}\lambda_k + j$, $(1 \leq i \leq m, 1 \leq j \leq \lambda_i)$. Then,

$$\begin{cases} \dot{\varepsilon}_1^i(t) = L_i \varepsilon_2^i(t) - L_i a_1^i \varepsilon_1^i(t) \\ \quad + L_i a_1^i \displaystyle\int_{t-\eta_i(t)}^{t} \dot{\varepsilon}_1^i(s)ds + \frac{\tilde{b}_1^i}{L_i^{\lambda_1^i - 1}}, \\ \quad \vdots \\ \dot{\varepsilon}_{\lambda_i-1}^i(t) = L_i \varepsilon_{\lambda_i}^i(t) - L_i a_{\lambda_i-1}^i \varepsilon_1^i(t) \\ \quad + L_i a_{\lambda_i-1}^i \displaystyle\int_{t-\eta_i(t)}^{t} \dot{\varepsilon}_1^i(s)ds + \frac{\tilde{b}_{\lambda_i-1}^i}{L_i^{\lambda_{\lambda_i-1}^i - 1}}, \\ \dot{\varepsilon}_{\lambda_i}^i(t) = -L_i a_{\lambda_i}^i \varepsilon_1^i(t) + L_i a_{\lambda_i}^i \displaystyle\int_{t-\eta_i(t)}^{t} \dot{\varepsilon}_1^i(s)ds \\ \quad + \frac{\tilde{b}_{\lambda_i}^i}{L_i^{\lambda_{\lambda_i}^i - 1}}, \quad i = 1, \ldots, m. \end{cases} \tag{7}$$

Remark 3. t_k^i denotes the sampling instant, and $\eta_i(t)$ is a time-varying delay with bound h_i. The transformation (6) is used to represent the error system (5) as a continuous time system (7) with successive delay components. The problem of continuous observer design can be solved based on this model.

Now, we give the following result for the system (2).

Theorem 1. *Consider the system (2) with the condition (3). If L_i satisfy $L_i > \max\{1, l_1, 8\lambda_{\lambda_i}^i l_1 \bar{p}_2^i, L_{i-1}\}$, and $a_j^i > 0$ $(1 \leq i \leq m, 1 \leq j \leq \lambda_i)$ are given such that there exists a symmetric positive definite matrix P such that*

$$\bar{A}^\top P + P\bar{A} \leq -I, \tag{8}$$

and

$$h_i < \min\left\{\frac{1}{4L_i(3 + \lambda_1^i)^2(1 + {a_1^i}^2)}, \frac{1}{16L_i\lambda_i\bar{\lambda}_i^2\bar{a}_i}, \frac{1}{2L_i\sqrt{(3 + \lambda_1^i)a_1^i}}\right\}, \quad i = 1, \ldots, m, \tag{9}$$

then, the system (4) is a global exponential stable observer for the system (2), where $L_0 \geq 1$, $\bar{A} = diag\{\bar{A}_1, \ldots, \bar{A}_m\}$, $P = diag\{P_1, \ldots, P_m\}$, $\underline{\lambda}_i = \lambda_{\min}(P_i)$, $\bar{\lambda}_i = \lambda_{\max}(P_i)$, $\bar{\lambda} = \max_{\{1 \leq i \leq m\}}\{\bar{\lambda}_i\}$, $\bar{a}_i = \max_{\{1 \leq j \leq \lambda_i\}}\{(a_j^i)^2\}$, $\bar{p}_2^i = \max_{\{1 \leq j \leq \lambda_i, 1 \leq r \leq \lambda_i\}}\{|P_{j,r}^i|\}$, $(1 \leq i \leq m)$, $P_{j,r}^i$ is the element of P_i at the jth line and rth column, and $\bar{A}_i = \begin{bmatrix} -a_1^i & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{\lambda_i-1}^i & 0 & \cdots & 1 \\ -a_{\lambda_i}^i & 0 & \cdots & 0 \end{bmatrix}$.

Proof. Consider the positive definite function

$$V_1(t) = \varepsilon(t)^\top P \varepsilon(t) = \sum_{i=1}^{m} \varepsilon^i(t)^\top P_i \varepsilon^i(t),$$

where $\varepsilon(t) = [\varepsilon^1(t)^\top, \ldots, \varepsilon^m(t)^\top]^\top$, $\varepsilon^i(t) = [\varepsilon_1^i(t), \ldots, \varepsilon_{\lambda_i}^i(t)]^\top$, $(1 \leq i \leq m)$. Then, the derivative of $V_1(t)$ along the system (7) is given by

$$\frac{d}{dt}V_1(t)|_{(7)} = \sum_{i=1}^{m} L_i \varepsilon^i(t)^\top (\bar{A}_i^\top P_i + P_i\bar{A}_i)\varepsilon^i(t) + 2\sum_{i=1}^{m} L_i(a_1^i, a_2^i, \ldots, a_{\lambda_i}^i)\left(\int_{t-\eta_i(t)}^{t} \dot{\varepsilon}_1^i(s)ds\right) P_i \varepsilon^i(t)$$

$$
\begin{aligned}
&+2\sum_{i=1}^{m}\sum_{r=1}^{\lambda_i}\sum_{j=1}^{\lambda_i}\frac{\tilde{b}_j^i}{L_i^{\lambda_j^i-1}}\varepsilon_j^i(t)P_{j,r}^i\\
&\leq -\sum_{i=1}^{m}L_i\varepsilon^i(t)^\top\varepsilon^i(t)+\frac{1}{4}\sum_{i=1}^{m}L_i\varepsilon^i(t)^\top\varepsilon^i(t)\\
&+4\sum_{i=1}^{m}L_i(a_1^i,a_2^i,\ldots,a_{\lambda_i}^i)\\
&\times\int_{t-\eta_i(t)}^{t}\dot{\varepsilon}_1^i(s)dsP_iP_i(a_1^i,a_2^i,\ldots,a_{\lambda_i}^i)^\top\int_{t-\eta_i(t)}^{t}\dot{\varepsilon}_1^i(s)ds\\
&+2l_1\sum_{i=1}^{m}\sum_{r=1}^{\lambda_i}\sum_{j=1}^{\lambda_i}|\varepsilon_j^i(t)P_{j,r}^i|\times\left(\sum_{k=1}^{j}\sum_{a=1}^{\lambda_b}\sum_{b=1}^{i-1}|\varepsilon_a^b|+|\varepsilon_k^i|\right)\\
&\leq -\sum_{i=1}^{m}\left(\frac{3}{4}L_i-2\lambda_{\lambda_i}^i l_1\bar{p}_2^i\right)\varepsilon^i(t)^\top\varepsilon^i(t)\\
&+4\sum_{i=1}^{m}L_i\lambda_i\bar{\lambda}_i^2\bar{a}_i\left(\int_{t-\eta_i(t)}^{t}\dot{\varepsilon}_1^i(s)ds\right)^2.
\end{aligned}
$$

Note that $L_i > \{8\lambda_{\lambda_i}^i l_1\bar{p}_2^i\}$. Then, we have

$$
\begin{aligned}
\frac{d}{dt}V_1(t)|_{(7)} \leq& -\frac{1}{2}\sum_{i=1}^{m}L_i\varepsilon^i(t)^\top\varepsilon^i(t)+4\sum_{i=1}^{m}L_i\lambda_i\bar{\lambda}_i^2\bar{a}_i\\
&\times\left(\int_{t-\eta_i(t)}^{t}\dot{\varepsilon}_1^i(s)ds\right)^2.
\end{aligned}
\tag{10}
$$

By Lemma 1 in Gu (2000), we have

$$
\left|\int_{t-\eta_i(t)}^{t}\dot{\varepsilon}_1^i(s)ds\right|^2 \leq h_i\int_{t-h_i}^{t}\dot{\varepsilon}_1^i(s)^2ds. \tag{11}
$$

It follows from (10) and (11) that

$$
\begin{aligned}
\frac{d}{dt}V_1(t)|_{(7)} \leq& -\frac{1}{2}\sum_{i=1}^{m}L_i\varepsilon^i(t)^\top\varepsilon^i(t)+4\sum_{i=1}^{m}L_i\lambda_i\bar{\lambda}_i^2\bar{a}_ih_i\\
&\times\int_{t-h_i}^{t}\dot{\varepsilon}_1^i(s)^2ds.
\end{aligned}
\tag{12}
$$

Consider the following auxiliary integral function:

$$
V_2(t)=\sum_{i=1}^{m}\int_{t-h_i}^{t}\int_{\rho}^{t}\dot{\varepsilon}_1^i(s)^2dsd\rho,\quad t\geq t_0+\bar{h},
$$

where $\bar{h}=\max_{1\leq i\leq m}\{h_i\}$. We have,

$$
\begin{aligned}
\frac{dV_2(t)}{dt} \leq& \sum_{i=1}^{m}L_i^2(3+\lambda_1^i)^2(1+{a_1^i}^2)h_i\varepsilon^i(t)^T\varepsilon^i(t)\\
&+\sum_{i=1}^{m}L_i^2(3+\lambda_1^i){a_1^i}^2h_i^2\int_{t-h_i}^{t}\dot{\varepsilon}_1^i(s)^2ds\\
&-\sum_{i=1}^{m}\int_{t-h_i}^{t}\dot{\varepsilon}_1^i(s)^2ds,\quad t\geq t_0+\bar{h},
\end{aligned}
$$

and

$$
V_2(t)\leq\sum_{i=1}^{m}h_i\int_{t-h_i}^{t}\dot{\varepsilon}_1^i(s)^2ds. \tag{13}
$$

Construct the following Lyapunov–Krasovskii function:

$$
V(t)=V_1(t)+V_2(t),\quad t\geq t_0+\bar{h}. \tag{14}
$$

From (12), (13) and (9), we have

$$
\begin{aligned}
\frac{dV(t)}{dt}|_{(7)} \leq& -\sum_{i=1}^{m}\left(\frac{1}{2}-(3+\lambda_1^i)^2(1+{a_1^i}^2)h_iL_i\right)\\
&\times L_i\varepsilon^i(t)^\top\varepsilon^i(t)+\sum_{i=1}^{m}(4L_i\lambda_i\bar{\lambda}_i^2\bar{a}_ih_i\\
&+(3+\lambda_1^i){a_1^i}^2h_i^2L_i^2-1)\int_{t-h_i}^{t}\dot{\varepsilon}_1^i(s)^2ds,\\
\leq& -\frac{\underline{L}}{4\bar{\lambda}}V(t),\quad t\geq t_0+\bar{h},
\end{aligned}
$$

where $\underline{L}=\min_{\{1\leq i\leq m\}}\{L_i\}$. Then, $V(t)\leq\exp(-\frac{\underline{L}}{4\bar{\lambda}}(t-t_0-\bar{h}))V(t_0+\bar{h})$, $t\geq t_0+\bar{h}$. Since the nonlinear terms in the system (2) and (4) satisfy the global Lipschitz conditions (3), then, the solutions of (2) and (4) exist and are continuous on $[t_0,t_0+\bar{h}]$. Therefore, there exists a non-decreasing function $N:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ such that $\|\hat{x}(t)-x(t)\|\leq\exp(-\frac{\underline{L}}{4\bar{\lambda}}(t-t_0-\bar{h}))N(\|x_0\|,\|\hat{x}_0\|)$ for any $x_0\in\mathbb{R}^n,\hat{x}_0\in\mathbb{R}^n$. Thus, the system (4) is a global exponential stable observer for the system (2).

3. Numerical simulation

In this section, we use an example to show the effectiveness of our high gain observer design for nonlinear systems with sampled and time delay measurements. Consider the following multi-output nonlinear system (Shim et al., 2001):

$$
\begin{cases}
\dot{x}_1(t)=x_2(t)+0.01u(t),\\
\dot{x}_2(t)=-x_1(t)+0.1(1-x_1^2(t))x_2(t)+0.1x_2(t)u(t),\\
\dot{x}_3(t)=x_4(t)+0.01x_2(t)x_3(t)\exp(u(t)),\\
\dot{x}_4(t)=-x_3(t)+0.1(1-x_3^2(t))x_4(t)+\dfrac{1}{1+(x_2(t)x_4(t))^2}u(t),\\
y_1(t)=x_1(t),\\
y_2(t)=x_3(t),
\end{cases}
$$

where $x(t)=(x_1(t),x_2(t),x_3(t),x_4(t))^T$, which is in the form of (1) with $m=2$ and $x^1(t)=(x_1(t),x_2(t))^T$, and $x^2(t)=(x_3(t),x_4(t))^T$. By (4), the observer is given by

$$
\begin{aligned}
&\dot{\hat{x}}_1(t)=\hat{x}_2(t)+0.01u(t)+3L_1(y_1(t_k^1)-\hat{x}_1(t_k^1)),\\
&\dot{\hat{x}}_2(t)=-\hat{x}_1(t)+0.1(1-\hat{x}_1^2(t))\hat{x}_2(t)+0.1\hat{x}_2(t)u(t)\\
&\quad t\in[t_k^1+\tau_k^1,\ t_{k+1}^1+\tau_{k+1}^1),\ k\geq 0,\\
&\dot{\hat{x}}_3(t)=\hat{x}_4(t)+0.01\hat{x}_2(t)\hat{x}_3(t)\exp(u(t))\\
&\qquad+2L_2(y_2(t_k^2)-\hat{x}_3(t_k^2)),\\
&\dot{\hat{x}}_4(t)=-\hat{x}_3(t)+0.1(1-\hat{x}_3^2(t))\hat{x}_4(t)\\
&\qquad+\frac{1}{1+(\hat{x}_2(t)\hat{x}_4(t))^2}u(t)+L_2^2(y_2(t_k^2)-\hat{x}_3(t_k^2)),\\
&\qquad t\in[t_k^2+\tau_k^2,\ t_{k+1}^2+\tau_{k+1}^2),\ k\geq 0,\\
&\hat{x}_i(t_{k+1}^2+\tau_{k+1}^2)=\lim_{t\to t_{k+1}^2+\tau_{k+1}^{2\,-}}\hat{x}_i(t),\quad i=3,4,
\end{aligned}
$$

where $t_k^1=kT_1-(1.1\cdot rand)T_1$ and $t_k^2=kT_2-(1.5\cdot rand)T_2$, *rand* is a random number in the interval $[0,1]$, τ_k^1 and τ_k^2 denote the transmission delays, T_1 and T_2 are two positive real constants and will be given later. By simple computation, $P=\mathrm{diag}\{P_1,P_2\}$, where $P_1=\begin{bmatrix}0.8917 & -0.5695\\ -0.5695 & 1.1735\end{bmatrix}$, $P_2=\begin{bmatrix}0.5062 & -0.5052\\ -0.5052 & 1.5124\end{bmatrix}$. Then $\lambda_{\max}(P)=1.7223$, $\lambda_{\min}(P)=0.2963$. The other parameters are given as $l=1.6$, $L_1=40$, $L_2=90$. τ_k^1 and τ_k^2 are simulated by random numbers in the interval $[0,1.5T_1]$ and $[0,1.8T_2]$. From the

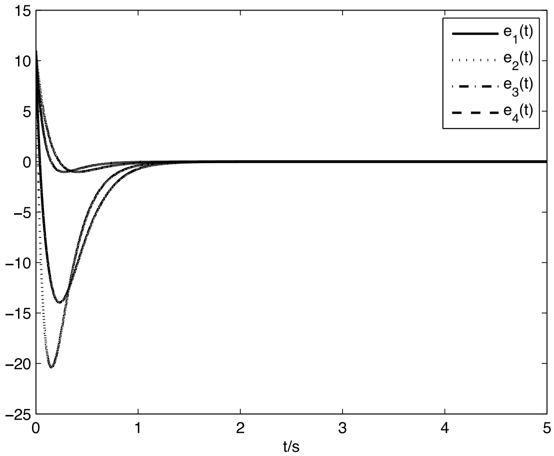

Fig. 1. Trajectories of the error states $e_i(t)(1 \le i \le 4)$ with $\hat{x}(0)$, $T_1 = 1.0 \times 10^{-5}$ s and $T_2 = 0.5 \times 10^{-5}$ s.

condition (9), we have $h_1 = 3.3 \times 10^{-5}$ s and $h_2 = 1.5 \times 10^{-5}$ s. Let $T_1 = 1.0 \times 10^{-5}$ s and $T_2 = 0.5 \times 10^{-5}$ s. Fig. 1 shows the simulation results with the initial condition of observer $\hat{x}(0) = [-10, -10, -10, -10]$.

4. Conclusion

In this paper, continuous observers were designed for a class of multi-output nonlinear systems with multi-rate sampled and delayed output measurements. The time delay might be larger or less than the sampling intervals. Sufficient conditions were presented to ensure global exponential stability of the observation errors by constructing a Lyapunov–Krasovskii function.

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