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GLOBAL DYNAMICS OF UNBALANCED DELTA-MODULATED FEEDBACK-CONTROLLED DISCRETE-TIME SYSTEMS

Rudong Gai[1] Guanrong Chen[2] and Xiaohua Xia[3]

[1]Department of Basic Science
Liaoning Technical University, Fuxin, 123000, P. R. China

[2]Department of Electronic Engineering
City University of Hong Kong, Hong Kong SAR, P. R. China

[3]Department of Electrical, Electronic and Computer Engineering
University of Pretoria, Pretoria, 0002, South Africa

Abstract.[1] In this paper, a control policy called Unbalanced Δ-Modulated Feedback (UDMF) is proposed. For one-dimensional discrete-time systems with a parameter $0 < \mid a \mid \leq 1$, we show that a system of Type II has only two fixed points and the set of fixed points is globally attracting. Compared with systems of Type II, the evolutions of systems of Type I are much more complicated. For $0 < a < 1$, systems of Type I have no fixed points. Moreover, using a constructive method, we prove that there is a denumerable set of rate value $\gamma = \frac{\Delta_2}{\Delta_1}$. Corresponding to each parameter γ of the denumerable set, systems of Type I have no periodic orbits and, in this case, every orbit is dense in the state interval $[-\Delta_1,\ \Delta_2)$. To each of the other rate values of γ, systems of Type I all have an unique periodic orbit. In particular, the structural property of the periodic motion is robust; that is, there exists an interval including this value γ such that all parameters in this interval are corresponding to those periodic orbits of the same structural property. For the case of $a = 1$, all points in the interval $[-\Delta_1,\ \Delta_2)$ are $n-$periodic with $n \geq 3$ when γ is a rational number, and every orbit is dense in the interval $[-\Delta_1,\ \Delta_2)$ when γ is an irrational number. Moreover, every such unique periodic orbit is globally attracting for both types of systems.

Keywords. Unbalanced Δ-modulated feedback, Periodic orbit, Global attractor.

AMS (MOS) subject classification:

1 Introduction

As is well known, even a one-dimensional nonlinear system may have very complicated dynamics [2, 3, 6, 9].

In this paper, the following discrete-time nonlinear system is considered:

$$x_{n+1} = ax_n + u, \tag{1}$$

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under the so-called Unbalanced Δ-Modulated Feedback (UDMF)

$$u = \Delta(ax) \overset{\text{def}}{=} \begin{cases} -\Delta_1, & ax \geq 0, \\ \Delta_2, & ax < 0, \end{cases} \tag{2}$$

where Δ_1 and Δ_2 are given positive real numbers, $\Delta_1 \neq \Delta_2$.

The study of UDMF not only has theoretical significance, but also has practical importance [1, 11, 5, 4, 7]. A UDMF control system may be considered as a special switching control system. A practical example of this control strategy is the Δ-modulated transmitting power control of a mobile unit in the Direct Sequence Code Division Multiple Access (DS-CDMA) cellular networks [1], where the "state" x is the error of the unit's power level of the mobile received at the base station with respect to the desired value. In this application, the control strategy stems from the observation that if the level of the received power is higher than the desired level, then it is decreased by Δ (dB); if lower, then it is increased by the same amount. There is only one parameter, Δ, and the power increment is either Δ or $-\Delta$, which can be stored at the base station or the active mobile unit. The base station only needs to send 1 or -1 to command the increase or decrease of the power level; namely, only one bit of datum is needed for implementing the Delta-modulated control. The requirement of one bit for transmitting power control is the well-known standard IS-95 [10]. Generally speaking, Δ-modulation provides a common method for converting analog signals to digital ones, which is also called Sigma-Delta ($\Sigma\Delta$) modulation in the field of electronic circuits. Today, Δ-modulation has been widely used in digital electronics and telecommunications. The main interest in Δ-modulation for digital electronics includes de-modulation schemes, statistical properties of the digital outputs as well as the complex dynamics involved.

It should be noted that Δ-modulated control is a special case of UDMF, *e.g.*, the *balanced* case with $\Delta_1 = \Delta_2$. In the same application area of transmitting power control, it has witnessed the flexibility of unbalanced Δ-modulated feedback in, *i.e.*, [12, 13]. All these motivate a careful study of system (1)-(2) to be carried out in the present paper. The case of $\Delta_1 = \Delta_2$ has been studied in [15, 14, 8]. In the present paper, we further focus on the important issue of unbalanced Δ-modulated feedback: $\Delta_1 \neq \Delta_2$. Define $\gamma = \frac{\Delta_2}{\Delta_1}$. Then, $\gamma \neq 1$. For convenience in the subsequent discussions, and without loss of generality, we assume $\Delta_1 = 1$ in this paper.

In the following, systems (1) is referred to as a system of Type I when $a > 0$, and system of Type II when $a < 0$, respectively. Moreover, denote

$$f_a(x) = ax + \Delta(ax). \tag{3}$$

We will only consider the case when the parameter $0 <| a |\leq 1$. We will show that a system of Type II has only two fixed points and the set of fixed points is globally attracting. For $0 < a < 1$, systems of Type I have no fixed points, and there is a denumerable set of values for the ratio $\gamma = \frac{\Delta_2}{\Delta_1}$, and for each parameter γ of the denumerable set, systems of Type I have no periodic orbits and, in this case, every orbit is dense in the state interval $[-\Delta_1,\ \Delta_2)$. To each of the other rate values of γ, systems of Type I all have an unique periodic orbit. The structural property of the periodic motion is robust; that is, there exists an interval including this value γ such that all parameters in this interval are corresponding to those periodic orbits of the same structural property. For the case of $a = 1$, all points in the interval $[-\Delta_1,\ \Delta_2)$ are $n-$periodic with $n \geq 3$ when γ is a rational number, and every orbit is dense in the interval $[-\Delta_1,\ \Delta_2)$ when γ is an irrational number. Moreover, every such unique periodic orbit is globally attracting for both types of systems.

2 Basic properties of systems in the case of $0 <\mid a \mid\leq 1$

We firstly prove the following lemma.

Lemma 1 *Suppose* $0 <\mid a \mid\leq 1$. *Then, the interval* $[-1,\ \gamma)$ *is a global attractor,* i.e., *every orbit will eventually move into this interval. In particular, a point* x *is a periodic point of system (1) only if* $x \in [-1,\ \gamma)$.

Proof 1 Let $V(x) = x^2$. For system (1) of Type I, when $x \geq 0$, since

$$V(f_a(x)) - V(x) = (ax-1)^2 - x^2 = -(1-a^2)x^2 - 2ax + 1, \tag{4}$$

one has

$$V(f_a(x)) - V(x) = \begin{cases} \geq 0, & x \in \left[0,\ \frac{1}{1+a}\right], \\ < 0, & x \in \left(\frac{1}{1+a},\ +\infty\right). \end{cases} \tag{5}$$

Similarly, when $x < 0$,

$$V(f_a(x)) - V(x) = \begin{cases} \geq 0, & x \in \left[-\frac{\gamma}{1+a},\ 0\right), \\ < 0, & x \in \left(-\infty,\ -\frac{\gamma}{1+a}\right). \end{cases} \tag{6}$$

Combining (5) and (6) shows that $V(f_a(x)) \geq V(x)$ if and only if $x \in \left[-\frac{\gamma}{1+a},\ \frac{1}{1+a}\right]$. Furthermore, since

$$f_a\left(\left[-\frac{\gamma}{1+a},\ \frac{1}{1+a}\right]\right) = \left[-1,\ -\frac{1}{1+a}\right] \bigcup \left[\frac{\gamma}{1+a}, \gamma\right) \subset [-1,\ \gamma)\,,$$

and

$$f_a\left([-1,\ \gamma)\right) = [-1,\ a\gamma - 1) \cup [\gamma - a,\ \gamma) \subset [-1,\ \gamma),$$

it shows that the interval $x \in [-1,\ \gamma)$ is a global attractor and, simultaneously, that there is no periodic point with period $n \geq 2$ out of the interval

$[-1,\ \gamma)$ for system (1) of Type I. Since (ref. Theorem 2 bellow) system (1) of Type I has no fixed point, the first part of the theorem holds true.

Similarly, for system (1) of Type II, one can verify that $V(f_a(x)) \geq V(x)$ if and only if $x \in \left[-\frac{1}{1-a},\ \frac{\gamma}{1-a}\right]$. Since, on one hand, $\left[-\frac{1}{1-a},\ \frac{\gamma}{1-a}\right] \subset [-1,\ \gamma)$, so

$$f_a\left(\left[-\frac{1}{1-a}, \frac{\gamma}{1-a}\right]\right) = \left[-1,\ -\frac{1}{1-a}\right] \bigcup \left[\frac{\gamma}{1-a},\ \gamma\right) \subset [-1,\ \gamma)\,,$$

and

$$f_a\left([-1,\ \gamma)\right) = [-1,\ -(1+a)] \cup [(1+a)\gamma,\ \gamma) \subset [-1,\ \gamma),$$

and on the other hand, system (1) of Type II has only two fixed points, $\left\{-\frac{1}{1-a},\ \frac{\gamma}{1-a}\right\}$ (ref. Theorem 1 below), there is no periodic point in system (1) of Type II out of the interval $[-1,\ \gamma)$.

2.1 Systems of Type II

First, we focus on system (1) of Type II.

Theorem 1 *Suppose $-1 \leq a < 0$. Then,*

1. except for two fixed points, $-\frac{1}{1-a}$ and $\frac{\gamma}{1-a}$, system (1) of Type II has no periodic orbits;

2. the set of fixed points $\left\{-\frac{1}{1-a},\ \frac{\gamma}{1-a}\right\}$ is globally attracting.

Proof 2 1. It can be easily verified that the two points $-\frac{1}{1-a}$ and $\frac{\gamma}{1-a}$ are fixed points of system (1) of Type II, and that system (1) of Type II has no other fixed points.

We now prove that system (1) of Type II has no n-periodic orbit for any $n > 1$. According to Lemma 1, system (1) of Type II has no periodic points in the interval $(-\infty,\ +\infty) \setminus [-1,\ \gamma)$. Furthermore, since

$$f_a((0,\ \gamma)) = ((1+a)\gamma,\ \gamma) \subset (0,\ \gamma) \tag{7}$$

and

$$f_a[-1,\ 0]) = (-1,\ -(1+a)] \subset [-1,\ 0], \tag{8}$$

it follows that f_a is a contraction mapping on the interval $[-1,\ \gamma)$. This property of the mapping implies that system (1) of Type II has no n- periodic orbit for any $n > 1$.

2. Combining (7) and (8) shows that f_a is a contraction mapping on the intervals $[-1,\ 0]$ and $(0,\ \gamma)$, respectively. Hence, combining this fact with Lemma 1 shows that assertion 2 is true.

2.2 Systems of Type I

Denote the sets of natural numbers and of positive even numbers by **N** and **PE**, respectively, and let $\mathbf{PE}_0$=$\mathbf{PE} \cap \{0\}$. Also, define

$$
\begin{aligned}
&\mathbf{T}_0 = \left\{ \{t_l,\ 0 \le l \le N\} \middle| t_l \in \mathbf{N},\ N \in \mathbf{PE}_0 \right\},\\
&\mathbf{T} = \left\{ \{t_l,\ 0 \le l \le N\} \middle| t_l \in \mathbf{N},\ N \in \mathbf{PE} \right\},\\
&\mathbf{P}_l(z) = 1 + z + \cdots + z^l,\\
&Q_{l+1} = (1 + a + \cdots + a^l)\gamma - a^{l+1},\\
&R_{l+1} = a^{l+1}\gamma - (1 + a + \cdots + a^l),\\
&\underline{B}_l = \frac{(1-a)a^l}{1-a^l},\\
&\overline{B}_{l+1} = \frac{(1-a)a^l}{1-a^l+(1-a)a^{l+1}},\\
&\underline{p}_1 = m + k(m+1) = \overline{p}_1,\\
&\underline{p}_0 = m + 1 + \underline{p}_1 = \overline{p}_2,\\
&\underline{q}_0 = m + 1 + km = \overline{q}_2,\\
&\underline{q}_1 = m + \underline{q}_0 = \overline{q}_1,\\
&a_1 = a^{\underline{p}_1} = \overline{a}_1,\\
&a_0 = a^{\underline{p}_0} = \overline{a}_2,\\
&\underline{b}_1 = a^{k(m+1)}Q_{m-1} + \mathbf{P}_{k-1}(a^{m+1})Q_m = \overline{b}_1,\\
&\underline{b}_0 = a^{m+1}\underline{b}_1 + Q_m = \overline{b}_2,\\
&\underline{d}_0 = a^{km}Q_m + \mathbf{P}_{k-1}(a^m)Q_{m-1} = \underline{d}_2,\\
&\underline{d}_1 = a^m\underline{d}_0 + Q_{m-1} = \underline{d}_1,
\end{aligned}
$$

where $m \ge 2$, $l \ge 0$ and $k \ge 1$.

Proposition 1 *1. Both sequences $\underline{B}_k$ and $\overline{B}_k$ are monotonously decreasing when k tends to infinity, and the inequalities $\underline{B}_k < \overline{B}_k < \underline{B}_{k-1}$ hold true for all $k \ge 1$.*

2. $Q_k \le 0$ if and only if $\gamma \le \underline{B}_k$, and $R_k \le 0$ if and only if $\gamma \le \frac{1}{\underline{B}_k}$.

Proof 3 The first part of assertions 1 can be easily verified by the definitions of $\underline{B}_k$ and $\overline{B}_k$.

The second part of assertion 1 is implied by the following inequalities:

$$
\begin{aligned}
\frac{1}{\underline{B}_k} &= \frac{1}{\overline{B}_k} + \frac{1-a^{k+1}}{a^k},\\
\frac{1}{\overline{B}_k} &= \frac{1}{\underline{B}_{k-1}} + a.
\end{aligned}
$$

The second part of assertion 2 follows from the equalities $Q_k = a^k\left(\frac{\gamma}{\underline{B}_k} - 1\right)$ and $R_k = a^k\left(\gamma - \frac{1}{\underline{B}_k}\right)$.

Remark 1 *From the above two groups of equalities, one can get the following useful relations about* $\underline{B}_k$ *and* $\overline{B}_k$*:*

$$\frac{1}{\underline{B}_k} = \frac{1}{\underline{B}_{k-1}} + \frac{1}{a^k}, \tag{9}$$

$$\frac{1}{\overline{B}_k} = \frac{1}{\overline{B}_{k-1}} + \frac{1}{a^{k-1}} \tag{10}$$

$$\frac{1}{\overline{B}_k} = \frac{1}{\underline{B}_{k-1}} + a. \tag{11}$$

We will have that, compared with system (1) of Type II, the dynamics of system (1) of Type I are much complicated.

For the subsequent discussions, a property of the mapping $f_a(x)$ is firstly given.

Lemma 2 *Suppose* $0 < a \leq 1$*. Then, the mapping* $f_a(x)$ *is invertible in the interval* $[-1,\ \gamma)$ *for any* $\gamma > 0$*.*

Proof 4 First of all, it is easily seen from the definition of $f_a(x)$ that the mapping $f_a(x)$ is monotonously increasing in both $(-\infty,\ 0)$ and $[0,\ +\infty)$. Therefore, for any $x,\ y \in (-\infty,\ 0)$ or $x,\ y \in [0,\ +\infty)$, $f_a(x) \neq f_a(y)$ if $x \neq y$. Hence, when $x \neq y$, $f_a(x) = f_a(y)$ only if x and y are not both in $(-\infty,\ 0)$ or $[0,\ +\infty)$. Without loss of generality, assume $x \in (-\infty,\ 0)$ and $y \in [0,\ +\infty)$ satisfy the equality $f_a(x) = f_a(y)$. Then, according to (1), we have $a(y - x) = 1 + \gamma$, which shows clearly that the distance between x and y is larger than $1 + \gamma$, namely, x and y are not both in the interval $[-1,\ \gamma)$. The proof is completed.

Theorem 2 *1. Suppose* $0 < a < 1$*. Then, when system (1) of Type I has an unique periodic orbit, the set of periodic points is globally attracting.*

2. Suppose $0 < a \leq 1$ *and system (1) of Type I have no periodic orbit for some* $\gamma > 0$*. Then, every orbit is dense in the interval* $[-1,\ \gamma)$*, namely, for every point* $x \in (-\infty,\ +\infty)$*, the* ω *limit set* $\omega_{f_a}(x) = [-1,\ \gamma)$*.*

Proof 5 1. First of all, Lemma 1 shows that every orbit will eventually move into the interval $[-1,\ \gamma)$ and then will not go out again. Hence, we can focus our discussions in this interval. For convenience, we use $\omega_{f_a}(U)$ to denote the $\omega-$limit set of mapping f_a on a set U.

Let $x_l, 1 \leq l \leq L$, be all the points in the unique periodic orbit and be ordered in magnitude. Denote $x_{L+1} = \gamma$ and $x_0 = -1$ if -1 is not a periodic point.

For any given $1 \leq l \leq L$, if $f_a^s([x_l,\ x_{l+1}))$ are always in $[-1,\ 0)$ or $[0,\ \gamma)$ for all $0 \leq s < L$, then, since $f_a^L(x_l) = x_l$ and $0 < a < 1$ means $f_a^L(x)$ is contractive, we have $f_a^L([x_l,\ x_{l+1})) \subset [x_l,\ x_{l+1})$. This implies that the point x_l is an accumulation point of all orbits starting from or passing through the interval $[x_l,\ x_{l+1})$, namely, $x_l \in \omega_{f_a}([x_l,\ x_{l+1}))$.

If 0 is an interior point of $f_a^s([x_l,\ x_{l+1}))$ for some $0 \le s < L$, then, since L is finite, the times of the cases of 0 being in the interior of $f_a^s([x_l,\ x_{l+1}))$ are also finite. This implies that there must be $x_l < x_l^{(1)} < x_{l+1}$ such that $f_a^s([x_l,\ x_l^{(1)}))$ is always in $[-1,\ 0)$ or $[0,\ \gamma)$ for all $0 \le s < L$. Of course, there is also $f_a^L([x_l,\ x_l^{(1)})) \subset [x_l,\ x_l^{(1)})$, that is, $x_l \in \omega_{f_a}([x_l,\ x_l^{(1)}))$. Take $x_l^{(1)}$ to be the least upper bound that has the above characteristics. Then, by the definition of mapping $f_a(x)$, there must be $f_a^{s_1}(x_l^{(1)}) = 0$ for some $0 \le s_1 < L$. Since there is no other periodic point between x_l and $x_l^{(1)}$, we have $f_a^{s_1+1}([x_l,\ x_l^{(1)})) = [x_L,\ \gamma)$, which implies also $x_l \in \omega_{f_a}([x_L,\ \gamma))$.

If $f_a^s([x_l^{(1)},\ x_{l+1}])$ is always in $[-1,\ 0)$ or $[0,\ \gamma)$ for all $0 \le s < L$, then, certainly there is an $x_{l+1} \in \omega_{f_a}([x_l^{(1)},\ x_{l+1}])$. If it is not the case, then there is an index s such that 0 is an interior of $f_a^s([x_l^{(1)},\ x_{l+1}])$. Let s_2 be the first index with the above characteristics and $x_l^{(2)}$ be the point satisfying the equation $f_a^{(s_2)}(x_l^{(2)}) = 0$. Since there is no periodic point in the interval $[x_l^{(1)},\ x_{l+1})$, certainly $f_a^{s_2+1}([x_l,\ x_l^{(2)})) \subset [x_L,\ \gamma)$. Thus, $x_l \in \omega_{f_a}([x_L,\ \gamma))$ implies $x_l \in \omega_{f_a}([x_l^{(1)},\ x_l^{(2)}))$. Hence, $x_l \in \omega_{f_a}([x_l,\ x_l^{(2)}))$.

Repeat the above procedure. If there is an $x_{l+1} = x_l^{(k+1)}$ for some $k \ge 1$, then $x_l \in \omega_{f_a}([x_l^{(1)},\ x_{l+1}))$; therefore, there should be an $x_l \in \omega_{f_a}([x_l,\ x_{l+1}))$. Otherwise, there are indexes s_k and points x_l^k, $k = 1,\ 2,\ \cdots$, with $x_l \in \omega_{f_a}([x_l^{(k)},\ x_l^{(k+1)}))$. Since the sequence $x_l^{(k)}$ is monotonously increasing and bounded, it has a limit. Denote the limit by x_l^*. If $x_l^* = x_{l+1}$, then there still has $x_l \in \omega_{f_a}([x_l,\ x_{l+1}))$. If $x_l^* < x_{l+1}$, then there must be $x_l \in \omega_{f_a}([x_l,\ x_l^*))$ and $x_{l+1} \in \omega_{f_a}([x_l^*,\ x_{l+1}])$; otherwise, there will be a contradiction.

Up to now, we have proved that for any pair of neighboring periodic points, x_l and x_{l+1}, the $\omega-$limit sets $\omega_{f_a}([x_l,\ x_l^*))$ and $\omega_{f_a}([x_l^*,\ x_{l+1}])$ are included in the set of periodic points. We have known that any periodic orbit must include points in both intervals $[-1,\ 0)$ and $[0,\ \gamma)$. If we denote x_- as the greatest periodic point in $[-1,\ 0)$ and x_+ the smallest periodic point in $[0,\ \gamma)$, respectively, then it can be easily verified that $\omega_{f_a}([x_-,\ 0)) = \omega_{f_a}([x_L,\ \gamma))$ and, when $x_+ > 0$, $\omega_{f_a}([0,\ x_+)) = \omega_{f_a}([-1,\ x_1))$. Thus, we have completed the proof of part 1.

2. Let $M(U)$ denote the Lebesgue measure of set U. Then, by the definition of mapping $f_a(x)$, it can be easily verified that when U is an interval, $M(f_a(U)) = aM(U)$. Furthermore, it can be verified that this conclusion is also true when U is the union of any set of finite intervals. Besides, the equality $M(f_a(U)) = aM(U)$ and the invertibility of mapping $f_a(x)$ in the interval $[-1,\ \gamma)$ together imply $M(f_a^{-1}(U)) = a^{-1}M(U)$.

According to Lemma 1, any orbit $\{f_a^l(x),\ l = 0,\ 1,\ \cdots\}$ is bounded;

therefore, $\omega_{f_a}(x)$ is not empty. If for some $x \in [-1, \gamma)$, $\omega_{f_a}(x)$ is not dense in the interval $[-1,\ \gamma)$, then according to the definition of denseness, there exists an open $\delta-$neighborhood of some point x_0 and a non-negative integer L such that $f_a^l(x) \notin (x_0-\delta,\ x_0+\delta)$ for all $l \geq L$. Without loss of generality, assume $L = 0$. If this is not the case, one can replace x by $f_a^L(x)$. Take δ as the least upper bound that satisfies $f_a^l(x) \notin (x_0-\delta,\ x_0+\delta)$ (if $x_0-\delta = -1$ or $x_0+\delta = \gamma$, then replace $(x_0-\delta,\ x_0+\delta)$ by $f_a((x_0-\delta,\ x_0+\delta))$). Then, $x_0-\delta$ or $x_0+\delta$ either belongs to the orbit $\{f_a^l(x),\ l = 1,\ 2,\ \cdots\}$ or is an accumulation point of the orbit. Here, we assume that $x_0+\delta$ has the above characteristics (the case of $x_0-\delta$ can be similarly discussed). Obviously, the invertibility of the mapping $f_a(x)$ in $[-1,\ \gamma)$ implies $f_a^s(x_0+\delta) \notin V \stackrel{def}{=} \bigcup_{l=0}^{\infty} f_a^l((x_0-\delta,\ x_0+\delta))$, $0 \leq s < +\infty$. But, the equality $M(f_a^{-s}((x_0-\delta,\ x_0+\delta))) = 2a^{-s}\delta,\ s \geq 0$, implies that, when $a < 1$, there is a finite positive integer s such that $2a^{-s}\delta \geq 1+\gamma$; therefore, there must be an $s \geq 1$ such that $x \in f_a^{-s}((x_0-\delta,\ x_0+\delta))$, that is, $f_a^s(x) \in (x_0-\delta,\ x_0+\delta)$.

For $a = 1$, since $M(f_a^s((x_0-\delta,\ x_0+\delta))) = 2\delta$ for all $s \geq 0$, it is necessary that $0 \in V$. Otherwise, $f_a^s((x_0-\delta,\ x_0+\delta))$ is always an interval for any $s \geq 0$, and $\bigcup_{s=0}^{\infty} M(f_a^s((x_0-\delta,\ x_0+\delta))) = \infty$; therefore, there must be $l_2 > l_1 \geq 0$ such that $f_a^{l_1}((x_0-\delta,\ x_0+\delta)) \cap f_a^{l_2}((x_0-\delta,\ x_0+\delta)) \neq \emptyset$. Hence,

1) if $f_a^{l_1}((x_0-\delta,\ x_0+\delta)) = f_a^{l_2}((x_0-\delta,\ x_0+\delta))$, then every point in the interval $f_a^{l_1}((x_0-\delta,\ x_0+\delta))$ is periodic, which contracts the precondition of part 2;

2) if $f_a^{l_1}((x_0-\delta,\ x_0+\delta)) \neq f_a^{l_2}((x_0-\delta,\ x_0+\delta))$, then it can be easily verified that $f_a^{l_2}(x_0+\delta) \in f_a^{l_1}((x_0-\delta,\ x_0+\delta))$; thus, with respect to the definition of δ, there also is $f_a^{l_2}(x) \in f_a^{l_1}((x_0-\delta,\ x_0+\delta)) \subset V$.

The above discussion also shows that, for any subinterval $V_i \subset V$, there must exist $0 \leq l < +\infty$ such that $0 \in f_a^l(V_i)$.

Let $l_1 \geq 0$ be the first index that satisfies $0 \in f_a^{l_1}((x_0-\delta,\ x_0+\delta))$. Here, it is obvious that $f_a^{l_1}((x_0-\delta,\ x_0+\delta)) = (f_a^{l_1}(x_0-\delta),\ f_a^{l_1}(x_0+\delta))$. According to the above analysis, in the sequence of sets $f_a^l((0,\ f_a^{l_1}(x_0+\delta)))$, $l = 1,\ 2,\ \cdots$, there are also sets that contain the origin. Let l_2 be the first index that satisfies $0 \in f_a^{l_2}((0,\ f_a^{l_1}(x_0+\delta))) = (f_a^{l_1+l_2}(0),\ f_a^{l_1+l_2}(x_0+\delta))$. Obviously, $f_a^{l_1}(x_0+\delta) > f_a^{l_1+l_2}(x_0+\delta) > 0$. This shows that $f_a^{l_1+l_2}(x_0+\delta) \in V$. Thus, we arrive at a contradiction. The proof is therefore completed.

Theorem 3 *Suppose* $0 < a < 1$. *Then,*

1. system (1) of Type I has no fixed points;

2. when $\gamma \in [\underline{B}_{n-1},\ \overline{B}_{n-1})$ *for some* $n > 2$, *or* $\frac{1}{\gamma} \in (\underline{B}_{n-1},\ \overline{B}_{n-1}]$ *for some* $n \geq 2$, *system (1) of Type I has a unique periodic orbit and its period is* n.

Proof 6 1. A point $\overline{x}$ is a fixed point of system (1) of Type I if and only if the following equality is satisfied:

$$\overline{x} = a\overline{x} + \Delta(\overline{x}) \tag{12}$$

that is,

$$(1-a)\overline{x} = \Delta(\overline{x}).$$

Since $0 < a < 1$, (12) holds true only if the signs of $\overline{x}$ and $\Delta(\overline{x})$ are the same. However, according to the definition of $\Delta(x)$, this is impossible.

2. First, consider the first situation of part 2.

It is not hard to verify that the orbit starting from the point defined below,

$$x^* = \frac{1}{1-a^n} Q_{n-1}, \tag{13}$$

is an n-periodic orbit of system (1) of Type I when $\gamma \in [\underline{B}_{n-1},\ \overline{B}_{n-1})$.

In the following, we prove that the n-periodic orbit passing through the point x^* is unique.

Firstly, Lemma 1 shows that any orbit of system (1) of Type I will ultimately move into the interval $[-1,\ \gamma)$. Thus, we focus on the properties of mappings $f_a^n(x)$ in the interval $[-1,\ \gamma)$.

Secondly, consider a group of real numbers: $-\frac{\gamma}{\underline{B}_k}$, $k = 1,\ 2,\ \cdots,\ n-1$. Since $\gamma \geq \underline{B}_{n-1}$, we have $-\frac{\gamma}{\underline{B}_{n-1}} \leq -1$. Hence, by defining $z_n = -1$, $z_{k+1} = -\frac{\gamma}{\underline{B}_k}$, $k = 1,\ 2,\ \cdots,\ n-2$, $z_1 = 0$ and $z_0 = \gamma$, we have

$$\begin{array}{rcl} f_a\left([z_{k+1},\ z_k)\right) & = & [z_k,\ z_{k-1}),\ 1 \leq k < n \\ f_a^k\left([z_{k+1},\ z_k)\right) & = & [z_1,\ z_0) = [0,\ \gamma), \\ f_a([0,\ \gamma)) & = & [z_n,\ a\gamma - 1). \end{array} \tag{14}$$

By equality (11) and the precondition $\gamma < \overline{B}_{n-1}$, we also have

$$z_{n-1} - f_a(\gamma) = -\frac{\gamma}{\underline{B}_{n-2}} - a\gamma + 1 = 1 - \frac{\gamma}{\overline{B}_{n-1}} > 0. \tag{15}$$

This shows $f_a([0,\ \gamma)) \subset [z_n,\ z_{n-1})$. Hence, we conclude that $f_a^n(x)$ is a contraction mapping in each of the n intervals $[z_k,\ z_{k-1})$, $1 \leq k \leq n$. Clearly, if mapping $f_a^n(x)$ has a fixed point in the interval $[z_k,\ z_{k-1})$, then this fixed point must be unique.

Lastly, we show that those n points of the n-periodic orbit starting from x^* belong to the subintervals $[z_{n-k+1},\ z_{n-k})$, $1 \leq k \leq n$, respectively. In fact, the precondition $\gamma \in [\underline{B}_{n-1},\ \overline{B}_{n-1})$ assures that $x^* \in [0,\ \gamma)$, and (14)and (15) clearly show that $f_a^k(x^*) \in [z_{n-k+1},\ z_{n-k})$, $1 \leq k \leq n-1$.

We next discuss the second situation about the parameter γ. First, one can similarly verify that the orbit starting from the point defined by

$$x^* = \frac{1}{1-a^n} R_{n-1} \tag{16}$$

is an n-periodic orbit.

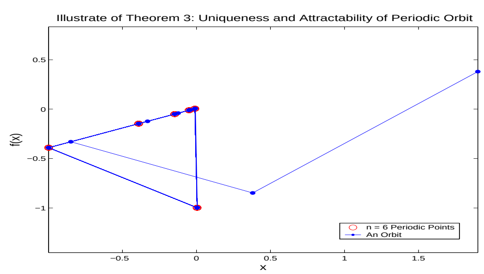
To prove that this periodic orbit is unique, we use the same method as above. Define $n+1$ numbers as follows: $z_0 = -1$, $z_k = \frac{1}{\underline{B}_k}$, $k = 1,\ 2,\ \cdots,\ n$. It can be easily verified that the first system of the relations in (14) holds true. Besides,

$$
\begin{aligned}
f_a([z_k,\ z_{k+1})) &= [z_{k-1},\ z_k),\ 1 \le k < n, &\quad (17)\\
f_a([z_0,\ z_1)) &= f_a([-1,\ 0)) = [\gamma - a,\ \gamma). &\quad (18)
\end{aligned}
$$

Thus, according to (10) and the precondition $\gamma < \frac{1}{\underline{B}_{n-1}} = z_n$, we have

$$
f_a(-1) - z_{n-1} = \gamma - a - \frac{1}{\underline{B}_{n-2}} = \gamma - \frac{1}{\overline{B}_{n-1}} \ge 0. \qquad (19)
$$

From (17)-(19), it is not hard to verify that $f_a^n(x)$ is a contraction mapping in each subinterval $[z_k,\ z_{k+1})$, $0 \le k \le n-1$. This implies that the n-periodic orbit starting from x^* is unique and globally attracting, completing the proof.



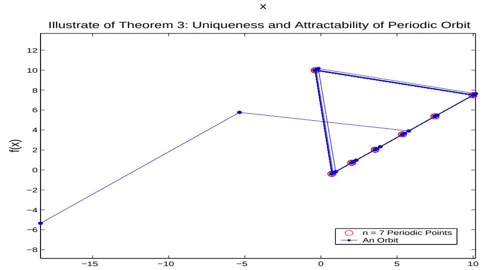


Fig. 1: $a = 0.4$, $n = 6$, $\gamma \in [0.0062,\ 0.0157)$

Fig. 2: $a = 0.85$, $n = 7$, $\gamma^{-1} \in (0.0908,\ 0.1086]$

Remark 2 *Theorem 3 characterizes system (1) of Type I when the value of the rate parameter* γ *is located in subinterval* $\left[\underline{B}_{n-1},\ \overline{B}_{n-1}\right)$ *or* $\left[\frac{1}{\overline{B}_{n-1}},\ \frac{1}{\underline{B}_{n-1}}\right)$. *Here, we denote the union of these subintervals by* U. *To the parameter region*

$(0,\ +\infty)$ of γ, it is easily seen that there is an inter-interval $\left[\overline{B}_n,\ \underline{B}_{n-1}\right)$ between the two neighboring subintervals $\left[\underline{B}_n,\ \overline{B}_n\right)$ and $\left[\underline{B}_{n-1},\ \overline{B}_{n-1}\right)$. Likewise, $\left[\frac{1}{\underline{B}_{n-1}},\ \frac{1}{\overline{B}_n}\right)$ is the inter-interval between the two neighboring subintervals $\left[\frac{1}{\overline{B}_{n-1}},\ \frac{1}{\underline{B}_{n-1}}\right)$ and $\left[\frac{1}{\overline{B}_n},\ \frac{1}{\underline{B}_n}\right)$. If we denote the union of these inter-intervals by V, then $(0,\ +\infty) = U\bigcup V$.

In the following, we first prepare some preliminaries for further discussing the characteristics of the dynamical evolution of system (1) of Type I in V.

Lemma 3 *Suppose $0 < a < 1$ and $\gamma \in \left[\overline{B}_m,\ \underline{B}_{m-1}\right)$ for some $m \geq 1$. Then,*

1. the point $x^* = -a^{-m}Q_{m-1} \in (0,\ \gamma]$ and $x^* = \gamma$ only if $\gamma = \overline{B}_m$;

2. for every point $x \in [0,\ \gamma)$ and every $1 \leq l \leq m-1$, one has $f_a^l(x) < 0$. Also $f_a^m(x) < 0$ if and only if $x < x^*$, and in this case, $f_a^{m+1}(x) > 0$;

3. when $\gamma \in \left(\overline{B}_m,\ \underline{B}_{m-1}\right)$, for any $0 \leq x < x^*$ and $x^* \leq y < \gamma$, one has $f_a^{m+1}(x) > f_a^m(y) \geq 0$;

4. for some $l \geq 0$, $f_a^{l(m+1)}(0) \geq x^*$ if and only if

$$\gamma \geq \frac{(1-a)a^{m-1}}{1 - a^{m-1} + \frac{\boldsymbol{P}_{l-1}(a^{m+1})}{\boldsymbol{P}_l(a^{m+1})}a^m}. \tag{20}$$

Proof 7 1. Conclusion 1 is just a corollary of the equalities $Q_{m-1} = a^{m-1}\left(\frac{\gamma}{\underline{B}_{m-1}} - 1\right)$ and $\gamma - x^* = \frac{1}{a}\left(\frac{\gamma}{\overline{B}_m} - 1\right)$.

2. According to the definition of $f_a(x)$, we have

$$\begin{aligned} f([0,\ \gamma)) &= [-1,\ a\gamma - 1), \\ f^2([0,\ \gamma)) &= [\gamma - a,\ (1+a^2)\gamma - a), \\ &\vdots \\ f^{m-1}([0,\ \gamma)) &= \left[a^{m-2}\left(\frac{\gamma}{\underline{B}_{m-2}} - 1\right),\ a^{m-2}\left(\frac{\gamma}{\overline{B}_{m-1}} - 1\right)\right). \end{aligned} \tag{21}$$

Clearly, $f^{m-1}([0,\ \gamma)) \subset [-1,\ 0)$, or equivalently, $f_a^l(x) < 0$ for every $x \in [0,\ \gamma)$ and all $1 \leq l \leq m-1$. Moreover,

$$f^m([0,\ \gamma)) = \left[a^{m-1}\left(\frac{\gamma}{\underline{B}_{m-1}} - 1\right),\ a^{m-1}\left(\frac{\gamma}{\overline{B}_m} - 1\right)\right). \tag{22}$$

Equality (22) shows that $f_a^m(0) < 0$ and $f_a^m(\gamma) \geq 0$. Furthermore, it can be easily verified that and $f_a^m(x) < 0$ if and only if $x < x^*$, and $f_a^{m+1}(x) = a^{m+1}x + a^m\left(\frac{\gamma}{\underline{B}_m} - 1\right) > 0$.

3. Let $0 \leq x < x^*$ and $x^* \leq y < \gamma$. Then, we have

$$\begin{aligned} f_a^{m+1}(x) - x &= Q_m - (1 - a^{m+1})x \\ &> Q_m + a^{-m}(1 - a^{m+1})Q_{m-1} \\ &= \gamma + a^{-m}Q_{m-1} > 0, \end{aligned}$$

$$y - f_a^m(y) = (1 - a^m)y - Q_{m-1} > 0,$$

and

$$\begin{aligned} f_a^{m+1}(x) - f_a^m(y) &= a^m(ax - y) + a^{m-1}(1 - a) + a^{m-1}\gamma \\ &> a^{m-1}(1 - a)(1 + \gamma). \end{aligned}$$

This complete the proof.

Remark 3 *To continue our study of the main issue, let us first have a discussion on relative prime numbers. Let $i > j$ be given positive integers. Then, there are integers $k \geq 1$ and $j > r \geq 0$ such that $i = kj + r$. In this case, it is not hard to verify that i and j are relative prime, namely,* $\gcd(i,\ j) = 1$, *if and only if* $\gcd(j,\ r) = 1$. *If $r > 1$, then we can repeat the same operation on j and r. Thus, one can see that there must exist integers $N \geq 1$, r_l and t_l, $1 \leq l \leq N$, such that $i = t_N r_N + r_{N-1}$ and $r_{l+1} = t_l r_l + r_{l-1}$, where $t_l \geq 1$, $r_N = j$ and $r_0 = 1$. It can then be verified that* $\gcd(i,\ j) = 1$ *is equivalent to* $\gcd(r_l,\ r_{l-1}) = 1$, $1 \leq l \leq N$. *On the contrary, suppose $N \geq 0$ and take $r_0 = 1$, $r_1 \geq 1$ and $t_l \geq 1$, $0 \leq l \leq N$. Then, when $r_{l+1} = t_l r_l + r_{l-1}$, it is also true that* $\gcd(r_l,\ r_{l-1}) = 1$ *for every $0 \leq l \leq N$, so that $r_{-1} = 0$.*

In the following, we always take $r_0 = 1$, $r_1 \geq 1$ and $r_{l+1} = t_l r_l + r_{l-1}$, $t_l \geq 1$, $0 \leq l \leq N$. If t_l, $0 \leq l \leq N$, have the above characteristics, then, the group of positive integers is called a coprime structure. Here, we want to point out an apparent but non-trivial fact. Suppose that t_l, $0 \leq l \leq N$, is a coprime structure of a coprime pair i and j. Then, except for the case of $N = 0$ and $t_0 = 1$, take a group of positive integers, $t_l^{'}$, $0 \leq l \leq L$, as follows:

1. when $t_0 \geq 2$, take $L = N + 1$, $t_0^{'} = 1$, $t_1^{'} = t_0 - 1$ and $t_{l+1}^{'} = t_l$, $1 \leq l \leq N$;

2. when $t_0 = 1$ and $N > 0$, take $L = N - 1$, $t_0^{'} = t_0 + t_1$ and $t_l^{'} = t_{l+1}$, $1 \leq l \leq N - 1$. Then, it can be easily verified that the group of positive integers that we took is also a coprime structure of i and j.

Lemma 4 *Suppose $\gamma \in \left[\overline{B}_m,\ \underline{B}_{m-1}\right)$ for some positive integer $m \geq 2$. Then, a positive integer n is the prime period of a periodic orbit of system (1) of Type I only if $n = i(m + 1) + jm$ for some positive integers i, j satisfying* $\gcd(i,\ j) = 1$.

In particular, system (1) of Type I has an n-periodic orbit in which i points, denoted by x_s, $1 \leq s \leq i$, are in the interval $[0,\ x^*)$, and j points, denoted by y_t, $1 \leq t \leq j$, in the interval $[x^*,\ \gamma)$, only if

1. when $i = kj + r$ for some $k \geq 1$,

$$Y_j = \begin{bmatrix} 0 & a^{\underline{p}_1} E_{j-r} \\ a^{\underline{p}_0} E_r & 0 \end{bmatrix} Y_j + \begin{pmatrix} \underline{b}_1 \boldsymbol{1}_{j-r} \\ \underline{b}_0 \boldsymbol{1}_r \end{pmatrix}; \tag{23}$$

2. when $j = ki + r$ for some $k \geq 1$,

$$X_i = \begin{bmatrix} 0 & a^{\underline{q}_1} E_r \\ a^{\underline{q}_0} E_{i-r} & 0 \end{bmatrix} X_i + \begin{pmatrix} \underline{d}_1 \boldsymbol{1}_r \\ \underline{d}_0 \boldsymbol{1}_{i-r} \end{pmatrix}, \tag{24}$$

where $X_i = (x_1,\ x_2,\ \cdots,\ x_i)^T$, $Y_j = (y_1, y_2, \cdots, y_j)^T$ *and* $\mathbf{1}_s = (1,\ 1,\ \cdots,\ 1)^T\ \in R^s$.

Proof 8 First of all, respect to the monotonicity of the mapping $f_a(x)$ in each interval of $[-1,\ 0)$ and $[0,\ \gamma)$, any periodic orbit of system (1) of Type I must simultaneously include some points in $[-1,\ 0)$ and $[0,\ \gamma)$.

The conclusion in part 1 of Lemma 3 and (22) together show that if n is the prime period of a periodic orbit, then there exist non-negative integers i and j, with $i+j>0$, such that $n = i(m+1)+jm$, where $i+j$ is the number of points that are in the n-periodic orbit and belong to the interval $[0,\ \gamma)$. Specifically, i is the number of the points in the interval $[0,\ x^*)$ and j is the number of the points in $[x^*,\ \gamma)$.

We next prove that $\gcd(i,\ j) = 1$.

i) Suppose $i = 0$, *i.e.*, $n = jm$. Let $y_l, 1 \le l \le j$, be the j points of an n-periodic orbit that belong to the interval $[0,\ \gamma)$. Without loss of generality, assume $y_1 < y_2 < \cdots < y_j$. Then, it is obvious that

$$0 \le f^m(y_1) < f^m(y_2) < \cdots < f^m(y_j).$$

These inequalities show that $f^m(y_l) = y_l, 1 \le l \le j$, namely,

$$y_l = \frac{a^{m-1}}{1-a^m}\left(\frac{\gamma}{\underline{B}_{m-1}} - 1\right), \quad 1 \le l \le j.$$

But this is impossible, since $\gamma < \underline{B}_{m-1}$.

ii) Suppose $j = 0$, that is, $n = i(m+1)$. Let $x_l, 1 \le l \le i$, be the i points of an n-period orbit that belong to the interval $[0,\ \gamma)$, with $x_1 < x_2 < \cdots < x_i$. Then,

$$0 < f^{m+1}(x_1) < f^{m+1}(x_2) < \cdots < f^{m+1}(x_i).$$

These inequalities show that $f^{m+1}(x_l) = x_l, 1 \le l \le i$, namely,

$$x_l = \frac{a^m}{1-a^{m+1}}\left(\frac{\gamma}{\underline{B}_m} - 1\right) = \frac{1}{1-a^{m+1}}Q_m.$$

But this is also impossible, since $\gamma \ge \overline{B}_m$, which implies that

$$x_l - \gamma \quad = \quad \frac{Q_m - (1-a^{m+1})\gamma}{1-a^{m+1}} = \frac{a^m}{1-a^{m+1}}\left(\frac{\gamma}{\overline{B}_m} - 1\right) \ge 0.$$

This is contradictory to Lemma 1.

Thus, we conclude that, if $n = i(m+1)+jm$ is a period, then it must be true that $i,\ j \ge 1$. Without loss of generality, let $x_1 < x_2 < \cdots < x_i < y_1 < \cdots < y_j$. To arrive at a contradiction, assume $c = \gcd(i,\ j) > 1$.

iii) For the case of $i > j$, if we denote i by $kj+r$, then, as we have known, the residual item r also has factor c.

Assume $r = 0$. Then, according to Lemma 3, we have $f^m(x_s) < 0, 1 \leq s \leq i$, $f^m(y_t) \geq 0, 1 \leq t \leq j$, and

$$0 \leq f^m(y_1) < f^m(y_2) < \cdots < f^m(y_j) < f^{m+1}(x_1) < \cdots < f^{m+1}(x_i). \quad (25)$$

Inequalities (25) imply that, for $1 \leq l \leq j$,

$$\begin{array}{rcl} f^{m+s(m+1)}(y_l) & = & x_{sj+l},\ 0 \leq s \leq k-1, \\ f^{\underline{p}_1}(y_l) & = & a^{\underline{p}_1} y_l + b_1 = y_l, \end{array} \quad (26)$$

which shows clearly that the points $y_t, 1 \leq t \leq j$, have the same period $k(m+1) + m < kj(m+1) + jm$.

For the case of $r \neq 0$. it can be easily verified that (26) still holds true except the last equality. From these equalities, we get

$$\begin{array}{ll} f^{\underline{p}_1}(y_{r+l}) = a^{\underline{p}_1} y_{r+l} + \underline{b}_1 = y_l, & 1 \leq l \leq j-r, \\ f^{\underline{p}_0}(y_l) = a^{\underline{p}_0} y_l + \underline{b}_0 = y_{j-r+l}, & 1 \leq l \leq r, \end{array}$$

which are the condition (23) of the lemma.

The above analysis shows that, for any $1 \leq s,\ t \leq j$, $y_t = f^N(y_s)$ only if $N = N_0 \underline{p}_0 + N_1 \underline{p}_1$ for some non-negative integers N_0 and N_1 satisfying $N_0 + N_1 \geq 1$. Now, denote j and r by $cj^{'}$ and $cr^{'}$, respectively. Thus, (23) can be rewritten as follows:

$$\begin{array}{rcll} f^{\underline{p}_1}(y_{cr'+l}) & = & y_l, & 1 \leq l \leq c(j^{'} - r^{'}), \\ f^{\underline{p}_0}(y_l) & = & y_{c(j'-r')+l}, & 1 \leq l \leq cr^{'}. \end{array}$$

The above shows clearly that $y_t = f^{\underline{p}_1}(y_s)$ or $y_t = f^{\underline{p}_0}(y_s)$ only if $\mid t - s \mid$ is a multiple of c. This implies that the j points do not belong simultaneously to one periodic orbit if $c > 1$. Thus, the necessity of condition 1 has been proved.

iv) For the case of $j > i > 0$, denote $j = ki + r$ as above.

If $r = 0$, then there will be

$$f^{m+1+km}(x_l) \quad = \quad a^{m+1+km} x_l + a^{km} Q_m + \mathbf{P}_{k-1}(a^m) Q_{m-1} = x_l,$$

which shows that the period of x_l is $m + 1 + km < n$ for each $1 \leq l \leq i$.

For the case of $r \neq 0$, it can be verified that the following equalities hold for all $0 \leq s \leq k-1$ and $1 \leq l \leq i$:

$$f^{m+1+sm}(x_l) = a^{m+1+sm} x_l + a^{sm} Q_m + \mathbf{P}_{s-1}(a^m) Q_{m-1} = y_{(k-s-1)i+r+l}. \quad (27)$$

Hence, one can further obtain

$$\begin{array}{rl} f^{\underline{q}_0}(x_l) = a^{\underline{q}_0} x_l + \underline{d}_0 = x_{r+l}, & 1 \leq l \leq i-r, \\ f^{\underline{q}_1}(x_{i-r+l}) = a^{\underline{q}_1} x_{i-r+l} + \underline{d}_1 = x_l, & 1 \leq l \leq r, \end{array} \quad (28)$$

which are the same as (24). We have known that $c = \gcd(i,\ j) > 1$ implies that r has also devisor c. Denote i and r by $ci^{'}$ and $cr^{'}$, respectively. Then, according to (24), we have

$$\begin{array}{rcl} f^{\underline{q}_1}(x_l) & = & a^{\underline{q}_1}x_l + \underline{d}_1 = x_{cr'+l},\ 1 \le l \le c(i^{'} - r^{'}), \\ f^{\underline{q}_0}(x_{c(i'-r')+l}) & = & a^{\underline{q}_0}x_{c(i'-r')+l} + \underline{d}_0 = x_l,\ 1 \le l \le r, \end{array}$$

which shows clearly that $x_t = f^{\underline{q}_1}(x_s)$ or $x_t = f^{\underline{q}_0}(x_s)$ only if $\mid t - s \mid$ is a multiple of c. This implies that the i points x_l, $1 \le l \le i$, do not belong to the same periodic orbit.

v) It can be easily verified that $i = j$ only if $i = j = 1$.

This proof is completed.

Corollary 1 *For every $m \ge 1$, when $\gamma = \overline{B}_m$, system (1) of Type I has not periodic orbits.*

Proof 9 The conclusion directly follows from Lemma's 3 and 4.

Let $Y_{r_N + r_{N+1}} \in \mathbf{R}^{r_N + r_{N+1}}$ be a given vector. Then, the vector $Y_{r_N + r_{N+1}}$ can be expressed as

$$\begin{array}{l} Y_{r_{N-2l}+r_{N-2l+1}} = \begin{pmatrix} Y^{(1)}_{r_{N-2l}} \\ \vdots \\ Y^{(t_{N-2l})}_{r_{N-2l}} \\ Y_{r_{N-2l-1}+r_{N-2l}} \end{pmatrix}, \\ Y_{r_{N-2l-1}+r_{N-2l}} = \begin{pmatrix} Y_{r_{N-2l-2}+r_{N-2l-1}} \\ Y^{(1)}_{r_{N-2l-1}} \\ \vdots \\ Y^{(t_{N-2l-1})}_{r_{N-2l-1}} \end{pmatrix}, \end{array} \tag{29}$$

or as

$$\begin{array}{l} Y_{r_{N-2l}+r_{N-2l+1}} = \begin{pmatrix} Y_{r_{N-2l-1}+r_{N-2l}} \\ Y^{(1)}_{r_{N-2l}} \\ \vdots \\ Y^{(t_l)}_{r_{N-2l}} \end{pmatrix}, \\ Y_{r_{N-2l-1}+r_{N-2l}} = \begin{pmatrix} Y^{(1)}_{r_{N-2l-1}} \\ \vdots \\ Y^{(t_{N-2l-1})}_{r_{N-2l-1}} \\ Y_{r_{N-2l-2}+r_{N-2l-1}} \end{pmatrix}, \end{array} \tag{30}$$

where $Y^{(\beta)}_{r_\alpha} \in R^{r_\alpha}$, $1 \le \beta \le t_\alpha$.

The above expressions can be illustrated with the following example. Let $N = 3$, $r_0 = 1$, $r_1 = 7$, $r_2 = 1 \times r_1 + 1 = 8$, $r_3 = 2 \times r_2 + r_1 = 23$ and

$r_4 = 3 \times r_3 + r_2 = 77$. In this case, one can verify that any given vector $Y_{100} \in R^{100}$ can be expressed as

$$Y_{100} = \left(Y_{23}^{T(1)},\ Y_{23}^{T(2)},\ Y_{23}^{T(3)},\ Y_{7}^{T(1)},\ Y_{1+7}^{T},\ Y_{8}^{T(1)},\ Y_{8}^{T(2)}\right)^T$$

and

$$Y_{100} = \left(Y_{8}^{T(1)},\ Y_{8}^{T(2)},\ Y_{1+7}^{T},\ Y_{7}^{T(1)},\ Y_{23}^{T(1)},\ Y_{23}^{T(2)},\ Y_{23}^{T(3)}\right)^T.$$

Proposition 2 *Suppose $N \geq 0$ is a non-negative integer, a_0, a_1, b_0 and b_1 are real numbers with $0 < a_0,\ a_1 < 1$, and*

$$A_{r_N+r_{N+1}} \overset{def}{=} \begin{bmatrix} 0 & a_1 E_{r_{N+1}} \\ a_0 E_{r_N} & 0 \end{bmatrix}. \tag{31}$$

Then, the system of linear equations

$$(E_{r_N+r_{N+1}} - A_{r_N+r_{N+1}})Y_{r_N+r_{N+1}} = \begin{pmatrix} b_1 \boldsymbol{1}_{r_{N+1}} \\ b_0 \boldsymbol{1}_{r_N} \end{pmatrix} \tag{32}$$

has a unique solution. When the vector $Y_{r_N+r_{N+1}}$ is expressed in the form of (29), the system of linear equations (32) can be solved by a recursive formula as follows:

$$\begin{gathered} \overline{A}_{r_{N-2l+1}+r_{N-2(l-1)}} Y_{r_{N-2l+1}+r_{N-2(l-1)}} = \begin{pmatrix} b_{2l-1} \boldsymbol{1}_{r_{N-2l+1}} \\ b_{2l} \boldsymbol{1}_{r_{N-2(l-1)}} \end{pmatrix},\ 2 \leq 2l \leq N, \\ Y_{r_{N-2(l-1)}}^{(s)} = a_{2l-1} Y_{r_{N-2(l-1)}}^{(s+1)} + b_{2l-1} \boldsymbol{1}_{r_{N-2(l-1)}},\ 1 \leq s \leq t_{N-2(l-1)}, \end{gathered}$$

$$\begin{gathered} \overline{A}_{r_{N-2l}+r_{N-2l+1}} Y_{r_{N-2l}+r_{N-2l+1}} = \begin{pmatrix} b_{2l+1} \boldsymbol{1}_{r_{N-2l+1}} \\ b_{2l} \boldsymbol{1}_{r_{N-2l}} \end{pmatrix},\ 2 \leq 2l \leq N, \\ Y_{r_{N-2l+1}}^{(s)} = a_{2l} Y_{r_{N-2l+1}}^{(s-1)} + b_{2l} \boldsymbol{1}_{r_{N-2l+1}},\ 1 \leq s \leq t_{N-2l+1}, \end{gathered} \tag{33}$$

where $\overline{A}_{r_l+r_{l+1}} \overset{ref}{=} E_{r_l+r_{l+1}} - A_{r_l+r_{l+1}}$, $Y_{r_{N-2(l-1)}}^{(t_{N-2(l-1)}+1)}$ consists of the first $r_{N-2(l-1)}$ components of the vector $Y_{r_{N-2l+1}+r_{N-2(l-1)}}$, and $Y_{r_{N-2l+1}}^{(0)}$ consists of the last r_{N-2l+1} components of the vector $Y_{r_{N-2l}+r_{N-2l+1}}$, with

$$A_{r_{N-2l+1}+r_{N-2(l-1)}} = \begin{bmatrix} 0 & a_{2l-1} E_{r_{N-2l+1}} \\ a_{2l} E_{r_{N-2(l-1)}} & 0 \end{bmatrix}, \tag{34}$$

$$A_{r_{N-2l}+r_{N-2l+1}} = \begin{bmatrix} 0 & a_{2l+1} E_{r_{N-2l+1}} \\ a_{2l} E_{r_{N-2l}} & 0 \end{bmatrix}, \tag{35}$$

$$a_{s+2} = a_s a_{s+1}^{t_{N-s}},\ 0 \leq s \leq N, \tag{36}$$

$$b_{s+2} = b_s + b_{s+1} a_s \boldsymbol{P}_{t_{N-s}-1}(a_{s+1}). \tag{37}$$

When N is an odd integer, the above procedure ends at $2l-1=N$, and the system of linear equations (32) reduces to the following special form:

$$\left(E_{1+r_1}-\begin{bmatrix}0 & a_N\\ a_{N+1}E_{r_1} & 0\end{bmatrix}\right)Y_{1+r_1}=\begin{pmatrix}b_N\\ b_{N+1}\mathbf{1}_{r_1}\end{pmatrix}. \tag{38}$$

Moreover, the solution of (38) is obtained as follows:

$$\begin{aligned}Y_{1+r_1}^{(1)}&=\tfrac{1}{1-a_N a_{N+1}^{r_1}}b_{N+2}\\ Y_{1+r_1}^{(s+1)}&=a_{N+1}Y_{1+r_1}^{(s)}+b_{N+1},\ 1\le l\le r_1.\end{aligned} \tag{39}$$

When N is zero or an even integer, the above procedure ends at $2l=N$, and the system of linear equations (32) reduces to the following special form:

$$\left(E_{1+r_1}-\begin{bmatrix}0 & a_{N+1}E_{r_1}\\ a_N & 0\end{bmatrix}\right)Y_{1+r_1}=\begin{pmatrix}b_{N+1}\mathbf{1}_{r_1}\\ b_N\end{pmatrix}, \tag{40}$$

whose solutions are given by

$$\begin{aligned}Y_{1+r_1}^{(1+r_1)}&=\tfrac{1}{1-a_N a_{N+1}^{r_1}}b_{N+2},\\ Y_{1+r_1}^{(l)}&=a_{N+1}Y_{1+r_1}^{(l+1)}+b_{N+1},\ 1\le l\le r_1.\end{aligned} \tag{41}$$

A proof of Proposition 2 is given in Appendix *I*.

Property 1 *Suppose $m\ge 2$, $k\ge 1$, and $N\in \boldsymbol{PE}_0$, $\{t_l,\ 0\le l\le N\}\in \boldsymbol{T}_0$ and $r_{l+1}=t_l r_l+r_{l-1}$, $1\le l\le N$.*

1. If $\underline{p}_{l+2}=\underline{p}_l+t_{N-l}\underline{p}_{l+1}$, $0\le l\le N$, and the value $n=i(m+1)+jm$ is evaluated with $j=r_N+r_{N+1}$ and $i=kj+r_N$, then $n=\underline{p}_{N+2}$.

2. If $\overline{p}_{l+3}=\overline{p}_{l+1}+t_{N-l}\overline{p}_{l+2}$, $0\le l\le N$, and the value $n=i(m+1)+jm$ is evaluated with $j=r_N+r_{N+1}$ and $i=kj+r_{N+1}$, then $n=\overline{p}_{N+3}$.

3. If $\underline{q}_{l+2}=\underline{q}_l+t_{N-l}\underline{q}_{l+1}$, $0\le l\le N$, and the value $n=i(m+1)+jm$ is evaluated with $i=r_N+r_{N+1}$ and $j=ki+r_N$, then $n=\underline{q}_{N+2}$.

4. If $\overline{q}_{l+3}=\overline{q}_{l+1}+t_{N-l}\overline{q}_{l+2}$, $0\le l\le N$, and the value $n=i(m+1)+jm$ is evaluated with $i=r_N+r_{N+1}$ and $j=ki+r_{N+1}$, then $n=\overline{q}_{N+3}$.

A proof of Property 1 is given in *Appendix II*.

Property 2 *Suppose $\gamma\in\left(\overline{B}_m,\ \underline{B}_{m-1}\right)$ for some positive integer $m\ge 2$, and $\underline{b}_{s+2}$ are defined by (36) and $\underline{p}_s$ is defined as before. Then, for any non-negative integer N, $\{t_s,\ 0\le s\le N\}\in \boldsymbol{T}_0$, and $0\le 2l\le N$, we have*

1. The following inequalities hold:

$$\frac{\underline{b}_{2l}}{1-a^{\underline{p}_{2l}}} > \frac{\underline{b}_{2l+2}}{1-a^{\underline{p}_{2l+2}}}, \tag{42}$$

$$\frac{\underline{b}_{2l+1}}{1-a^{\underline{p}_{2l+1}}} < \frac{\underline{b}_{2l+3}}{1-a^{\underline{p}_{2l+3}}}, \tag{43}$$

$$\frac{\underline{b}_{2l+2}}{1-a^{\underline{p}_{2l+2}}} - \frac{\underline{b}_{2l+1}}{1-a^{\underline{p}_{2l+1}}} = \frac{\boldsymbol{P}_k(a^{m+1})Q_m-(a^{\underline{p}_1}\gamma+\underline{b}_1)}{(1-a^{\underline{p}_{2l+1}})(1-a^{\underline{p}_{2l+2}})}>0. \tag{44}$$

2. For given $N \in \boldsymbol{PE}_0$ and $\{t_l,\ 0 \le l \le N\} \in \boldsymbol{T}_0$, except for $\frac{1}{1-a^{\underline{p}_2}}\underline{b}_2$ with $t_0 = 1$, there exist $L \in \boldsymbol{N}$ and a group of positive integers $\{t_l^{'},\ 0 \le l \le L\}$, such that

$$\frac{a^{\underline{p}^{'}_{L+1}}}{1-a^{\underline{p}^{'}_{L+2}}}\underline{b}^{'}_{L+2} + \underline{b}^{'}_{L+1} = \frac{1}{1-a^{\underline{p}_{N+2}}}\underline{b}_{N+2}, \tag{45}$$

$$\frac{1}{1-a^{\underline{p}^{'}_{L+2}}}\underline{b}^{'}_{L+2} = \frac{a^{\underline{p}_{N+1}}}{1-a^{\underline{p}_{N+2}}}\underline{b}_{N+2} + \underline{b}_{N+1}, \tag{46}$$

and vice versa, where $\underline{p}^{'}_0 = \underline{p}_0$, $\underline{p}^{'}_1 = \underline{p}_1$, $\underline{p}^{'}_{l+2} = \underline{p}^{'}_l + t^{'}_{L-l}\underline{p}^{'}_{l+1}$ and $\underline{b}^{'}_{l+2} = \underline{b}^{'}_l + \underline{b}^{'}_{l+1}a^{\underline{p}^{'}_l}\boldsymbol{P}_{t^{'}_{L-l}-1}(a^{\underline{p}^{'}_{l+1}})$, $0 \le l \le L$.

3. For given $N \in \boldsymbol{PE}_0$ and $\{t_l,\ 0 \le l \le N\} \in \boldsymbol{T}_0$, except for $\frac{1}{1-a^{\overline{p}_3}}\overline{b}_3$ with $t_0 = 1$, there exist $L \in \boldsymbol{N}$ and a group of positive integers $\{t_l^{'},\ 0 \le l \le L\}$, such that

$$\frac{a^{\overline{p}^{'}_{L+2}}}{1-a^{\overline{p}^{'}_{L+3}}}\overline{b}^{'}_{L+3} + \overline{b}^{'}_{L+2} = \frac{1}{1-a^{\overline{p}_{N+3}}}\overline{b}_{N+3}, \tag{47}$$

$$\frac{1}{1-a^{\overline{p}^{'}_{L+3}}}\overline{b}^{'}_{L+3} = \frac{a^{\overline{p}_{N+2}}}{1-a^{\overline{p}_{N+3}}}\overline{b}_{N+3} + \overline{b}_{N+2}, \tag{48}$$

and vice versa, where $\overline{p}_{s+3} \overset{def}{=} \overline{p}_{s+1} + t_{N-s}\overline{p}_{s+2}$, $0 \le s \le N$, $\overline{p}^{'}_1 = \overline{p}_1$, $\overline{p}^{'}_2 = \overline{p}_2$, $\overline{p}^{'}_{l+3} = \overline{p}^{'}_{l+1} + t^{'}_{L-l}\overline{p}^{'}_{l+2}$ and $\overline{b}^{'}_{l+3} = \overline{b}^{'}_{l+1} + \overline{b}^{'}_{l+2}a^{\overline{p}^{'}_{l+1}}\boldsymbol{P}_{t^{'}_{L-l}-1}(a^{\overline{p}^{'}_{l+2}})$, $0 \le l \le L$.

A proof of Property 2 is given in *Appendix III*.

Property 3 *When Y_{1+r_1} is the solution vector of (38), its first component $Y^{(1)}_{1+r_1} = y_{r_{N+1}}$. When Y_{1+r_1} is the solution vector of (40), its last component $Y^{(1+r_1)}_{1+r_1} = y_{r_{N+1}+1}$.*

Proof 10 For $N = 1$, the expression of (29) shows that there is $t_1 r_1$ components of $Y_{r_1+r_2}$ ahead of $Y^{(1)}_{1+r_1}$, since $r_2 = t_1 r_1 + r_0$, $r_0 = 1$, and $Y^{(1)}_{1+r_1}$ is the first component of $y_{r_{N+1}}$, therefore, the first assertion holds true for $N = 1$. Assume $Y^{(1)}_{1+r_1} = y_{r_{N+1}}$ for some odd integer N. For $N+2$, by assumption, there are $r_{N+1} - 1$ components of $Y_{r_N + r_{N+1}}$ ahead of $Y^{(1)}_{1+r_1}$, and from the expression (29), we know that there are $t_{N+2}r_{N+2}$ components ahead of $Y_{r_N+r_{N+1}}$. Hence, there are $t_{N+2}r_{N+2} + r_{N+1} - 1 = r_{N+3} - 1$ components ahead of $Y^{(1)}_{1+r_1}$ in the vector $Y_{r_{N+2}+r_{N+3}}$.

The second assertion can be similarly discussed. This completes the proof.

From Proposition 2, we can see that to solve the system of linear equations (32) one should distinguish two cases: $i = kj + r$ and $j = ki + r$. For each of the two cases, it should be distinguished two situations: $j > 2r$ and $j < 2r$ (similarly, $i > 2r$ and $i < 2r$), which are corresponding to (32) and respectively to the initial system of linear equations with the coefficient matrix in the following form

$$A_{r_N + r_{N+1}} = \begin{bmatrix} 0 & a_1 E_{r_N} \\ a_0 E_{r_{N+1}} & 0 \end{bmatrix}. \tag{49}$$

Besides, it seems to be necessary to consider N being an odd and an even (or zero) integers, separately, but it will be seen that the case of N being an odd integer can be replaced by the case of $N = 0$ or an even integer. Thus, we only need to consider four cases besides the above two special cases.

Theorem 4 *Suppose $\gamma \in \left(\overline{B}_m,\ \underline{B}_{m-1}\right)$ for some positive integers $m \geq 2$ and $N \geq 1$. Then:*

1. A necessary and sufficient condition for system (1) of Type I to have $n = (k+1)(m+1)+m$-periodic orbits, in which $k+1$ points belong to $[0,\ x^*)$ and one to $[x^*,\ \gamma)$, is

$$\frac{(1-a)a^{m-1}}{1-a^{m-1}+\frac{\boldsymbol{P}_k(a^{m+1})}{\boldsymbol{P}_{k+1}(a^{m+1})}(1-a)a^m} \leq \gamma < \frac{(1-a)a^{m-1}}{1-a^{m-1}+\frac{\boldsymbol{P}_{k-1}(a^{m+1})+a^{\underline{p}_1}}{\boldsymbol{P}_k(a^{m+1})+a^{\underline{p}_1}}(1-a)a^m}. \tag{50}$$

2. A necessary and sufficient condition for system (1) of Type I to have $n = m+1+(k+1)m$-periodic orbits, in which one point belongs to $[0,\ x^*)$ and $k+1$ points to $[x^*,\ \gamma)$, is

$$\frac{(1-a)a^{m-1}}{1-a^{m-1}+\frac{a^{km}}{\boldsymbol{P}_k(a^m)+a^{\underline{q}_0}}(1-a)a^m} \leq \gamma < \frac{(1-a)a^{m-1}}{1-a^{m-1}+\frac{a^{(k+1)m}}{\boldsymbol{P}_{k+1}(a^m)}(1-a)a^m}. \tag{51}$$

3. Suppose $n = i(m+1)+jm$ and $i = kj+r$ for some positive integers $m \geq 2$, $k \geq 1$, i, j and r, with $\gcd(i,\ j) = 1$. Then, system (1) of Type I has an n-periodic orbit, in which i points are in the interval $[0,\ x^*)$ and j points in the interval $[x^*,\ \gamma)$, if and only if

3.1 when $j > 2r$, there exist $N \in \boldsymbol{PE}_0$ and $\{t_l,\ 0 \leq l \leq N\} \in \boldsymbol{T}_0$, such that $j = r_N + r_{N+1}$, $i = kj + r_N$ and

$$\begin{cases} \frac{1}{1-a^n}\underline{b}_{N+2} & \geq & \boldsymbol{P}_k(a^{m+1})Q_m, \\ \frac{a^{\underline{p}_{N+1}}}{1-a^n}\underline{b}_{N+2} + \underline{b}_{N+1} & < & a^{\underline{p}_1}\gamma + \underline{b}_1; \end{cases} \tag{52}$$

3.2 when $j < 2r$, there exist $N \in \boldsymbol{PE}_0$ and $\{t_l,\ 0 \leq l \leq N\} \in \boldsymbol{T}_0$, such that $j = r_N + r_{N+1}$, $i = kj + r_{N+1}$ and

$$\begin{cases} \frac{1}{1-a^n}\overline{b}_{N+3} & < & a^{\overline{p}_1}\gamma + \overline{b}_1, \\ \frac{a^{\overline{p}_{N+2}}}{1-a^n}\overline{b}_{N+3} + \overline{b}_{N+2} & \geq & \boldsymbol{P}_k(a^{m+1})Q_m. \end{cases} \tag{53}$$

4. Suppose $n = i(m+1)+jm$ and $j = ki+r$ for some positive integers $m \geq 2$, $k \geq 1$, i, j and r, with $\gcd(i,\ j) = 1$. Then, *4.1 when $i > 2r$, system (1) of Type I has an n-periodic orbit, in which i points are in the interval $[0,\ x^*)$ and j points in the interval $[x^*,\ \gamma)$, if and only if there exist $N \in \boldsymbol{PE}_0$ and $\{t_l,\ 0 \le l \le N\} \in \boldsymbol{T}_0$, such that $i = r_N + r_{N+1}$, $j = ki + r_N$ and*

$$\begin{cases} \frac{1}{1-a^n}\underline{d}_{N+2} & \geq & \underline{d}_0, \\ \frac{a^{\underline{q}_{N+1}}}{1-a^n}\underline{d}_{N+2} + \underline{d}_{N+1} & < & a^{(k+1)m}\gamma + \boldsymbol{P}_k(a^m)Q_{m-1}; \end{cases} \tag{54}$$

4.2 when $i < 2r$, system (1) of Type I has an n-periodic orbit, in which i points are in the interval $[0,\ x^*)$ and j points in the interval $[x^*,\ \gamma)$, if and only if there exist $N \in \boldsymbol{PE}_0$ and $\{t_l,\ 0 \le l \le N\} \in \boldsymbol{T}_0$, such that $i = r_N + r_{N+1}$, $j = ki + r_{N+1}$ and

$$\begin{cases} \frac{1}{1-a^n}\overline{d}_{N+3} & < & a^{(k+1)m}\gamma + \boldsymbol{P}_k(a^m)Q_{m-1} \\ \frac{a^{\overline{q}_{N+2}}}{1-a^n}\overline{d}_{N+3} + \overline{d}_{N+2} & \geq & \overline{d}_2. \end{cases} \tag{55}$$

In particular, when one of the above conditions holds true, system (1) of Type I has a unique n-periodic orbit, which is globally attracting.

Proof 11 In the following, both x_l and y_l are the same as that in Lemma 4.

1. According to Lemma 3, it can be easily verified that $y_1 \in [x^*,\ \gamma)$ is a periodic point with the stated characteristics only if the point is given by

$$y_1 = \frac{\underline{b}_0}{1-a^n}. \tag{56}$$

Thus,

$$\begin{aligned} y_1 \ge x^* \quad &\Rightarrow \quad a^{(k+1)(m+1)}Q_{m-1} + \mathbf{P}_k(a^{m+1})Q_m + a^{-m}(1-a^n)Q_{m-1} \ge 0 \\ &\Rightarrow \quad a^m\mathbf{P}_k(a^{m+1})Q_m + Q_{m-1} \ge 0, \end{aligned}$$

which is just the first inequality of (50). On the other hand, from the precondition $y_1 \ge x^*$ and the first part of (26), we know that $f_a^{\underline{p}_1}(y_1) = x_k < x^*$; thus, we have

$$\begin{aligned} f_a^{\underline{p}_1}(y_1) - x^* \quad &< \quad 0 \\ &\Rightarrow \quad a^m(a^{\underline{p}_1}Q_m + \underline{b}_1) + (1-a^n)Q_{m-1} < 0 \\ &\Leftrightarrow \quad a^m\left(a^{\underline{p}_1} + \mathbf{P}_{k-1}(a^{m+1})\right)Q_m + (1 + a^{\underline{p}_1} - a^n)Q_{m-1} < 0 \\ &\Leftrightarrow \quad a^m\left(a^{\underline{p}_1} + \mathbf{P}_{k-1}(a^{m+1})\right)\gamma + \left(a^{\underline{p}_1} + \mathbf{P}_k(a^{m+1})\right)Q_{m-1} < 0. \end{aligned}$$

It is not hard to verify that the above is equivalent to the second inequality of (50).

Next, we prove the sufficiency of conditions (50). Firstly, it can be easily verified that, for any given positive real number c, the function

$$\frac{t}{c+t}$$

is strictly increasing in the interval $(0,\ +\infty)$. Hence,

$$\frac{\mathbf{P}_{k-1}(a^{m+1})+a^{\underline{p}_1}}{\mathbf{P}_k(a^{m+1})+a^{\underline{p}_1}} > \frac{\mathbf{P}_{k-1}(a^{m+1})}{\mathbf{P}_k(a^{m+1})} > \frac{a^{m+1}\mathbf{P}_{k-2}(a^{m+1})}{a^{m+1}\mathbf{P}_{k-1}(a^{m+1})} > \cdots > \frac{1}{\mathbf{P}_1(a^{m+1})} \quad (57)$$

and

$$\frac{\mathbf{P}_{k-1}(a^{m+1})+a^{\underline{p}_1}}{\mathbf{P}_k(a^{m+1})+a^{\underline{p}_1}} < \frac{\mathbf{P}_{k-1}(a^{m+1})+a^{k(m+1)}}{\mathbf{P}_k(a^{m+1})+a^{k(m+1)}} < \frac{\mathbf{P}_k(a^{m+1})}{\mathbf{P}_{k+1}(a^{m+1})}. \quad (58)$$

As a straightforward corollary of (57) and (58), we have

$$f_a^{k(m+1)}(0) = \mathbf{P}_{k-1}(a^{m+1})Q_m < x^*, \quad (59)$$
$$f_a^{(k+1)(m+1)}(0) = \mathbf{P}_k(a^{m+1})Q_m > x^*. \quad (60)$$

In the following, we prove that the point given by formula (56) is n-periodic when condition (50) is satisfied.

As a matter of fact, the conclusion in part 2 of Lemma 3 and the first inequality of (50) together imply that

$$f_a^m(y_1) = \frac{a^m\underline{b}_0}{1-a^n} + Q_{m-1} = \frac{1}{1-a^n}\left(a^m\mathbf{P}_k(a^{m+1})Q_m + Q_{m-1}\right) \geq 0;$$

that is, $y_1 \in [x^*,\ \gamma)$. On the other hand, according to the conclusions in parts 3 and 4 of Lemma 3, for every $0 < l \leq k$,

$$f_a^{l(m+1)}(0) > f_a^{l(m+1)}\left(f_a^m(y_1)\right) \geq 0.$$

Hence, we have

$$\begin{aligned} f_a^{\underline{p}_0}(y_1) &= f_a^n(y_1) \\ &= \frac{a^{(k+1)(m+1)}}{1-a^n}\left(a^m\mathbf{P}_k(a^{m+1})Q_m + Q_{m-1}\right) + \mathbf{P}_k(a^{m+1})Q_m \\ &= y_1. \end{aligned}$$

The above proof shows clearly that system (1) of Type I has a unique periodic orbit passing through the point defined by (56).

2. This assertion can be discussed in the same way as above.

Before proving assertions $3-4$, we firstly have an explanation about the conditions in parts $3-4$. One can see that all the conditions in parts $3-4$ are related to zero or positive even integers for N. The reason is that, with respect to Remark 3, two positive integers i and j are coprime if and only if there exists a coprime structure $t_l,\ 0 \leq l \leq N$, with N being zero or a positive even integer. Therefore, there is no need to discuss the case of N being odd integers.

3. Necessity of condition 3.1. Assume that system (1) of Type I has an n-periodic orbit, in which i points are in the interval $[0,\ x^*)$ and j points in the interval $[x^*,\ \gamma)$. Here, we take the expression of $Y_{r_N+r_{N+1}} =$

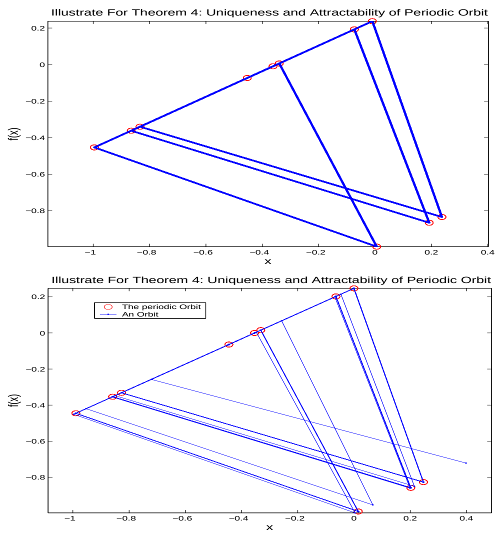


Fig. 3–4: $a = 0.7$, $m = 3$, $k = 1$, $0.2416 \le \gamma < 0.2474$ and $y_1 = 0.2934$

$(y_1,\ y_2,\ \cdots,\ y_j)^T$ as in the form of (29). According to Lemma 4, $Y_{r_N + r_{N+1}}$ should be a solution of the system of linear equations (32) with $a_0 = a^{\underline{p}_0}$, $a_1 = a^{\underline{p}_1}$, $b_0 = \underline{b}_0$ and $b_1 = \underline{b}_1$. Moreover, by Properties 1 and 2, we have

$$\begin{aligned} y_{r_1+1}^{(r_1+1)} &= \frac{1}{1 - a_N a_{N+1}^{r_1}} \left[\underline{b}_N + \underline{b}_{N+1} a_N \mathbf{P}_{r_1-1}(a_{N+1}) \right] = \frac{1}{1-a^n} \underline{b}_{N+2}, \\ y_{r_1+1}^{(l)} &= a_{N+1} y_{r_1+1}^{(l+1)} + \underline{b}_{N+1} = a^{\underline{p}_{N+1}} y_{r_1+1}^{(l+1)} + \underline{b}_{N+1},\ 1 \le l \le r_1, \end{aligned} \tag{61}$$

where $y_{r_1+1}^{(r_1+1)} = y_{r_{N+1}+1}$ and $y_{r_{N+1}} = y_{r_1+1}^{(r_1)} = a^{\underline{p}_{N+1}} y_{r_1+1}^{(r_1+1)} + \underline{b}_{N+1}$. By assumption, $y_1 \ge x^*$, hence,

$$y_{r_{N+1}+1} = a^{\underline{p}_0} y_1 + \underline{b}_0 \ge a^{\underline{p}_0} x^* + \underline{b}_0 = \mathbf{P}_k(a^{m+1}) Q_m,$$

which is just the first part of (52). On the other hand, the restriction condition $y_j < \gamma$ and the relation $y_{r_{N+1}} = a^{\underline{p}_1} y_j + \underline{b}_1$ together show clearly that the second part of (52) is also necessary.

Sufficiency of condition 3.1. We divide the proof into several parts.

1) $y_{r_{N+1}+1} = a^{\underline{p}_0} y_1 + \underline{b}_0 \ge \mathbf{P}_k(a^{m+1}) Q_m$ implies $y_1 \ge x^*$.

2) $a^{\underline{p}_1} y_j + \underline{b}_1 = a^{\underline{p}_{N+1}} y_{r_{N+1}+1} + \underline{b}_{N+1} < a^{\underline{p}_1}\gamma + \underline{b}_1$ implies $y_j < \gamma$.

3) According to the first assertion of Property 2, it is known that condition (52) implies $y_{r_{N+1}+1} > y_{r_{N+1}}$. Moreover, this inequality and the second part of (52) together ensure that the inequality $y_{l+1} > y_l$ holds true for all $r_{N+1} + 1 - r_1 \le l \le r_{N+1}$.

4) In the following, we prove that vector $Y_{r_N + r_{N+1}}$ is ordered in magnitude. A vector $Y_r \in R^r$ will be denoted as $Y_r = \left(Y_r^{(1)},\ Y_r^{(2)},\ \cdots,\ Y_r^{(r)} \right)^T$.

As to vector $Y_{r_s}^{(l)}$, $0 \leq l \leq t_s$, $1 \leq s \leq N+1$, we will denote it by $Y_{r_s}^{(l)} = \left(Y_{r_s}^{(l,1)},\ Y_{r_s}^{(l,2)},\ \cdots,\ Y_{r_s}^{(l,r_s)}\right)^T$.

Firstly, from 3)), we know that the vector Y_{1+r_1} is ordered in magnitude.

Secondly, according to the second part of (33), it can be easily seen that each of $\{Y_{r_1}^{(s)},\ 0 \leq s \leq t_1\}$ is also ordered in magnitude. Thus, if $Y_{r_1}^{(1,1)} > Y_{r_1}^{(0,r_1)}$, that is, the first component of vector $Y_{r_1}^{(1)}$, which is just behind Y_{1+r_1}, is greater than the last component of vector Y_{1+r_1}, then, for all $1 \leq l \leq t_1$,

$$Y_{r_1}^{(l+1,1)} - Y_{r_1}^{(l,r_1)} \quad = \quad a^{\underline{p}_N}\left(Y_{r_1}^{(l,1)} - Y_{1+r_1}^{(l-1,r_1)}\right) > 0,$$

which implies that vector $Y_{r_1+r_2}$ is also ordered in magnitude. Now, compare $Y_{r_1}^{(1,1)}$ and $Y_{r_1}^{(0,r_1)}$ with their values. Since $Y_{r_1}^{(0,r_1)}$ is just the component $Y_{1+r_1}^{(1+r_1)}$, according to (40), we have $Y_{1+r_1}^{(1+r_1)} = a_N Y_{1+r_1}^{(1)} + \underline{b}_N$. Hence, from the second part of (33) with $2l = N$, we have

$$Y_{r_1}^{(1,1)} - Y_{r_1}^{(0,r_1)} = a^{\underline{p}_N} Y_{r_1}^{(0,1)} + \underline{b}_N - a^{\underline{p}_N} Y_{1+r_1}^{(1)} - \underline{b}_N = a^{\underline{p}_N}\left(Y_{1+r_1}^{(2)} - Y_{1+r_1}^{(1)}\right) > 0.$$

Thirdly, as to vector $Y_{r_2+r_3}$, the second part of (33) and the above conclusion on $Y_{r_1+r_2}$ together imply that each of $Y_{r_2}^{(l)}, 1 \leq l \leq t_2$, is ordered in magnitude. Besides, by a reason similar to the above, this vector $Y_{r_2+r_3}$ is also ordered in magnitude if $Y_{r_2}^{(t_2+1,1)} > Y_{r_2}^{(t_2,r_2)}$. This inequality can be verified as follows: on one hand, the first part of (33) shows that $Y_{r_2}^{(t_2+1,1)} = a_{N-1} Y_{r_1+r_2}^{(r_2+1)} + \underline{b}_{N-1}$; on the other hand, from the second part of (33), we have $Y_{r_2}^{(t_2,r_2)} = a_{N-1} Y_{r_2}^{(t_2+1,r_2)} + \underline{b}_{N-1}$, in which $Y_{r_2}^{(t_2+1,r_2)} = Y_{r_2+r_3}^{(r_2)}$. Hence, the inequality $Y_{r_2}^{(t_2+1,1)} > Y_{r_2}^{(t_2,r_2)}$ holds true.

Finally, we assume that $Y_{r_{N-2l}+r_{N-2l+1}}$ is ordered in magnitude for some $2l \leq N$. We then prove that both vectors $Y_{r_{N-2l+1}+r_{N-2l+2}}$ and $Y_{r_{N-2l+2}+r_{N-2l+3}}$ are ordered in magnitude. Combining the above conclusions with Proposition 2, it is not hard to verify that vector $Y_{r_{N-2l+1}+r_{N-2l+2}}$ is ordered in magnitude if $Y_{r_{N-2l+1}}^{(1,1)} > Y_{r_{N-2l+1}}^{(0,r_{N-2l+1})}$. According to (29) and (33),

$$Y_{r_{N-2l+1}}^{(1,1)} = a_{2l} Y_{r_{N-2l+1}}^{(0,1)} + \underline{b}_{2l}$$

and

$$Y_{r_{N-2l+1}}^{(0,r_{N-2l+1})} = a_{2l} Y_{r_{N-2l}+r_{N-2l+1}}^{(r_{N-2l})} + \underline{b}_{2l},$$

where

$$Y_{r_{N-2l+1}}^{(0,1)} = Y_{r_{N-2l}+r_{N-2l+1}}^{(r_{N-2l+1})} = Y_{r_{N-2l}+r_{N-2l+1}}^{(r_{N-2l}+1)}.$$

Thus, by assumption, inequality $Y_{r_{N-2l+1}}^{(1,1)} > Y_{r_{N-2l+1}}^{(0,r_{N-2l+1})}$ holds true. As to vector $Y_{r_{N-2l+2}+r_{N-2l+3}}$, from (29) and (33), we have

$$Y_{r_{N-2l+2}}^{(t_{N-2l+2}+1,1)} = a_{2l-1} Y_{r_{N-2l+1}+r_{N-2l+2}}^{(r_{N-2l+1}+1)} + \underline{b}_{2l-1}$$

and

$$Y_{r_{N-2l+2}}^{(t_{N-2l+2},r_{N-2l+2})} = a_{2l-1} Y_{r_{N-2l+2}}^{(t_{N-2l+2}+1,r_{N-2l+2})} + \underline{b}_{2l-1},$$

where

$$Y_{r_{N-2l+2}}^{(t_{N-2l+2}+1,r_{N-2l+2})} = Y_{r_{N-2l+1}+r_{N-2l+2}}^{(r_{N-2l+2})}.$$

We have proved that vector $Y_{r_{N-2l+1}+r_{N-2l+2}}$ is ordered in magnitude. So, $Y_{r_{N-2l+2}}^{(t_{N-2l+2}+1,1)} > Y_{r_{N-2l+2}}^{(t_{N-2l+2}+1,r_{N-2l+2})}$, which implies that vector $Y_{r_{N-2l+2}+r_{N-2l+3}}$ is also ordered in magnitude.

5) Combining 1), 2) and 4), we conclude that $x^* \leq y_l < \gamma$. In the following, we further prove that $f_a^{\underline{p}_1}(y_l) = a^{\underline{p}_1} y_l + \underline{b}_1$ for all $1 \leq l \leq j$ and $f_a^{\underline{p}_0}(y_l) = a^{\underline{p}_0} y_l + \underline{b}_0$ for $1 \leq l \leq r_N$.

Firstly, since

$$a^{\underline{p}_0} y_1 + \underline{b}_0 = a^{m+1}(a^{\underline{p}_1} y_1 + \underline{b}_1) + Q_m = a\left[a^m (a^{\underline{p}_1} y_1 + \underline{b}_1) + Q_{m-1}\right] + \gamma,$$

it follows that

$$\gamma > y_{r_{N+1}+1} = a^{\underline{p}_0} y_1 + \underline{b}_0 \geq \mathbf{P}_k(a^{m+1}) Q_m,$$

which implies $a^m (a^{\underline{p}_1} y_1 + \underline{b}_1) + Q_{m-1} < 0$. Furthermore,

$$x^* > a^{\underline{p}_1} y_1 + \underline{b}_1 \geq \mathbf{P}_{k-1}(a^{m+1}) Q_m. \tag{62}$$

Generally, utilizing (62), we can prove the following equalities:

$$\begin{aligned}
&a^{(l+1)(m+1)}(a^m y_1 + Q_{m-1}) + \mathbf{P}_l(a^{m+1}) Q_m \\
&= a^{m+1}\left(a^{l(m+1)}(a^m y_1 + Q_{m-1}) + \mathbf{P}_{l-1}(a^{m+1}) Q_m\right) + Q_m \\
&= a\left[a^m\left(a^{l(m+1)}(a^m y_1 + Q_{m-1}) + \mathbf{P}_{l-1}(a^{m+1}) Q_m\right) + Q_{m-1}\right] + \gamma.
\end{aligned}$$

Using induction, we can verify the following inequalities:

$$x^* > a^{l(m+1)}(a^m y_1 + Q_{m-1}) + \mathbf{P}_{l-1}(a^{m+1}) Q_m \geq \mathbf{P}_{l-1}(a^{m+1}) Q_m, \tag{63}$$

for all $0 \leq l \leq k$, where we stipulate $\mathbf{P}_{-1}(a^{m+1}) = 0$. By (63), we get $f_a^{l(m+1)}(0) = \mathbf{P}_{l-1}(a^{m+1}) Q_m$ for every $0 \leq l \leq k+1$. This conclusion and part 2) of Lemma 3 together show that, for each $1 \leq l \leq j$, the following inequalities hold for all $0 \leq l \leq k$:

$$\begin{array}{rcl}
f_a^{l(m+1)}(0) & \leq & f_a^{m+l(m+1)}(y_l) \\
 & = & a^{m+l(m+1)} y_l + a^{l(m+1)} Q_{m-1} + \mathbf{P}_{l-1}(a^{m+1}) Q_m \\
 & < & f_a^{(l+1)(m+1)}(0).
\end{array} \tag{64}$$

An obvious conclusion followed from (64) is $f_a^{\underline{p}_1}(y_l) = a^{\underline{p}_1} y_l + \underline{b}_1$, $1 \leq l \leq j$. Besides, according to (23), when $1 \leq l \leq r_N$, $f_a^{\underline{p}_1 + m}(y_l) \geq 0$,

$$y_{r_{N+1}+l} = a^{\underline{p}_0} y_l + \underline{b}_0 = a\left(f_a^m\left(f_a^{\underline{p}_1}(y_l)\right)\right) + \gamma.$$

The above implies $f_a^{\underline{p}_1}(y_l) < x^*$, which is obviously equivalent to $f_a^{\underline{p}_0}(y_l) = a^{\underline{p}_0} y_l + \underline{b}_0$, $1 \leq l \leq r_N$.

6) Now, we prove that the group $\{y_l,\ 1 \leq l \leq j\}$ belongs to an n-periodic orbit. At first, it is not hard to see that (23) defines a one-to-one mapping in the group:

$$\mathcal{F}(y_l) = \begin{cases} f_a^{\underline{p}_0}(y_l), & 1 \leq l \leq r_N \\ f_a^{\underline{p}_1}(y_l), & r_N < l \leq j. \end{cases}$$

This implies that there must be a positive integer N_l such that $y_l = f_a^{N_l}(y_l)$ for each $1 \leq l \leq j$, *i.e.*, each point in $\{y_l,\ 1 \leq l \leq j\}$ is a periodic point. Without loss of generality, we take N_l as the smallest positive integer that satisfies the equality $y_l = f_a^{N_l}(y_l)$. Clearly, this stipulation and conclusion 1) of Property 2 together imply that $N_l \leq r_N \underline{p}_0 + r_{N+1} \underline{p}_1 = n$.

If there is an $N_l < n$ for some $1 \leq l \leq j$, then there must be $1 \leq s \leq j$ and $s \neq l$ such that $f_a^t(y_l) \neq y_s$ and $f_a^t(y_s) \neq y_l$ for all $t \geq 1$. This leads the system of linear equations (32) to be separated into some independent subsystems. But this is impossible according to Proposition 2. Therefore, the group $\{y_l,\ 1 \leq l \leq j\}$ must belong to the same n-periodic orbit. Furthermore, the uniqueness of the solution of (32) ensures that the periodic orbit is unique.

Necessity of 3.2. For notational convenience, in the case of $i = kj + r_{N+1}$, we use $N+1$ instead of N in (31) and take the vector $Y_{r_N + r_{N+1}}$ in the form of (30). Let $a_1 = a^{\underline{p}_1}$, $a_2 = a^{\underline{p}_0}$ and replace b_1 and b_2 by $\overline{b}_1$ and $\overline{b}_2$ in the first part of (33), respectively. Then, (33) becomes

$$\overline{A}_{r_N + r_{N+1}} Y_{r_N + r_{N+1}} = \begin{pmatrix} \overline{b}_1 \mathbf{1}_{r_N} \\ \overline{b}_2 \mathbf{1}_{r_{N+1}} \end{pmatrix}, \tag{65}$$

where

$$A_{r_N + r_{N+1}} = \begin{bmatrix} 0 & a_1 E_{r_N} \\ a_2 E_{r_{N+1}} & 0 \end{bmatrix}.$$

Clearly, the system of linear equations (65) is the same as (23) with the above parameters. Thus, when N is zero or a positive even integer, we meet the special situation (38) and its solution is given by

$$\begin{gathered} Y_{1+r_1}^{(1)} = \tfrac{1}{1 - a^{\overline{q}_{N+1} + r_1 \overline{q}_{N+2}}} \left[\overline{b}_{N+1} + \overline{b}_{N+2} \overline{a}_{N+1} \mathbf{P}_{r_1 - 1}(\overline{a}_{N+2})\right] = \tfrac{1}{1 - a^n} \overline{b}_{N+3}, \\ Y_{1+r_1}^{(l+1)} = a^{\overline{p}_{N+2}} Y_{1+r_1}^{(l)} + \overline{b}_{N+2},\ 1 \leq l \leq r_1. \end{gathered} \tag{66}$$

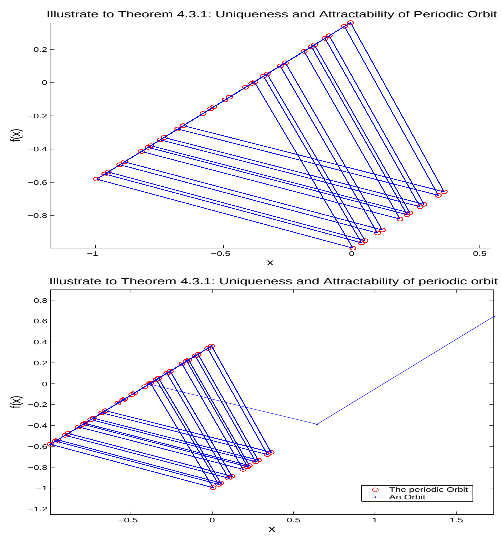


Fig. 5–6: $a = 0.95$, $m = 3$, $k = 1$, $N = 2$, $t_s = 1$, $j > 2r$, $0.3651 \le \gamma < 0.3657$ and $Y_{r_1+1}^{(r_1+1)} = 0.3385$

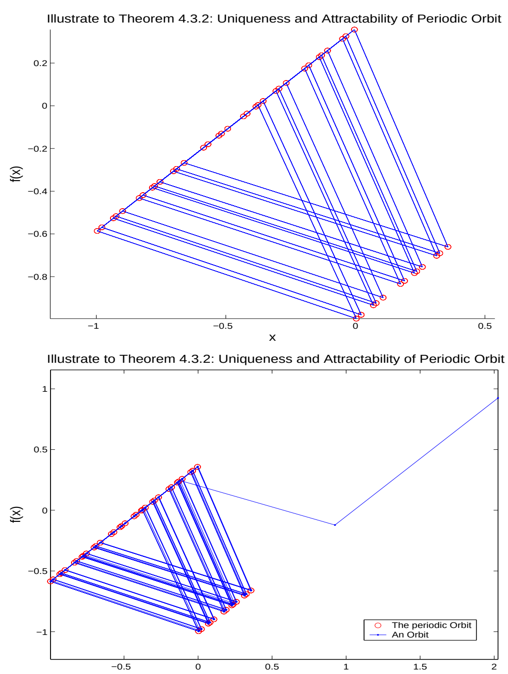


Fig. 7–8: $a = 0.95$, $m = 3$, $k = 1$, $N = 2$, $t_s = 1$, $j < 2r$, $0.3604 \le \gamma < 0.3609$ and $Y_{r_1+1}^{(1)} = 0.2585$

Besides, by comparing expression (29) with (30), and utilizing Property 1, it is not hard to verify that $Y_{1+r_1}^{(1+r_1)} = y_{r_N+1}$ and $Y_{1+r_1}^{(1)} = y_{r_N}$. Thus, similar to conclusion 3.1 of this theorem, conclusion 3.2 holds true.

4. Necessity of condition 4.1. Suppose that system (1) of Type I has an n-periodic orbit, in which i points are in the interval $[0,\ x^*)$ and j points in the interval $[x^*,\ \gamma)$. Let the vector $X_{r_N+r_{N+1}} = X_i$ be expressed as in the form of (29). Let $a_0 = a^{\underline{q}_0}$, $a_1 = a^{\underline{q}_1}$ and replace b_0 and b_1 by $\underline{d}_0$ and $\underline{d}_1$, respectively. Then, according to Lemma 4 and Proposition 2, when N is zero or a positive even integer, we have

$$\begin{aligned} X_{1+r_1}^{(1+r_1)} &= \tfrac{1}{1-a_N a_{N+1}^{r_1}} \underline{d}_{N+2} = \tfrac{1}{1-a^n} \underline{d}_{N+2}, \\ X_{1+r_1}^{(l)} &= a_{N+1} X_{1+r_1}^{(l+1)} + \underline{d}_{N+1} = a^{\underline{q}_{N+1}} X_{1+r_1}^{(l+1)} + \underline{d}_{N+1},\ 1 \le l \le r_1. \end{aligned} \tag{67}$$

By Property 3, we know that $X_{1+r_1}^{(1+r_1)} = x_{r_{N+1}+1}$, and Lemma 4 shows $x_{r_{N+1}+1} = a^{\underline{q}_0} x_1 + \underline{d}_0$. Thus, the inequality $x_1 \ge 0$ implies the first inequality of (54). On the other hand, it can be verified that the necessity $x_i < x^*$ and the relation $x_{r_{N+1}} = a^{\underline{q}_1} x_i + \underline{d}_1$ together imply also the second part of (54).

Sufficiency of condition 4.1. We again separate the proof into several parts.

1) According to Lemma 4 and conditions (54), we get $x_i < x^*$ and $x_1 \ge 0$.

2) Inequalities (54) and the following inequality

$$\underline{d}_0 - \left(a^{(k+1)m}\gamma + \mathbf{P}_k(a^m) Q_{m-1}\right) = a^{km}\left((1-a^m)\gamma - (1-a)Q_{m-1}\right) > 0 \tag{68}$$

together imply $x_{r_{N+1}} < x_{r_{N+1}+1}$. Thus, in the same way as the proof of 3.1, it can be verified that vector $X_{r_N+r_{N+1}}$ is ordered in magnitude, with $0 \le x_l < x^*$, $1 \le l \le i$.

3) In the following, we prove that $f_a^{\underline{q}_0}(x_l) = a^{\underline{q}_0} x_l + \underline{d}_0$ for all $1 \le l \le i$ and that $f_a^{\underline{q}_1}(x_{r_N+l}) = a^{\underline{q}_1} x_{r_N+l} + \underline{d}_1$ for $1 \le l \le r_{N+1}$.

Firstly, the two inequalities, $x_1 \ge 0$ and (68), imply $\underline{d}_0 > 0$. Moreover, by $Q_{m-1} < 0$, we get

$$a^{\underline{q}_0} x_1 + \underline{d}_0 = a^{(k-1)m}\left[a^m(a^{m+1}x_1 + Q_m) + Q_{m-1}\right] + \mathbf{P}_{k-2}(a^m) Q_{m-1} > 0,$$

which implies

$$f_a^{m+1}(x_1) = a^{m+1}x_1 + Q_m > x^*.$$

Generally, for $1 \le s \le k-1$, utilizing the equalities

$$\begin{aligned} a^{\underline{q}_0} x_1 + \underline{d}_0 \ &= \ a^{sm}\left[a^{(k-s)m}(a^{m+1}x_1 + Q_m) + \mathbf{P}_{k-s-1}(a^m) Q_{m-1}\right] \\ &+ \ \mathbf{P}_{s-1}(a^m) Q_{m-1}, \end{aligned}$$

we get

$$a^{(k-s)m}(a^{m+1}x_1 + Q_m) + \mathbf{P}_{k-s-1}(a^m) Q_{m-1} > x^*. \tag{69}$$

On the other hand, the first part of conditions (54) shows that $a^{\underline{p}_1}x_i + \underline{d}_1 < a^{(k+1)m}\gamma + \mathbf{P}_k(a^m)Q_{m-1}$. By comparing this inequality with (69), it is obvious that the equality $f_a^{\underline{q}_0}(x_l) = a^{\underline{q}_0}x_l + \underline{d}_0$ is true for every $1 \le l \le i$. Besides, since $f_a^{\underline{q}_0}(x_{r_{N+1}+l}) > f_a^{\underline{q}_0}(x_l) = x_{r_{N+1}+1} \ge x^*$, $1 \le l \le r_{N+1}$, we have, $f_a^{\underline{q}_1}(x_{r_{N+1}+l}) = a^{\underline{q}_1}x_{r_{N+1}+1} + \underline{d}_1$.

4) Thus, we conclude, from conclusion 3) and (24), that every x_l is a periodic point. In the same way as the proof of 3.1, it can be proved that the period of x_l is n.

4.2 As done in the proof of conclusions 3.2, we use $N+1$ instead of N and let the vector $X_{r_N+r_{N+1}}$ be in the form of (30). Let $a_1 = a^{\overline{q}_1}$, $a_2 = a^{\overline{q}_0}$ and replace $\underline{d}_1$ and $\underline{d}_2$ by $\overline{d}_1$ and $\overline{d}_2$ in (65), respectively. We get

$$\overline{A}_{r_N+r_{N+1}}X_{r_N+r_{N+1}} = \begin{pmatrix} \overline{d}_1\mathbf{1}_{r_N} \\ \overline{d}_2\mathbf{1}_{r_{N+1}} \end{pmatrix}. \tag{70}$$

Clearly, the system of linear equations (70) is also the same as (24) with the above parameters. Thus, when N is zero or a positive even integer, according to the recursive algorithm of Proposition 2, the system of linear equations (70) reduces to the special situation (40) and its solution is as follows:

$$X_{1+r_1}^{(1)} = \tfrac{1}{1-a^n}\overline{d}_{N+3},$$
$$X_{1+r_1}^{(l+1)} = a^{\overline{q}_{N+2}}X_{1+r_1}^{(l)} + \overline{d}_{N+2},\ 1 \le l \le r_1,$$

where the equality $X_{1+r_1}^{(1)} = x_{r_{N-1}}$ still holds true. Thus, similar to conclusion 3.2 of this theorem, condition (55) is necessary and sufficient. This completes the proof.

Remark 4 *Since* $\gamma \in \left[\frac{1}{\underline{B}_{m-1}},\ \frac{1}{\overline{B}_m}\right)$ *is the same as* $\frac{1}{\gamma} \in \left(\overline{B}_m,\ \underline{B}_{m-1}\right]$, *we can equivalently transform the discussion of the case* $\gamma \in \left[\frac{1}{\underline{B}_{m-1}},\ \frac{1}{\overline{B}_m}\right)$ *into the case of* $\frac{1}{\gamma} \in \left(\overline{B}_m,\ \underline{B}_{m-1}\right]$ *for system (1) with parameter* $\frac{1}{\gamma}$. *As a result, we can obtain conclusions similar to the above.*

In the following, we discuss the distribution characteristics of the periodic orbits.

Corollary 2 *For any given* $\{t_l,\ 0 \le l \le N\} \in \boldsymbol{T}_0$, *the set of parameters* γ, *corresponding to every periodic orbit expressed by (52)-(55), is a left-closed and right-open interval.*

A proof of Corollary 2 is given in *Appendix IV*.

Theorem 5 *1. Let*

$$\underline{i}(m,\ k) \quad = \quad \frac{(1-a)a^{m-1}}{1 - a^{m-1} + \frac{a^{\underline{p}_1} + \boldsymbol{P}_{k-1}(a^{m+1})}{a^{\underline{p}_1} + \boldsymbol{P}_k(a^{m+1})}(1-a)a^m}, \tag{71}$$

$$\bar{i}(m,\ k) \quad = \quad \frac{(1-a)a^{m-1}}{1 - a^{m-1} + \frac{\boldsymbol{P}_{k-1}(a^{m+1})}{\boldsymbol{P}_k(a^{m+1})}(1-a)a^m}, \tag{72}$$

and

$$\gamma_1(m,\ k) = \left\{ \gamma \middle| \begin{array}{c} \gamma = \sup\left\{ \gamma \middle| \frac{a^{\underline{p}_{N+1}}}{1-a^{\underline{p}_{N+2}}} \underline{b}_{N+2} + \underline{b}_{N+1} < \alpha(m,\ k) \right\} \ \textit{or} \\ \gamma = \sup\left\{ \gamma \middle| \frac{\overline{b}_{N+3}}{1-a^{\overline{p}_{N+3}}} < \alpha(m,\ k) \right\}, \\ \{t_l,\ 0 \le l \le N\} \in \boldsymbol{T}_0 \end{array} \right\}, \tag{73}$$

where $\alpha(m,\ k) = a^{\underline{p}_1}\gamma + \underline{b}_1$. *Then, for any given* $k \ge 1$, $j > r \ge 1$, $i = kj + r$, $\gcd(i,\ j) = 1$ *and* $n = i(m+1) + jm$, *a necessary and sufficient condition for system (1) of Type I to have an* n*-periodic orbit, in which* i *points belong to* $[0,\ x^*)$ *and* j *points to* $[x^*,\ \gamma)$, *is* $\gamma \in \big(\underline{i}(m,\ k),\ \overline{i}(m,\ k)\big) \setminus \gamma_1(m,\ k)$.

2. Let

$$\underline{j}(m,\ k) = \frac{(1-a)a^{m-1}}{1 - a^{m-1} + \frac{a^{km}}{\boldsymbol{P}_k(a^m)}(1-a)a^m}, \tag{74}$$

$$\overline{j}(m,\ k) = \frac{(1-a)a^{m-1}}{1 - a^{m-1} + \frac{a^{km}}{\boldsymbol{P}_k(a^m) + a^{\underline{q}_0}}(1-a)a^m}, \tag{75}$$

and

$$\gamma_2(m,\ k) = \left\{ \gamma \middle| \begin{array}{c} \gamma = \sup\left\{ \gamma \middle| \frac{a^{\underline{q}_{N+1}}}{1-a^n} \underline{d}_{N+2} + \underline{d}_{N+1} < \beta(m,\ k) \right\} \ \textit{or} \\ \gamma = \sup\left\{ \gamma \middle| \frac{\overline{d}_{N+3}}{1-a^n} < \beta(m,\ k) \right\}, \\ \{t_l,\ 0 \le l \le N\} \in \boldsymbol{T}_0 \end{array} \right\}, \tag{76}$$

where $\beta(m,\ k) = a^{(k+1)m}\gamma + \boldsymbol{P}_k(a^m)Q_{m-1}$. *Then, for any given* $k \ge 1$, $i > r \ge 1$, $j = ki + r$, $\gcd(i,\ j) = 1$ *and* $n = i(m+1) + jm$, *a necessary and sufficient condition for system (1) of Type I to have an* n*-periodic orbit, in which* i *points belong to* $[0,\ x^*)$ *and* j *points to* $[x^*,\ \gamma)$, *is* $\gamma \in \big(\underline{j}(m,\ k),\ \overline{j}(m,\ k)\big) \setminus \gamma_2(m,\ k)$.

Proof 12 Since the two conclusions are similar, for simplicity in the following proof of the theorem, we only prove the first one.

We firstly prove that the parameter γ, corresponding to a periodic orbit determined by (52) and (53), belongs to $\big(\underline{i}(m,\ k),\ \overline{i}(m,\ k)\big)$.

Assume that $y_l \in [x^*,\ \gamma)$, $1 \le l \le j$, are j points of an n-periodic orbit. Then, according to Lemma 4, $y_{j-r+1} = a^{\underline{p}_0} y_1 + \underline{b}_0$. Since, by assumption, $y_1 \ge x^*$ and $y_{j-r+1} < \gamma$, we have

$$a^{\underline{p}_0} x^* + \underline{b}_0 = \mathbf{P}_k(a^{m+1})Q_m < \gamma \Leftrightarrow a^m \mathbf{P}_{k-1}(a^{m+1})\gamma + \mathbf{P}_k(a^{m+1})Q_{m-1} < 0.$$

It can be verified that the last inequality above is equivalent to the inequality $\gamma < \overline{i}(m,\ k)$. On the other hand, from the equality $y_1 = a^{\underline{p}_1} y_{r+1} + \underline{b}_1$, we have

$$\begin{aligned} & a^{\underline{p}_1}\gamma + a^{k(m+1)}Q_{m-1} + \mathbf{P}_{k-1}(a^{m+1})Q_m > x^* \\ & \Leftrightarrow a^m(a^{\underline{p}_1} + \mathbf{P}_{k-1}(a^{m+1}))\gamma + (a^{\underline{p}_1} + \mathbf{P}_k(a^{m+1}))Q_{m-1} > 0. \end{aligned}$$

This shows $\gamma > \underline{i}(m,\ k)$.

In the following, we investigate the distribution of the parameter γ corresponding to the periodic orbits given by (52) and (53). For this purpose, we need to know the evolution property of these points appearing in (52) and (53), which depend on t_l and N. For any given positive integers $t_l,\ 0 \le l \le N$, with N being zero or positive even, by simple differential operations, one can verify that both points $\frac{1}{1-a^{\underline{p}_{N+2}}}\underline{b}_{N+2}$ and $\frac{a^{\underline{p}_{N+1}}}{1-a^{\underline{p}_{N+2}}}\underline{b}_{N+2}+\underline{b}_{N+1}$ are monotonously decreasing when t_0 tends to infinity. Especially, due to Property 2, we have

$$\lim_{t_0\to\infty}\left(\frac{a^{\underline{p}_{N+1}}}{1-a^{\underline{p}_{N+2}}}\underline{b}_{N+2}+\underline{b}_{N+1}\right) \le \lim_{t_0\to\infty}\left(\frac{a^{\underline{p}_1}}{1-a^{\underline{p}_2}}\underline{b}_2+\underline{b}_1\right) \tag{77}$$

and

$$\begin{aligned}
\lim_{t_0\to\infty}\left(\frac{a^{\underline{p}_1}}{1-a^{\underline{p}_2}}\underline{b}_2+\underline{b}_1\right) &< (a^{\underline{p}_1}\gamma+\underline{b}_1)\\
&\Leftrightarrow \underline{b}_0+\frac{a^{\underline{p}_0}}{1-a^{\underline{p}_1}}\underline{b}_1-\gamma<0 \qquad (78)\\
&\Leftrightarrow \mathbf{P}_k(a^{m+1})Q_{m-1}+\mathbf{P}_{k-1}(a^{m+1})a^m\gamma<0\\
&\Leftrightarrow \gamma<\bar{i}(m,\ k).
\end{aligned}$$

On the other hand, to those periodic orbits expressed by (53), we have

$$\begin{aligned}
&\frac{a^{\overline{p}_{N+4}}}{1-a^{\overline{p}_{N+5}}}\overline{b}_{N+5}+\overline{b}_{N+4}-\frac{a^{\overline{p}_{N+2}}}{1-a^{\overline{p}_{N+3}}}\overline{b}_{N+3}-\overline{b}_{N+2}\\
&= a^{\overline{p}_{N+4}}\left(\frac{\overline{b}_{N+5}}{1-a^{\overline{p}_{N+5}}}-\frac{\overline{b}_{N+3}}{1-a^{\overline{p}_{N+3}}}\right)\\
&> 0
\end{aligned}$$

and

$$\lim_{t_0\to\infty}\frac{a^{\overline{p}_2}}{1-a^{\overline{p}_3}}\overline{b}_3+\overline{b}_2 \ \ge\ \mathbf{P}_k(a^{m+1})Q_m \Leftrightarrow \gamma \ge \underline{i}(m,\ k). \tag{79}$$

Besides, in the case of $t_0 = 1$ and $N = 0$, one can easily verify that

$$\frac{a^{\overline{p}_2}}{1-a^{\overline{p}_3}}\overline{b}_3+\overline{b}_2 = \frac{1}{1-a^{\underline{p}_2}}\underline{b}_2, \tag{80}$$

$$\frac{1}{1-a^{\overline{p}_3}}\overline{b}_3 = \frac{a^{\underline{p}_1}}{1-a^{\underline{p}_2}}\underline{b}_2+\underline{b}_1. \tag{81}$$

The above shows that, except for the periodic orbit with $N = 0$ and $t_0 = 1$, γ corresponding to the periodic orbits expressed by (52) and (53) distributes in two different parts of the interval $\big(\underline{i}(m,\ k),\ \bar{i}(m,\ k)\big)$.

For any given $\{t_l,\ 0 \le l \le N-2\} \in \mathbf{T}_0$, let the point $\frac{1}{1-a^{\underline{p}'_{N+2}}}\underline{b}'_{N+2}$ be determined by $t'_0 = 1$, $t'_1 \ge 1$ and $t'_{l+2} = t_l,\ 0 \le l \le N-2$. Then, by (42), we know that $\frac{1}{1-a^{\underline{p}_N}}\underline{b}_N > \frac{1}{1-a^{\underline{p}'_{N+2}}}\underline{b}'_{N+2}$. To simplify the notation, denote t'_1 by t below. Thus, we have

$$\begin{aligned}\frac{d}{dt}\frac{1}{1-a^{\underline{p}'_{N+2}}}\underline{b}'_{N+2} &= a^{\underline{p}_N}\frac{d}{dt}\frac{\underline{b}_{N+1}}{1-a^{\underline{p}_N+\underline{p}_{N+1}}}\\ &= -\frac{a^{2\underline{p}_N}\left[(1-a^{\underline{p}_{N-1}})\underline{b}_N-(1-a^{\underline{p}_N})\underline{b}_{N-1}\right]}{(1-a^{\underline{p}_N})(1-a^{\underline{p}_N+\underline{p}_{N+1}})}\frac{da^{\underline{p}_{N+1}}}{dt}\\ &> 0.\end{aligned}\tag{82}$$

This shows that the sequence $\left\{\frac{1}{1-a^{\underline{p}'_{N+2}}}\underline{b}'_{N+2},\ t'_1\geq 1\right\}$ is monotonously increasing when t'_1 tends to infinity, and

$$\begin{aligned}\lim_{t_1\to\infty}\frac{\underline{b}'_{N+2}}{1-a^{\underline{p}'_{N+2}}}-\frac{a^{\underline{p}_{N-1}}\underline{b}_N}{1-a^{\underline{p}_N}}-\underline{b}_{N-1} &= (1-a^{\underline{p}_{N-1}})\underline{b}_N-(1-a^{\underline{p}_N})\underline{b}_{N-1}\\ &= \mathbf{P}_k(a^{m+1})Q_m-(a^{\underline{p}_1}\gamma+\underline{b}_1).\end{aligned}\tag{83}$$

The result (83) shows clearly that the infimum of the set

$$\left\{\gamma\,\middle|\,\lim_{t_1\to\infty}\frac{\underline{b}'_{N+2}}{1-a^{\underline{p}'_{N+2}}}\geq \mathbf{P}_k(a^{m+1})Q_m\right\}$$

is the same as the supremum of the set

$$\left\{\gamma\,\middle|\,\frac{a^{\underline{p}_{N-1}}}{1-a^{\underline{p}_N}}\underline{b}_N+\underline{b}_{N-1}< a^{\underline{p}_1}\gamma+\underline{b}_1\right\},$$

namely,

$$\begin{aligned}&\inf\left\{\gamma\,\middle|\,\lim_{t_1\to\infty}\frac{\underline{b}'_{N+2}}{1-a^{\underline{p}'_{N+2}}}\geq \mathbf{P}_k(a^{m+1})Q_m\right\}\\ &=\sup\left\{\gamma\,\middle|\,\frac{a^{\underline{p}_{N-1}}}{1-a^{\underline{p}_N}}\underline{b}_N+\underline{b}_{N-1}< a^{\underline{p}_1}\gamma+\underline{b}_1\right\}.\end{aligned}\tag{84}$$

On the other hand, for the case of $N\geq 2$ and $t_0=1$, one can show, in the same way, that both points $\frac{1}{1-a^{\overline{p}_{N+3}}}\overline{b}_{N+3}$ and $\frac{a^{\overline{p}_{N+2}}}{1-a^{\overline{p}_{N+3}}}\overline{b}_{N+3}+\overline{b}_{N+2}$ are monotonously decreasing when t_1 tends to infinity. In particular,

$$\lim_{t_1\to\infty}\frac{1}{1-a^{\overline{p}_{N+3}}}\overline{b}_{N+3}=a^{\overline{p}_{N+1}}\overline{b}_N+\frac{1-a^{\overline{p}_{N+1}}+a^{\overline{p}_N+\overline{p}_{N+1}}}{1-a^{\overline{p}_{N+1}}}\overline{b}_{N+1}.\tag{85}$$

Hence, for any given $\{t_l,\ 0\leq l\leq N\}\in\mathbf{T}_0$, let $t'_0=1$, $t'_1\geq 1$ and $t'_{l+2}=t_l,\ 0\leq l\leq N$, so that

$$\begin{aligned}\frac{a^{\overline{p}_{N+2}}\overline{b}_{N+3}}{1-a^{\overline{p}_{N+3}}}+\overline{b}_{N+2}-\lim_{t_1\to\infty}\frac{\overline{b}'_{N+5}}{1-a^{\overline{p}'_{N+5}}} &= (1-a^{\overline{p}_{N+3}})\overline{b}_{N+2}-(1-a^{\overline{p}_{N+2}})\overline{b}_{N+3}\\ &= \mathbf{P}_k(a^{m+1})Q_m-(a^{\underline{p}_1}\gamma+\underline{b}_1).\end{aligned}\tag{86}$$

This implies that

$$\sup\left\{\gamma\left|\lim_{t_1\to\infty}\frac{\overline{b}'_{N+5}}{1-a^{\overline{p}'_{N+5}}}<a^{\overline{p}_1}\gamma+\overline{b}_1\right.\right\}$$
$$=\inf\left\{\gamma\left|\frac{a^{\overline{p}_{N+2}}}{1-a^{\overline{p}_{N+3}}}\overline{b}_{N+3}+\overline{b}_{N+2}\geq \mathbf{P}_k(a^{m+1})Q_m\right.\right\}. \qquad (87)$$

Thus, for any two different group of positive integers $t_l,\ 0\leq l\leq N$ and $t'_l,\ 0\leq l\leq L$, if

$$\inf\left\{\gamma\left|\frac{\underline{b}_{N+2}}{1-a^{\underline{p}_{N+2}}}\geq \mathbf{P}_k(a^{m+1})Q_m\right.\right\}$$
$$\neq\sup\left\{\gamma\left|\frac{a^{\underline{p}'_{L+1}}}{1-a^{\underline{p}'_{L+2}}}\underline{b}'_{L+2}+\underline{b}'_{L+1}<a^{\underline{p}_1}\gamma+\underline{b}_1\right.\right\},$$

then, according to (84), there must be a periodic orbit such that the corresponding parameter γ is between $\inf\left\{\gamma\left|\frac{\underline{b}_{N+2}}{1-a^{\underline{p}_{N+2}}}\geq \mathbf{P}_k(a^{m+1})Q_m\right.\right\}$ and $\sup\left\{\gamma\left|\frac{a^{\underline{p}'_{L+1}}}{1-a^{\underline{p}'_{L+2}}}\underline{b}'_{L+2}+\underline{b}'_{L+1}<a^{\underline{p}_1}\gamma+\underline{b}_1\right.\right\}$. For the case of

$$\inf\left\{\gamma\left|\frac{a^{\overline{p}_{N+2}}}{1-a^{\overline{p}_{N+3}}}\overline{b}_{N+3}+\overline{b}_{N+2}\geq \mathbf{P}_k(a^{m+1})Q_m\right.\right\}$$
$$\neq\sup\left\{\gamma\left|\frac{\overline{b}'_{L+3}}{1-a^{\overline{p}'_{L+3}}}<a^{\overline{p}_1}\gamma+\overline{b}_1\right.\right\},$$

we also have the same conclusion.

Assume $\gamma_0\in\big(\underline{i}(m,\ k),\ \overline{i}(m,\ k)\big)$ is not corresponding to any periodic orbit, and not equal to $\sup\left\{\gamma\left|\frac{a^{\underline{p}_{N+1}}}{1-a^{\underline{p}_{N+2}}}\underline{b}_{N+2}+\underline{b}_{N+1}<a^{\underline{p}_1}\gamma+\underline{b}_1\right.\right\}$ nor $\sup\left\{\gamma\left|\frac{\overline{b}_{N+3}}{1-a^{\overline{p}_{N+3}}}<a^{\overline{p}_1}\gamma+\overline{b}_1\right.\right\}$ for any given group of positive integers $t_l,\ 0\leq l\leq N$. Then, according to Corollary 2, every parameter interval determined by (52) and (53) is left-closed and right-open. Hence, there must be an open interval, denoted by $(\alpha,\ \beta)$, such that $\gamma_0\in(\alpha,\ \beta)$ and every $\gamma\in(\alpha,\ \beta)$ is not corresponding to any periodic orbit. But, from (78), (79), (84) and (87), we know that this case is impossible.

Assume $\gamma_0=\sup\left\{\gamma\left|\frac{a^{\underline{p}_{N+1}}}{1-a^{\underline{p}_{N+2}}}\underline{b}_{N+2}+\underline{b}_{N+1}<a^{\underline{p}_1}\gamma+\underline{b}_1\right.\right\}$ for some group of given positive integers $t_l,\ 0\leq l\leq N$, with N being zero or positive even, and it is corresponding to some periodic orbit. Then, by the definition of γ_0 and characteristics of the parameter interval corresponding to a periodic

orbit, there is a positive real number, $\gamma_1 > \gamma_0$, such that every parameter $\gamma \in [\gamma_0,\ \gamma_1)$ corresponds to the same periodic orbit. But equality (84) and

$$\lim_{t_1 \to \infty} \left(\frac{\underline{b}'_{N+2}}{1 - a^{\underline{p}'_{N+2}}} - \frac{a^{\underline{p}'_{N+1}} \underline{b}'_{N+2}}{1 - a^{\underline{p}'_{N+2}}} - \underline{b}'_{N+1} \right) = \mathbf{P}_k(a^{m+1}) Q_m - (a^{\underline{p}_1}\gamma + \underline{b}_1) \quad (88)$$

together show that such an interval does not exist.

Similarly, one can reach the same conclusion about

$$\gamma_0 = \sup \left\{ \gamma \left| \frac{\overline{b}_{N+3}}{1 - a^{\overline{p}_{N+3}}} < a^{\overline{p}_1}\gamma + \overline{b}_1 \right. \right\}.$$

With respect to the foregoing Theorems 3 - 5 and Corollary 1, one can immediately draw the following conclusion.

Corollary 3 *Suppose $0 < \gamma < a < 1$. Then, system (1) of Type I has no periodic orbits if and only if*

$$\gamma \in \{\overline{B}_m,\ m \in \boldsymbol{N}\} \bigcup \gamma_1(m,\ k) \bigcup \gamma_2(m,\ k). \quad (89)$$

In the rest of this paper, we discuss the characteristics of the dynamics of system (1) for the case of $a = 1$.

Theorem 6 *Suppose $a = 1$. Then, system (1) of Type I has the following properties:*

1. System (1) of Type I has no 2-periodic orbit.

2. For every positive integer $n > 2$, system (1) of Type I has an n-periodic orbit if and only if there are two positive integers, l and m, with $\gcd(l, m) = 1$, such that $n = l + m$ and $\gamma = \frac{l}{m}$. In particular, when the above condition is satisfied, all the points in the interval $[-1, \gamma)$ are n-periodic points.

3. Suppose $\gamma = \frac{l}{m}$ satisfies the conditions in part 2. Then, any periodic orbit has the following property: when the n points of the periodic orbit $\{x_i,\ 1 \le i \le n\}$ are sorted by their values, the distance between two neighboring points satisfies $\frac{1}{m} \le |x_{i+1} - x_i| < \frac{2}{m}$, $1 \le i < n$.

Proof 13 1. We have known that a point x is 2- periodic only if $\Delta(x)$and $\Delta(f_1(x))$ have different signs. Therefore, equality

$$f_1^2(x) = x + \Delta(x) + \Delta(f_1(x)) = x$$

holds true only if $\Delta(x) + \Delta(f_1(x)) = 0$, namely, $\Delta_1 = \Delta_2$. This contradicts the assumption of $\Delta_1 \neq \Delta_2$.

2. Necessity. First of all, we have known that any periodic orbit must include points in the two intervals $[-1,\ 0)$ and $[0,\ \gamma)$. Moreover, for any point $x \in [0,\ \gamma)$, the following equality holds true:

$$f_1^n(x) = x + m\gamma - l, \quad (90)$$

where l, m are positive integers and $n = l + m$. Therefore, a point $x \in [0,\ \gamma)$ is n-periodic only if $m\gamma - l = 0$, that is, $\gamma = \frac{l}{m}$.

Let $c = \gcd(l, m)$. We next prove $c = 1$.

Case one: Assume $\gamma = \frac{l}{m} < 1$. Let $m = Kl + r$. Since, for each $x \in [0,\ \gamma)$,

$$f_1^N(x) = x + \frac{(N-1)l}{m} - 1 = x - \frac{l+r}{m} < 0 \tag{91}$$

and

$$f_1^{1+N}(x) = x + \frac{Nl}{m} - 1 = x - \frac{r}{m}, \tag{92}$$

we know that a positive integer $n > 2$ is a period of system (1) of Type I with parameter $a = 1$ only if there are non-negative integers l_1 and l_2, $l_1 + l_2 \geq 1$, such that $n = l_1(2+N) + l_2(1+N)$. Here, $l_1 + l_2$ is obviously the number of points that belong to an n-periodic orbit and to the interval $[0,\ \gamma)$. In particular, l_1 is the number of the points that satisfy $f_1^{1+N}(x) < 0$ and l_2 is the number of the points that satisfy $f_1^{1+N}(x) \geq 0$, respectively. Besides, it is clear that $m = N(l_1 + l_2) + l_1$; therefore, $c > 1$ if and only if $l_1 = 0$, $l_2 > 1$ or $l_2 = 0$, $l_1 > 1$.

Assume $l_1 = 0$ and $l_2 > 1$. Let $x_1, x_2, \cdots, x_l$ be l positive points. Then, for each $1 \leq i \leq l$, by (91) and (92), we have $f_1^{1+N}(x_i) = x$. This shows clearly that the point x is not n-periodic. Similarly, one can prove that x is not n-periodic if $l_2 = 0$.

Case two: Assume $\gamma = \frac{l}{m} > 1$. Let $l = Km + r$.

For each $x \in [-1,\ 0)$, since

$$f_1(x) = x + \gamma \geq (N-1) + \frac{r}{m} > 0, \tag{93}$$

we have

$$f_1^N(x) = x + \gamma - (N-1) \geq \frac{r}{m} \tag{94}$$

and

$$f_1^{1+N}(x) = x + \frac{r}{m}. \tag{95}$$

Since $0 \leq \frac{r}{m} < 1$, we conclude that n is a period of system (1) of Type I with parameter $a = 1$ only if there are non-negative integers m_1 and m_2, $m_1 + m_2 \geq 1$, such that $n = m_1(2+N) + m_2(1+N)$. Here, $m_1 + m_2$ is obviously the number of points that belong to an n-periodic orbit and to the interval $[-1,\ 0)$. In particular, m_1 is the number of the points that satisfy $f_1^{1+N}(x) \geq 0$ and m_2 is the number of the points that satisfy $f_1^{1+N}(x) < 0$, respectively. Clearly, $l = N(m_1 + m_2) + m_1$; therefore, $c > 1$ if and only if $m_1 = 0$, $m_2 > 1$ or $m_2 = 0$, $m_1 > 1$.

Similar to the above proof, one can verify that the point x is not n-periodic in both of the two cases.

Sufficiency. We prove it by contradiction. Assume that there exists a point $x \in [-1,\ \gamma)$ that is not n periodic. Then, there should exist two positive integers, M and L, with $M+L=n$, $M \neq m$ and $L \neq l$, such that

$$f_1^n(x) = x + M\gamma - L. \tag{96}$$

i. If $M < m$, let $m - M = P$. Then, one must have that $L = l + P$, so that

$$f_1^n(x) = x + M\gamma - L = x - P(1+\gamma) < -(P-1)\gamma - P \leq -1. \tag{97}$$

ii. If $M > m$, let $M = m + P$. Then, one must have that $L = l - P$, so that

$$f_1^n(x) = x + M\gamma - L = x + P(1+\gamma) \geq (P-1) + P\gamma \geq \gamma. \tag{98}$$

Obviously, both of the above situations cannot occur.

3. For any point $x \in [-1,\ \gamma)$ and any positive integer $1 \leq k < n$, there exist two non-negative integers, $M \leq m$, $L \leq l$ and $M + L < n$, such that

$$f_1^k(x) = x + M\gamma - L.$$

Since $|M\gamma - L| \neq 0$, one must have $|M\gamma - L| = |\frac{Ml-mL}{m}| \geq \frac{1}{m}$. This shows that the distance between any two neighboring points is greater than $\frac{1}{m}$. If there are two neighboring points whose distance is equal to or greater than $\frac{2}{m}$, then we have

$$x_n - x_1 \geq \frac{n}{m} = \frac{m+l}{m} = 1 + \gamma.$$

But this is a contradiction, since $x_1,\ x_n \in [-1,\ \gamma)$. This completes the proof.

3 Appendices

Appendix I. Proof of Proposition 2

Firstly, it is not hard to verify that the coefficient matrix of the linear system of equations (31) is in the form of

$$\overline{A}_{r_N + r_{N+1}} = \begin{bmatrix} E_{r_N} & -a_1 E_{r_N} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & E_{r_N} & -a_1 F_{r_N \times r_{N-1}} & -a_1 F_{r_N} \\ 0 & \cdots & 0 & E_{r_{N-1}} & -a_1 F_{r_{N-1} \times r_N} \\ -a_0 E_{r_N} & 0 & \cdots & 0 & E_{r_N} \end{bmatrix}, \tag{99}$$

where

$$\begin{aligned} F_{r_N\times r_{N-1}} &= \begin{bmatrix} E_{r_{N-1}} \\ 0 \end{bmatrix}, \\ F_{r_{N-1}\times r_1} &= \begin{bmatrix} 0 & E_{r_{N-1}} \end{bmatrix}, \\ F_{r_N} &= \begin{bmatrix} 0 & 0 \\ E_{r_N-r_{N-1}} & 0 \end{bmatrix}. \end{aligned} \tag{100}$$

Thus, multiplying $\overline{A}_{r_N+r_{N+1}}$ by the elementary transform matrix

$$T_{r_N+r_{N+1}} = \begin{bmatrix} E_{r_N} & 0 & \cdots & & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & E_{r_N} & & \\ 0 & \cdots & 0 & E_{r_{N-1}} & 0 \\ a_0E_{r_N} & \cdots & a_0a_1^{t_N-1}E_{r_N} & 0 & E_{r_N} \end{bmatrix} \tag{101}$$

to the left side, one gets

$$T_{r_N+r_{N+1}}\overline{A}_{r_N+r_{N+1}} = \begin{bmatrix} E_{r_N} & -a_1E_{r_N} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & E_{r_N} & -a_1E_{r_N\times r_{N-1}} & -a_1F_{r_N} \\ 0 & \cdots & 0 & E_{r_{N-1}} & -a_1E_{r_{N-1}\times r_N} \\ 0 & \cdots & 0 & -a_0a_1^{t_N}E_{r_N\times r_{N-1}} & E_{r_N}-a_0a_1^{t_N}F_{r_N} \end{bmatrix} \tag{102}$$

It can be verified that

$$\begin{bmatrix} -a_1E_{r_N\times r_{N-1}} & -a_1F_{r_N} \end{bmatrix} = \begin{bmatrix} -a_1E_{r_N} & 0 \end{bmatrix} \tag{103}$$

and

$$\overline{A}_{r_{N-1}+r_N} = \begin{bmatrix} E_{r_{N-1}} & -a_1E_{r_{N-1}\times r_N} \\ -a_0a_1^{t_N}E_{r_N\times r_{N-1}} & E_{r_N}-a_0a_1^{t_N}F_{r_N} \end{bmatrix}, \tag{104}$$

which can be expressed in detail as follows:

$$\overline{A}_{r_{N-1}+r_N} = \begin{bmatrix} E_{r_{N-1}} & 0 & \cdots & 0 & -a_1E_{r_{N-1}} \\ -a_2H_{r_{N-2}\times r_{N-1}} & E_{r_{N-2}} & \ddots & 0 & \\ -a_2H_{r_{N-1}} & -a_2H_{r_{N-1}\times r_{N-2}} & E_{r_{N-1}} & \vdots & \\ 0 & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -a_2E_{r_{N-1}} & E_{r_{N-1}} \end{bmatrix} \tag{105}$$

where

$$
\begin{aligned}
H_{r_{N-2}\times r_{N-1}} &= \left[\begin{array}{cc} E_{r_{N-2}} & 0 \end{array}\right], \\
H_{r_{N-1}\times r_{N-2}} &= \left[\begin{array}{c} 0 \\ E_{r_{N-2}} \end{array}\right], \\
H_{r_{N-1}} &= \left[\begin{array}{cc} 0 & E_{r_{N-1}-r_{N-2}} \\ 0 & 0 \end{array}\right].
\end{aligned} \tag{106}
$$

Besides,

$$
T_{r_N+r_{N+1}}\left(\begin{array}{c} b_1\mathbf{1}_{r_{N+1}} \\ b_0\mathbf{1}_{r_N} \end{array}\right) = \left(\begin{array}{c} b_1\mathbf{1}_{r_{N+1}} \\ \left[b_0 + b_1 a_0 \mathbf{P}_{t_N-1}(a_1)\right]\mathbf{1}_{r_N} \end{array}\right) = \left(\begin{array}{c} b_1\mathbf{1}_{r_{N+1}} \\ b_2\mathbf{1}_{r_N} \end{array}\right) \tag{107}
$$

Hence, we can see that the system of linear equations (32) reduces to the system of linear equations (33) with $l = 1$.

Moreover, multiplying $\overline{A}_{r_{N-1}+r_N}$ by the elementary transform matrix

$$
T_{r_{N-1}+r_N} = \left[\begin{array}{ccccc}
E_{r_{N-1}} & 0 & a_1 a_2^{t_{N-1}-1} E_{r_{N-1}} & \cdots & a_1 E_{r_{N-1}} \\
0 & E_{r_{N-2}} & 0 & \cdots & 0 \\
\vdots & \ddots & E_{r_{N-1}} & \ddots & \vdots \\
 & & & \ddots & 0 \\
0 & \cdots & & 0 & E_{r_{N-1}}
\end{array}\right] \tag{108}
$$

to the left side, and noticing of the relation $a_{s+2} = a_s a_{s+1}^{t_{N-s}}$, one gets

$$
T_{r_{N-1}+r_N}\overline{A}_{r_{N-1}+r_N} =
\left[\begin{array}{ccccc}
E_{r_{N-1}} - a_3 H_{r_{N-1}} & -a_3 E_{r_{N-1}\times r_{N-2}} & 0 & \cdots & 0 \\
-a_2 E_{r_{N-2}\times r_{N-1}} & E_{r_{N-2}} & 0 & & \vdots \\
-a_2 H_{r_{N-1}} & -a_2 E_{r_{N-1}\times r_{N-2}} & E_{r_{N-1}} & \ddots & \\
 & & & \ddots & 0 \\
0 & \cdots & & 0 & E_{r_{N-1}}
\end{array}\right] \tag{109}
$$

where

$$
\left[-a_2 H_{r_{N-1}} \quad -a_2 E_{r_{N-1}\times r_{N-2}}\right] = \left[\begin{array}{cc} 0 & -a_2 E_{r_{N-1}} \end{array}\right], \tag{110}
$$

$$
\overline{A}_{r_{N-2}+r_{N-1}} = \left[\begin{array}{cc} E_{r_{N-1}} - a_3 H_{r_{N-1}} & -a_3 E_{r_{N-1}\times r_{N-2}} \\ -a_2 E_{r_{N-2}\times r_{N-1}} & E_{r_{N-2}} \end{array}\right], \tag{111}
$$

and

$$
\begin{aligned}
T_{r_{N-1}+r_N}\left(\begin{array}{c} b_1\mathbf{1}_{r_{N-1}} \\ b_2\mathbf{1}_{r_N} \end{array}\right) &= \left(\begin{array}{c} \left[b_1 + b_2 a_1 \mathbf{P}_{t_{N-1}-1}(a_2)\right]\mathbf{1}_{r_{N-1}} \\ b_2\mathbf{1}_{r_N} \end{array}\right) \\
&= \left(\begin{array}{c} b_3\mathbf{1}_{r_{N-1}} \\ b_2\mathbf{1}_{r_N} \end{array}\right).
\end{aligned} \tag{112}
$$

Clearly, the first subsystem of linear equations (33) is also translated into its equivalent form (33) with $l = 1$.

The general cases can be proved by induction without any technical difficulty. Thus, based on the above analysis and discussions, when N is odd, solving the system of linear equations (32) reduces to solving the recursive systems of linear equations (33), (33), and the program will end at (38). Similarly, when N is zero or an even integer, the recursive program will end at (41), where a_l and b_l are given by (36) and (37).

Appendix II. Proof of Property 1

Since the proof methods for assertions $1-4$ are totally the same, we only give a proof for assertion 1.

On one hand, we have

$$\begin{aligned} n &= i(m+1) + jm \\ &= (k(r_N + r_{N+1}) + r_N)(m+1) + (r_N + r_{N+1})m \\ &= r_N \underline{p}_0 + r_{N+1}\underline{p}_1, \end{aligned}$$

and on the other hand, by the stipulation on $t_0 = r_1$ and the definition of $\underline{p}_l$, we have

$$\begin{aligned} \underline{p}_{N+2} &= \underline{p}_N + t_0\underline{p}_{N+1} = \underline{p}_N + r_1\underline{p}_{N+1} \\ &= \underline{p}_N + r_1(\underline{p}_{N-1} + t_1\underline{p}_N) = r_1\underline{p}_{N-1} + r_2\underline{p}_N \\ &\vdots \\ &= r_{N-1}\underline{p}_1 + r_N\underline{p}_2 = r_{N-1}\underline{p}_1 + r_N(\underline{p}_0 + t_N\underline{p}_1) \\ &= r_N\underline{p}_0 + r_{N+1}\underline{p}_1. \end{aligned} \tag{113}$$

Thus, we have actually proved assertion 1. Other assertions can be similarly verified.

Appendix III. Proof of Property 2

1. First, we verify (44). For any $0 \le 2l \le N$, due to (36), (37) and the formula $\underline{p}_{s+2} = \underline{p}_s + t_{N-s}\underline{p}_{s+1}$, we have

$$\begin{aligned}
\frac{\underline{b}_{2l+2}}{1-a^{\underline{p}_{2l+2}}} - \frac{\underline{b}_{2l+1}}{1-a^{\underline{p}_{2l+1}}} &= \frac{(1-a^{\underline{p}_{2l+1}})\underline{b}_{2l+2} - (1-a^{\underline{p}_{2l+2}})\underline{b}_{2l+1}}{(1-a^{\underline{p}_{2l+1}})(1-a^{\underline{p}_{2l+2}})} \\
&= \frac{(1-a^{\underline{p}_{2l+1}})\underline{b}_{2l} - (1-a^{\underline{p}_{2l}})\underline{b}_{2l+1}}{(1-a^{\underline{p}_{2l+1}})(1-a^{\underline{p}_{2l+2}})} \\
&\vdots \\
&= \frac{(1-a^{\underline{p}_{1}})\underline{b}_{0} - (1-a^{\underline{p}_{0}})\underline{b}_{1}}{(1-a^{\underline{p}_{2l+1}})(1-a^{\underline{p}_{2l+2}})} \\
&= \frac{\mathbf{P}_k(a^{m+1})Q_m - (a^{\underline{p}_1}\gamma + \underline{b}_1)}{(1-a^{\underline{p}_{2l+1}})(1-a^{\underline{p}_{2l+2}})} \\
&> 0.
\end{aligned} \tag{114}$$

Thus, we have already proved inequality (44). Utilizing this result, assertions (42) and (43) can be easily verified as follows:

$$\begin{aligned}
\frac{\underline{b}_{2l}}{1-a^{\underline{p}_{2l}}} - \frac{\underline{b}_{2l+2}}{1-a^{\underline{p}_{2l+2}}} &= \frac{(1-a^{\underline{p}_{2l+2}})\underline{b}_{2l} - (1-a^{\underline{p}_{2l}})\underline{b}_{2l+2}}{(1-a^{\underline{p}_{2l}})(1-a^{\underline{p}_{2l+2}})} \\
&= \frac{(a^{\underline{p}_{2l}} - a^{\underline{p}_{2l+2}})\left[(1-a^{\underline{p}_{2l+1}})\underline{b}_{2l} - (1-a^{\underline{p}_{2l}})\underline{b}_{2l+1}\right]}{(1-a^{\underline{p}_{2l}})(1-a^{\underline{p}_{2l+1}})(1-a^{\underline{p}_{2l+2}})} \\
&> 0
\end{aligned} \tag{115}$$

and

$$\frac{\underline{b}_{2l+3}}{1-a^{\underline{p}_{2l+3}}} - \frac{\underline{b}_{2l+1}}{1-a^{\underline{p}_{2l+1}}} \tag{116}$$

$$\begin{aligned}
&= \frac{(1-a^{\underline{p}_{2l+1}})\underline{b}_{2l+3} - (1-a^{\underline{p}_{2l+3}})\underline{b}_{2l+1}}{(1-a^{\underline{p}_{2l+1}})(1-a^{\underline{p}_{2l+3}})} \\
&= \frac{(a^{\underline{p}_{2l+1}} - a^{\underline{p}_{2l+3}})\left[(1-a^{\underline{p}_{2l+1}})\underline{b}_{2l+2} - (1-a^{\underline{p}_{2l+2}})\underline{b}_{2l+1}\right]}{(1-a^{\underline{p}_{2l+1}})(1-a^{\underline{p}_{2l+2}})(1-a^{\underline{p}_{2l+3}})} \\
&> 0.
\end{aligned} \tag{117}$$

2. Denote t_0 by t. Then, since

$$\begin{aligned}
\frac{d}{dt}\frac{1}{1-a^{\underline{p}_2}}\underline{b}_2 &= \frac{1}{a^{\underline{p}_1}(1-a^{\underline{p}_1})}\frac{d}{da^{\underline{p}_2}}\left(\frac{\underline{b}_0(1-a^{\underline{p}_1}) + \underline{b}_1(a^{\underline{p}_0} - a^{\underline{p}_2})}{1-a^{\underline{p}_2}}\right) \\
&= \frac{a^{\underline{p}_2}\ln a}{a^{\underline{p}_1}(1-a^{\underline{p}_1})}\frac{\underline{b}_0(1-a^{\underline{p}_1}) - \underline{b}_1(1-a^{\underline{p}_0})}{(1-a^{\underline{p}_2})^2} \\
&< 0,
\end{aligned} \tag{118}$$

we know that $\frac{1}{1-a^{\underline{p}_2}}\underline{b}_2$ is monotonously decreasing when t tends to infinity. Thus, for $\frac{1}{1-a^{\underline{p}_2}}\underline{b}_2$ with $t_0 = 1$, according to this monotonicity and inequality (42), we have

$$\frac{a^{\underline{p}_2'}}{1-a^{\underline{p}_3'}}\underline{b}_3' + \underline{b}_2' < \frac{1}{1-a^{\underline{p}_2'}}\underline{b}_2' \le \frac{1}{1-a^{\underline{p}_2}}\underline{b}_2. \tag{119}$$

Generally, utilizing (42), we obtain

$$\frac{a^{\underline{p}_{L+1}'}}{1-a^{\underline{p}_{L+2}'}}\underline{b}_{L+2}' + \underline{b}_{L+1}' < \frac{1}{1-a^{\underline{p}_{L+1}'}}\underline{b}_{L+1}' < \frac{1}{1-a^{\underline{p}_2'}}\underline{b}_2' \le \frac{1}{1-a^{\underline{p}_2}}\underline{b}_2. \tag{120}$$

The above shows that there are no L and t_l', $0 \le l \le L$, such that the equality (45) is satisfied with $\frac{1}{1-a^{\underline{p}_2}}\underline{b}_2$ when $t_0 = 1$.

If $t_0 \ge 2$, we take $L = N+1$ and arrange an ordered set of positive integers $\{t_l',\ 0 \le l \le L\}$ as follows: $t_0' = 1$, $t_1' = t_0 - 1$ and $t_{l+1}' = t_l$, $1 \le l \le N$. With this choice, one can easily verify that $\underline{p}_{l+2}' = \underline{p}_{l+2}$ and $\underline{b}_{l+2}' = \underline{b}_{l+2}$, $0 \le l \le N-1$. Hence,

$$\underline{p}_{N+2} = \underline{p}_N + t_0\underline{p}_{N+1} = \underline{p}_N + (t_1'+1)\underline{p}_{N+1} = \underline{p}_{N+1}' + \underline{p}_{N+2}' = \underline{p}_{N+3}', \tag{121}$$

$$\begin{aligned} &\frac{1}{1-a^{\underline{p}_{L+2}'}}\underline{b}_{L+2}' \\ &= \frac{1}{1-a^{\underline{p}_{N+2}}}\left[\underline{b}_{N+1}' + a^{\underline{p}_{N+1}'}\underline{b}_{N+2}'\right] \\ &= \frac{1}{1-a^{\underline{p}_{N+2}}}\left[\underline{b}_{N+1} + a^{\underline{p}_{N+1}}\left(\underline{b}_N + \underline{b}_{N+1}a^{\underline{p}_N}\mathbf{P}_{t_1'-1}(a^{\underline{p}_{N+1}})\right)\right] \\ &= \frac{1}{1-a^{\underline{p}_{N+2}}}\left[\underline{b}_{N+1} + a^{\underline{p}_{N+1}}\left(\underline{b}_{N+2} - a^{\underline{p}_N+(t_0-1)\underline{p}_{N+1}}\underline{b}_{N+1}\right)\right] \\ &= \frac{\underline{p}_{N+1}}{1-a^{\underline{p}_{N+2}}}\underline{b}_{N+2} + \underline{b}_{N+1}, \end{aligned} \tag{122}$$

and

$$\begin{aligned} &\frac{a^{\underline{p}_{L+1}'}}{1-a^{\underline{p}_{L+2}'}}\underline{b}_{L+2}' + \underline{b}_{L+1}' \\ &= \frac{a^{\underline{p}_{N+2}'}}{1-a^{\underline{p}_{N+3}'}}\left(\underline{b}_{N+1}' + a^{\underline{p}_{N+1}'}\underline{b}_{N+2}'\right) + \underline{b}_{N+2}' \\ &= \frac{1}{1-a^{\underline{p}_{N+3}'}}\left(a^{\underline{p}_{N+2}'}\underline{b}_{N+1}' + \underline{b}_{N+2}'\right) \end{aligned}$$

$$= \frac{1}{1-a^{\underline{p}'_{N+3}}}\left[\underline{b}'_N + \underline{b}'_{N+1}\left(a^{\underline{p}'_{N+2}} + a^{\underline{p}'_N}\mathbf{P}_{t'_1-1}(a^{\underline{p}'_{N+1}})\right)\right] \tag{123}$$
$$= \frac{1}{1-a^{\underline{p}'_{N+3}}}\left[\underline{b}'_N + \underline{b}'_{N+1}a^{\underline{p}'_N}\mathbf{P}_{t'_1}(a^{\underline{p}'_{N+1}})\right]$$
$$= \frac{1}{1-a^{\underline{p}_{N+2}}}\underline{b}_{N+2}.$$

If $t_0 = 1$ and $N \in \mathbf{PE}$, then we take $L = N-1$ and arrange an ordered set of positive integers $\{t'_l,\ 0 \le l \le L\}$ as follows: $t'_0 = t_1 + 1$ and $t'_l = t_{l+1},\ 1 \le l \le L$. With this choice, we have $\underline{p}'_{l+2} = \underline{p}_{l+2}$ and $\underline{b}'_{l+2} = \underline{b}_{l+2},\ 0 \le l \le L-1$. Hence,

$$\underline{p}'_{L+2} = \underline{p}'_{N-1} + t'_0\underline{p}'_N = \underline{p}'_N + \underline{p}'_{N-1} + t_1\underline{p}'_N = \underline{p}_N + \underline{p}_{N+1} = \underline{p}_{N+2}, \tag{124}$$

$$\begin{aligned}
\frac{1}{1-a^{\underline{p}'_{L+2}}}\underline{b}'_{L+2} &= \frac{1}{1-a^{\underline{p}_{N+2}}}\left(\underline{b}_{N-1} + \underline{b}_N a^{\underline{p}_{N-1}}\mathbf{P}_{t_1}(a^{\underline{p}_N})\right)\\
&= \frac{1}{1-a^{\underline{p}_{N+2}}}\left(\underline{b}_{N+1} + \underline{b}_N a^{\underline{p}_{N+1}}\right)\\
&= \frac{1}{1-a^{\underline{p}_{N+2}}}\left[a^{\underline{p}_{N+1}}\underline{b}_{N+2} + (1-a^{\underline{p}_{N+2}})\underline{b}_{N+1}\right] \qquad (125)\\
&= \frac{a^{\underline{p}_{N+1}}}{1-a^{\underline{p}_{N+2}}}\underline{b}_{N+2} + \underline{b}_{N+1},
\end{aligned}$$

and

$$\begin{aligned}
\frac{a^{\underline{p}'_{L+1}}\underline{b}'_{L+2}}{1-a^{\underline{p}'_{L+2}}} + \underline{b}'_{L+1} &= \frac{a^{\underline{p}_N}}{1-a^{\underline{p}_{N+2}}}\left(\underline{b}_{N-1} + \underline{b}_N a^{\underline{p}_{N-1}}\mathbf{P}_{t_1}(a^{\underline{p}_N})\right) + \underline{b}_N\\
&= \frac{1}{1-a^{\underline{p}_{N+2}}}\left(a^{\underline{p}_N}\underline{b}_{N+1} + \underline{b}_N\right) \qquad (126)\\
&= \frac{1}{1-a^{\underline{p}_{N+2}}}\underline{b}_{N+2}.
\end{aligned}$$

The proof of conclusion 3 can be similarly carried out.

Appendix IV Proof of Corollary 2

Due to the similarity of the proofs, we only prove (52).

Firstly, by the definition of Q_l, it is obvious that the coefficients of γ in both of Q_{m-1} and Q_m are positive. Furthermore, with respect to (37), $\underline{b}_l$, $0 \le l \le N+2$, are linear compositions of $\underline{b}_l = \alpha_l(a)Q_m + \beta_l(a)Q_{m-1}$, where $\alpha_l(a) > 0$ and $\beta_l(a) > 0$. Since $\alpha_0(a) = \mathbf{P}_k(a^{m+1}) > (1-a^{\underline{p}_0})\mathbf{P}_k(a^{m+1})$, and by (37), $\alpha_{l+2}(a) > \alpha_l(a)$ for all $l \ge 0$, we can see that both of $\alpha_l(a) - (1 -$

$a^{\underline{p}_l})\mathbf{P}_k(a^{m+1})$ and $\beta_l(a)+a(\alpha_l(a)-(1-a^{\underline{p}_l})\mathbf{P}_k(a^{m+1}))$ are positive. Thus, we have

$$\begin{aligned}\frac{1}{1-a^{\underline{p}_{l+2}}}\underline{b}_{l+2} &\geq \mathbf{P}_k(a^{m+1})Q_m \Leftrightarrow \left[\alpha_{l+2}(a)-(1-a^{\underline{p}_{l+2}})\mathbf{P}_k(a^{m+1})\right]Q_m \\ &+ \beta_{l+2}(a)Q_{m-1}\geq 0;\end{aligned}$$

that is,

$$\gamma \geq a^{m-1}\left(\frac{1-a^{m-1}}{1-a}+\frac{\alpha_{l+2}(a)-(1-a^{\underline{p}_{l+2}})\mathbf{P}_k(a^{m+1})}{\beta_{l+2}(a)+a(\alpha_{l+2}(a)-(1-a^{\underline{p}_{l+2}})\mathbf{P}_k(a^{m+1}))}\right)^{-1}.$$

For $l\geq 0$, let

$$\begin{aligned}\mu_{l+2} &= a^{\underline{p}_{l+1}}\alpha_{l+2}(a)+(1-a^{\underline{p}_{l+2}})\left[\alpha_{l+1}(a)-\alpha_1(a)\right],\\ \nu_{l+2} &= a^{\underline{p}_{l+1}}\beta_{l+2}(a)+(1-a^{\underline{p}_{l+2}})\left[\beta_{l+1}(a)-\beta_1(a)\right].\end{aligned}$$

Obviously, $\nu_{l+2}>0$ for all $l\geq 0$. Besides, when $l=0$,

$$\mu_2 = a^{\underline{p}_1}\alpha_2(a) > a^{\underline{p}_1}(\mathbf{P}_k(a^{m+1}) > a^{\underline{p}_1}(1-a^{\underline{p}_2});$$

when $l\geq 1$,

$$\begin{aligned}\mu_{l+2} &= a^{\underline{p}_{l+1}}\alpha_{l+2}(a)+(1-a^{\underline{p}_{l+2}})\left[\alpha_{l+1}(a)-\alpha_1(a)\right]\\ &> (1-a^{\underline{p}_{l+2}})\left[\alpha_0(a)-\alpha_1(a)\right]\\ &> a^{\underline{p}_1}(1-a^{\underline{p}_{l+2}}).\end{aligned}$$

Hence, we have

$$\begin{aligned}&\tfrac{a^{\underline{p}_{l+1}}}{1-a^{\underline{p}_{l+2}}}\underline{b}_{l+2}+\underline{b}_{l+1}<a^{\underline{p}_1}\gamma+\underline{b}_1\\ \Leftrightarrow &\left(\tfrac{1-a^{m-1}}{1-a}+\tfrac{\mu_{l+2}-a^{\underline{p}_1}(1-a^{\underline{p}_{l+2}})}{\nu_{l+2}+a\mu_{l+2}}\right)\gamma-a^{m-1}<0\\ \Leftrightarrow &\,\gamma<a^{m-1}\left(\tfrac{1-a^{m-1}}{1-a}+\tfrac{\mu_{l+2}-a^{\underline{p}_1}(1-a^{\underline{p}_{l+2}})}{\nu_{l+2}+a\mu_{l+2}}\right)^{-1}.\end{aligned}$$

Assume that

$$\begin{aligned}&\sup\left\{\gamma\middle|\frac{a^{\underline{p}_{N+1}}}{1-a^{\underline{p}_{N+2}}}\underline{b}_{N+2}+\underline{b}_{N+1}<a^{\underline{p}_1}\gamma+\underline{b}_1\right\}\\ &\leq \inf\left\{\gamma\middle|\frac{\underline{b}_{N+2}}{1-a^{\underline{p}_{N+2}}}\geq \mathbf{P}_k(a^{m+1})Q_m\right\}.\end{aligned}$$

Then, when $\gamma=\sup\left\{\gamma\middle|\frac{a^{\underline{p}_{N+1}}}{1-a^{\underline{p}_{N+2}}}\underline{b}_{N+2}+\underline{b}_{N+1}<a^{\underline{p}_1}\gamma+\underline{b}_1\right\}$, we have $\frac{a^{\underline{p}_{N+1}}}{1-a^{\underline{p}_{N+2}}}\underline{b}_{N+2}+\underline{b}_{N+1}=a^{\underline{p}_1}\gamma+\underline{b}_1$ and $\frac{\underline{b}_{N+2}}{1-a^{\underline{p}_{N+2}}}\leq \mathbf{P}_k(a^{m+1})Q_m$; therefore,

$$\frac{1}{1-a^{\underline{p}_{l+2}}}\underline{b}_{l+2}-\frac{a^{\underline{p}_{l+1}}}{1-a^{\underline{p}_{l+2}}}\underline{b}_{l+2}-\underline{b}_{l+1}\leq \mathbf{P}_k(a^{m+1})Q_m-a^{\underline{p}_1}\gamma-\underline{b}_1.$$

But, according to (43), we know that, for any γ,

$$\begin{aligned}
&\frac{1}{1-a^{\underline{p}_{l+2}}}\underline{b}_{l+2} - \frac{a^{\underline{p}_{l+1}}}{1-a^{\underline{p}_{l+2}}}\underline{b}_{l+2} - \underline{b}_{l+1} \\
&= \frac{1}{1-a^{\underline{p}_{l+2}}}\left(\mathbf{P}_k(a^{m+1})Q_m - a^{\underline{p}_1}\gamma - \underline{b}_1\right) \\
&> \mathbf{P}_k(a^{m+1})Q_m - a^{\underline{p}_1}\gamma - \underline{b}_1,
\end{aligned}$$

which shows that the above assumption is not true. Thus, we have completed the proof.

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E-mail: gchen@ee.cityu.edu.hk (corresponding author)