Adaptive Model Predictive Control for Unconstrained Discrete-Time Linear Systems With Parametric Uncertainties

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***Abstract*—In this technical note, an adaptive model predictive control (MPC) is proposed for unconstrained discrete-time linear systems with parametric uncertainties. The control objective is reference tracking. The adaptive MPC is designed by combining an adaptive updating law for estimated parameters and a constrained MPC for an estimated system. It is proved theoretically that, with the proposed adaptive MPC, the closed-loop system is capable of tracking time-varying reference signals with ultimately bounded tracking errors, and the estimated parameters are bounded. Moreover, if the reference signals are constant, tracking errors of the closed-loop system can be stabilized asymptotically. Performances of the closed-loop system are demonstrated by a simulation example.**

***Index Terms*—Adaptive control, discrete-time systems, linear systems, model predictive control.**

I. Introduction

Model predictive control (MPC) is regarded as an effective approach to address process control problems (with or without constraints). For an overview on MPC, please refer to the survey paper [1]. The core content of MPC is to solve an optimization problem with respect to the control inputs at every sampling time. The leading factor that MPC can be widely applied is that its superior robustness with respect to external disturbances [2]. In MPC design, system outputs at the next several sampling times are predicted by using the system state equation, and are fed-back for calculating controls for the next sampling time. That is to say, the relative long term effect of disturbances are considered in control design, contributing to the superior robustness of MPC. Variations of classical MPC technique are developed in recent years. For example, MPC with disturbance feedback [3], MPC for switched nonlinear systems [4], and time-varying MPC [5]. Applications of MPC to various areas include energy generation [6], resource allocation [7], chemical process [8], flight control [9], and so on.

Although the inborn robustness of MPC with respect to external disturbances is usually satisfactory, its reactions to parametric uncertainties remains an open topic (at least theoretically). The reason is that, parametric uncertainties would lead to difficulties in predicting future states of the plant. An intuitive solution to the problem of MPC design with parametric uncertainties is to introduce adaptive schemes. Some representative researches include adaptive MPC based on persistent excitation [10], adaptive strategy for single loop MPC [11], adaptive MPC by using comparison model [12], [13], and adaptive MPC for continuous-time nonlinear system [14]. Recently, neural networks are introduced in adaptive MPC design to solve problems of system identification [15] and time-delay [16].

In this technical note, a new simple adaptive MPC is proposed for unconstrained discrete-time linear systems with parametric uncertainties. The proposed adaptive MPC is developed by combining an adaptive updating law for estimated parameters and a constrained MPC. The main *contribution* is that, by applying the adaptive updating law, the unconstrained MPC design with parametric uncertainties can be transformed into a constrained MPC design *without* parametric uncertainties. It can be proved theoretically that, with the proposed adaptive MPC, the tracking errors of the closed-loop system are ultimately bounded, and the estimated parameters are bounded. Moreover, the tracking errors can be stabilized asymptotically, if the reference signals are constant. An initial application of the proposed adaptive MPC to a hybrid energy system can be referred to [17].

The configuration of this technical note is arranged as following. Some mathematical preliminaries are provided in Section II. Main results of the adaptive MPC for unconstrained discrete-time linear system are proposed in Section III. A simulation example of the closed-loop system with the proposed adaptive MPC is presented in Section IV. Concluding remarks are given in the final section.

II. Preliminaries

Some mathematical preliminaries are necessary to facilitate the presentation of the main results.

Definition 1: Partial differentiation of a scalar multi-variable function with respect to a matrix is defined by

$$\frac{\partial f(x,\Theta)}{\partial \Theta} \triangleq \left[\frac{\partial f(x,\Theta)}{\partial \theta_{ji}}\right]_{n\times m} = \begin{bmatrix} \frac{\partial f}{\partial \theta_{11}} & \frac{\partial f}{\partial \theta_{21}} & \cdots & \frac{\partial f}{\partial \theta_{m1}} \\ \frac{\partial f}{\partial \theta_{12}} & \ddots & & \vdots \\ \vdots & & & \\ \frac{\partial f}{\partial \theta_{1n}} & \cdots & & \frac{\partial f}{\partial \theta_{mn}} \end{bmatrix} \tag{1}$$

where $f(x,\Theta) \in \mathbb{R}$, $\Theta = [\theta_{ij}]_{m\times n} \in \mathbb{R}^{m\times n}$.

Lemma 1: For a scalar function $f(x,y,\Theta) = x^T\Theta y$, its partial differentiation with respect to Θ could be calculated by

$$\frac{\partial f(x,y,\Theta)}{\partial \Theta} = yx^T \tag{2}$$

where $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$ are column vectors, and $\Theta = [\theta_{ij}]_{m\times n} \in \mathbb{R}^{m\times n}$.

Proof: Rewrite f into the form of sum

$$f = \sum_j^n \sum_i^m \theta_{ij} x_i y_j.$$

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According to Definition 1

$$\frac{\partial f(x,y,\Theta)}{\partial \Theta} = \begin{bmatrix} x_1y_1 & x_2y_1 & \cdots & x_my_1 \\ x_1y_2 & \ddots & & \vdots \\ \vdots & & & \\ x_1y_n & \cdots & & x_my_n \end{bmatrix} = yx^T$$

which proves the result given by (2). ◇

Some properties [18] of trace of matrix are necessary.

Property 1: For matrix A and B, and vectors x and y, all with proper dimensions, the following properties hold:

1) $\mathrm{tr}(AB) = \mathrm{tr}(BA)$;
2) $\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B)$;
3) $\mathrm{tr}(yx^T) = x^Ty$.

Definition 2: For vectors $a \triangleq [a_1,\ldots,a_n]^T$ and $b \triangleq [b_1,\ldots,b_n]^T$, the inequity $a < b$ (or $a \le b$) indicates that $a_i < b_i$ (or $a_i \le b_i$) for all $i = 1,\ldots,n$.

III. Adaptive MPC for Unconstrained Discrete-Time Linear Systems

In this technical note, the plant to be controlled is an unconstrained discrete-time linear system expressed by

$$\begin{cases} x_p(k+1) = A_px_p(k) + B_pu_p(k) \\ y_p(k) = C_px_p(k) \end{cases} \tag{3}$$

where $x_p(k) \in \mathbb{R}^n$, $u_p(k) \in \mathbb{R}^m$ and $y_p(k) \in \mathbb{R}^l$ are system state, input and output, respectively; $A_p \in \mathbb{R}^{n\times n}$ and $B_p \in \mathbb{R}^{n\times m}$ are uncertain constant matrices. In this technical note, it is supposed that the pair (A_p, B_p) is controllable. The *control objective* is reference tracking

$$y_p(k) \to r_s(k), \text{ as } k \to +\infty$$

where the reference signal is supposed to be bounded by $\|r_s\| \le \bar{r}_s$.

In this technical note, we consider only the state feedback problem. The system state $x_p(k)$ is fully available, and $C_p \in \mathbb{R}^{l\times n}$ is known accurately. Output feedback is beyond the scope of this research.

According to classical linear MPC design [2], the original system (3) can be transformed into an incremental form

$$\begin{cases} x(k+1) = Ax(k) + B\Delta u(k) \\ y(k) = Cx(k) \end{cases} \tag{4}$$

where $x(k) \triangleq [\Delta x_p(k)^T, y_p(k)^T]^T$, $\Delta x_p(k) \triangleq x_p(k) - x_p(k-1)$, $\Delta u(k) \triangleq u_p(k) - u_p(k-1)$, and $y(k) = y_p(k)$; and

$$A = \begin{bmatrix} A_p & 0_{n\times l} \\ C_pA_p & I_{l\times l} \end{bmatrix},\; B = \begin{bmatrix} B_p \\ C_pB_p \end{bmatrix},\; C = \begin{bmatrix} 0_{l\times n} & I_{l\times l} \end{bmatrix}.$$

Consequently, in the incremental system (4), matrices A and B can be regarded as uncertain constant parameters, and C is a known constant matrix. Controllability of (A, B) can be induced by controllability of (A_p, B_p).

Assumption 1: Denote $\Theta \triangleq [A, B]$. There exists a conservative bound for the uncertain constant parameters: $\|\Theta\| \le \bar{\Theta}$.

A. Adaptive Updating Law for Uncertain Parameters

Design an estimated system

$$\hat{x}(k+1) = \hat{A}(k)x(k) + \hat{B}(k)\Delta u(k) \tag{5}$$

where $\hat{A}(k)$ and $\hat{B}(k)$ are time-varying estimated parameters for uncertain constant matrices A and B; $\hat{x}(k)$ is the estimated state for the system state $x(k)$.

The system state and the estimated system state can be rewritten into more compact forms

$$x(k+1) = \Theta X(k)$$
$$\hat{x}(k+1) = \hat{\Theta}(k)X(k)$$

where $\hat{\Theta}(k) \triangleq [\hat{A}(k), \hat{B}(k)]$, and $X(k) \triangleq [x(k)^T, \Delta u(k)^T]^T$. Subtracting the above two equations yields

$$\tilde{x}(k+1) = \tilde{\Theta}(k)X(k)$$

where $\tilde{\Theta} \triangleq \Theta - \hat{\Theta}$, and $\tilde{x} \triangleq x - \hat{x}$.

Define a cost function for the estimated error $\tilde{x}$

$$J_x \triangleq \tilde{x}(k+1)^T\tilde{x}(k+1) = \left(x(k+1) - \hat{\Theta}(k)X(k)\right)^T \left(x(k+1) - \hat{\Theta}(k)X(k)\right).$$

The gradient of J_x with respect to $\hat{\Theta}$ can be obtained by

$$\nabla J_x(\hat{\Theta}) = \frac{\partial J_x}{\partial \hat{\Theta}} = -X(k)\left(x(k+1) - \hat{\Theta}(k)X(k)\right)^T = -X(k)\tilde{x}(k+1)^T$$

where the result of Lemma 1 can be used to calculate the partial differentiation. The gradient method [18] is employed herein to minimize the cost function

$$\begin{aligned}\hat{\Theta}(k+1) &= \hat{\Theta}(k) - \lambda\nabla J_x^T \\ &= \hat{\Theta}(k) + \lambda\left(x(k+1) - \hat{\Theta}(k)X(k)\right)X(k)^T \\ &= \hat{\Theta}(k) + \lambda\tilde{x}(k+1)X(k)^T \end{aligned} \tag{6}$$

where $\lambda > 0$ is the updating rate to be assigned.

Theorem 1: Consider the discrete-time linear system (3) with uncertain parameters A_p and B_p. Suppose that there exists a feasible control $u_p(k)$ (or $\Delta u(k)$) satisfying that

$$X(k)^TX(k) \le \frac{2-\alpha}{\lambda} \tag{7}$$

where $0 < \alpha < 2$ and $\lambda > 0$. Then, with the adaptive updating law (6), the following statements hold:

1) estimated parameter error $\tilde{\Theta}$ is ultimately bounded;
2) estimated state error $\tilde{x}$ is asymptotically stable.

Proof: Select a Lyapunov candidate $V_\theta(k) = \mathrm{tr}(\tilde{\Theta}(k)^T\tilde{\Theta}(k))$, where $\mathrm{tr}(\cdot)$ denotes the trace for matrix. It follows that:

$$\begin{aligned}V_\theta(k+1) &= \mathrm{tr}\left(\tilde{\Theta}(k+1)^T\tilde{\Theta}(k+1)\right) \\ &= \mathrm{tr}\left(\Theta^T\Theta - 2\Theta^T\hat{\Theta}(k+1) + \hat{\Theta}(k+1)^T\hat{\Theta}(k+1)\right)\end{aligned} \tag{8}$$

where the results in Property 1 are used. On the right side of the above equation, there are three terms. The first term is $\mathrm{tr}(\Theta^T\Theta)$; the second term can be calculated by

$$\mathrm{tr}\left(-2\Theta^T\hat{\Theta}(k+1)\right) = \mathrm{tr}\left(-2\Theta^T\hat{\Theta}(k) + 2\lambda\Theta^T\nabla J_x^T\right)$$

and the third term can be calculated by

$$\mathrm{tr}\left(\hat{\Theta}(k+1)^T\hat{\Theta}(k+1)\right) = \mathrm{tr}\left(\hat{\Theta}(k)^T\hat{\Theta}(k) - 2\lambda\hat{\Theta}(k)^T\nabla J_x^T + \lambda^2\nabla J_x\nabla J_x^T\right).$$

It then follows from (8) that:

$$\begin{aligned}V_\theta(k+1) &= \mathrm{tr}\left(\tilde{\Theta}(k)^T\tilde{\Theta}(k) + 2\lambda\tilde{\Theta}(k)^T\nabla J_x^T + \lambda^2\nabla J_x\nabla J_x^T\right) \\ &= V_\theta(k) + \mathrm{tr}\left(2\lambda\tilde{\Theta}(k)^T\nabla J_x^T + \lambda^2\nabla J_x\nabla J_x^T\right)\end{aligned}$$

where the second term on the right side can be calculated by

$$\begin{aligned}&\mathrm{tr}\left(2\lambda\tilde{\Theta}(k)^T\nabla J_x^T+\lambda^2\nabla J_x\nabla J_x^T\right)\\&=\lambda\mathrm{tr}\Big(-2\tilde{\Theta}(k)^T\tilde{x}(k+1)X^T(k)\\&\qquad+\lambda X(k)\tilde{x}^T(k+1)\tilde{x}(k+1)X(k)^T\Big)\\&=\lambda\Big(-2X^T(k)\tilde{\Theta}(k)^T\tilde{x}(k+1)\\&\qquad+\lambda X(k)^TX(k)\tilde{x}^T(k+1)\tilde{x}(k+1)\Big)\\&=\lambda\Big(-2\tilde{x}(k+1)^T\tilde{x}(k+1)\\&\qquad+\lambda X(k)^TX(k)\tilde{x}^T(k+1)\tilde{x}(k+1)\Big)\\&=\lambda\left(-2+\lambda X(k)^TX(k)\right)\tilde{x}^T(k+1)\tilde{x}(k+1)\\&\le-\lambda\alpha\tilde{x}^T(k+1)\tilde{x}(k+1).\end{aligned}$$

Consequently

$$V_\theta(k+1)\le V_\theta(k)-\lambda\alpha\tilde{x}^T(k+1)\tilde{x}(k+1) \tag{9}$$

implying that $V_\theta(k)$ decreases as k increases. It is obvious that $V_\theta(k)>0$; therefore, $\lim_{k\to+\infty}V_\theta(k)$ exists, and $\tilde{\Theta}(k)$ is ultimately bounded (but not necessarily converges to 0), which proves 1).

It follows from (9) that:

$$\begin{aligned}V_\theta(1)&\le V_\theta(0)-\lambda\alpha\tilde{x}^T(1)\tilde{x}(1)\\V_\theta(2)&\le V_\theta(1)-\lambda\alpha\tilde{x}^T(2)\tilde{x}(2)\\&\vdots\\V_\theta(k+1)&\le V_\theta(k)-\lambda\alpha\tilde{x}^T(k+1)\tilde{x}(k+1)\end{aligned}$$

indicating that

$$V_\theta(k+1)\le V_\theta(0)-\lambda\alpha\sum_{i=1}^{k+1}\tilde{x}^T(i)\tilde{x}(i).$$

Consequently

$$\lambda\alpha\sum_{i=1}^{+\infty}\tilde{x}^T(i)\tilde{x}(i)\le V_\theta(0)-\lim_{k\to+\infty}V_\theta(k)$$

implying that the infinite series $\sum_{i=1}^{+\infty}\tilde{x}^T(i)\tilde{x}(i)$ converges; hence $\tilde{x}\to 0$. This proves 2). $\diamondsuit$

Remark 1: A small enough λ can always be found such that (7) is satisfied with finite $X(k)$.

Remark 2: It is indicated from Theorem 1 that our next step is to find a model predictive control $\Delta u(k)$ satisfying the constraint (7).

B. MPC Design for the Estimated System

In this part, $\Delta u(k)$ is designed for the estimated system (5) subject to the constraint (7). Predictive equations of the estimated system (5) can be given by

$$\begin{aligned}\hat{x}(k+1|k)&=\hat{A}(k)x(k)+\hat{B}(k)\Delta u(k)\\\hat{x}(k+2|k)&=\hat{A}(k)\hat{x}(k+1|k)+\hat{B}(k)\Delta u(k+1|k)\\&\vdots\\\hat{x}(k+N_c|k)&=\hat{A}(k)\hat{x}(k+N_c-1|k)+\hat{B}(k)\Delta u(k+N_c-1|k)\\&\vdots\\\hat{x}(k+N_p|k)&=\hat{A}(k)\hat{x}(k+N_p-1|k)+\hat{B}(k)\Delta u(k+N_c-1|k).\end{aligned}$$

Suppose that the output of the estimated system is given by

$$\hat{y}(k)=C\hat{x}(k). \tag{10}$$

According to classical MPC design, the predictive equations for (5) can be written into a compact form

$$\hat{Y}(k)=\hat{F}(k)x(k)+\hat{\Phi}(k)\Delta U$$

where $\hat{Y}(k)\triangleq[\hat{y}^T(k+1|k),\hat{y}^T(k+2|k),\ldots,\hat{y}^T(k+N_p|k)]^T$, and $\Delta U\triangleq[\Delta u^T(k),\Delta u^T(k+1|k),\ldots,\Delta u^T(k+N_c-1|k)]^T$; predictive matrices are given by

$$\hat{F}(k)=\begin{bmatrix}C\hat{A}\\C\hat{A}^2\\\vdots\\C\hat{A}^{N_p}\end{bmatrix} \tag{11}$$

$$\hat{\Phi}(k)=\begin{bmatrix}C\hat{B}&0&\cdots&0\\C\hat{A}\hat{B}&C\hat{B}&\cdots&0\\\vdots&&&\vdots\\C\hat{A}^{N_p-1}\hat{B}&C\hat{A}^{N_p-2}\hat{B}&\cdots&C\hat{A}^{N_p-N_c}\hat{B}\end{bmatrix}. \tag{12}$$

The predictive reference signals are given by

$$R_s(k)=\left[r_s^T(k+1),r_s^T(k+2),\ldots,r_s^T(k+N_p)\right]^T$$

and the cost function is designed by

$$J_y=\left(\hat{Y}(k)-R_s(k)\right)^T\left(\hat{Y}(k)-R_s(k)\right)+\Delta U^T(k)\bar{R}\Delta U(k) \tag{13}$$

where $\bar{R}=\mathrm{diag}(\bar{r})_{N_c\times N_c}$ is a diagonal weight matrix with $\bar{r}>0$. To formulate the optimization problem, the cost function is further calculated by

$$\begin{aligned}J_y&=(\hat{F}x+\hat{\Phi}\Delta U-R_s)^T(\hat{F}x+\hat{\Phi}\Delta U-R_s)\\&\quad+\Delta U^T(k)\bar{R}\Delta U(k)\\&=(\hat{F}x-R_s)^T(\hat{F}x-R_s)+2\Delta U^T\hat{\Phi}^T(\hat{F}x-R_s)\\&\quad+\Delta U^T(\hat{\Phi}^T\hat{\Phi}+\bar{R})\Delta U\end{aligned}$$

where the first term on the right-hand side is independent on ΔU. It then follows that:

$$\min_{\Delta U}J_y\ \Rightarrow\ \min_{\Delta U}\hat{J}_y$$

where

$$\begin{aligned}\hat{J}_y&=2\Delta U^T\hat{\Phi}^T(\hat{F}x-R_s)+\Delta U^T(\hat{\Phi}^T\hat{\Phi}+\bar{R})\Delta U\\&=2\Delta U^T\hat{H}(k)+\Delta U^T\hat{E}(k)\Delta U\end{aligned} \tag{14}$$

and $\hat{H}\triangleq\hat{\Phi}^T(\hat{F}x-R_s)$, $\hat{E}\triangleq\hat{\Phi}^T\hat{\Phi}+\bar{R}$.

Constraint (7) can be re-formulated as follows:

$$\begin{aligned}&X(k)^TX(k)<\frac{2-\alpha}{\lambda}\\&\Leftrightarrow\ x(k)^Tx(k)+\Delta u^T(k)\Delta u(k)<\frac{2-\alpha}{\lambda}\\&\Leftrightarrow\ \Delta u^T(k)\Delta u(k)<\frac{2-\alpha}{\lambda}-x(k)^Tx(k)\\&\Leftrightarrow\ \|\Delta u(k)\|<\sqrt{\frac{2-\alpha}{\lambda}-x(k)^Tx(k)}\\&\Leftarrow\ \begin{cases}[I_{m\times m},0,\ldots,0]\Delta U<\frac{\sqrt{m\left(\frac{2-\alpha}{\lambda}-x(k)^Tx(k)\right)}}{m}\mathbf{1}_m\\[-I_{m\times m},0,\ldots,0]\Delta U<\frac{\sqrt{m\left(\frac{2-\alpha}{\lambda}-x(k)^Tx(k)\right)}}{m}\mathbf{1}_m\end{cases}\end{aligned}$$

which can be written into a compact form

$$M\Delta U\le\gamma \tag{15}$$

where

$$M = \begin{bmatrix} I_{m\times m} & 0 & \cdots & 0 \\ -I_{m\times m} & 0 & \cdots & 0 \end{bmatrix}$$
$$\gamma = \frac{\sqrt{m\left(\frac{2-\alpha}{\lambda} - x(k)^T x(k)\right)}}{m}\begin{bmatrix} \mathbf{1}_m \\ \mathbf{1}_m \end{bmatrix}$$
$$\mathbf{1}_m \triangleq [1,\ldots,1]^T \in \mathbb{R}^m.$$

To guarantee the stability of the closed-loop estimated system, we introduce a terminal constraint

$$\hat{y}(k+N_c|k) = r_s(k+N_c) \tag{16}$$

or equivalently, with respect to ΔU

$$M_e \Delta U = \gamma_e \tag{17}$$

where $M_e = C[\hat{A}^{N_c-1}\hat{B}, \hat{A}^{N_c-2}\hat{B},\ldots,\hat{B}]$, and $\gamma_e = r_s(k+N_c) - C\hat{A}^{N_c}x(k)$.

The optimization in MPC can be formulated by

$$\Delta U^*(k) = \arg\min_{\Delta U}(2\Delta U^T\hat{H} + \Delta U^T\hat{E}\Delta U) \tag{18}$$

subject to (5), (10), (15), and (17). The proposed adaptive MPC is then implemented by using receding horizon scheme

$$\Delta u(k) = [I_{m\times m}, 0, \ldots, 0]\Delta U^*(k). \tag{19}$$

Remark 3: In the above control algorithm, $\Delta u(k)$ should be calculated before the update of adaptive estimated parameter $\hat{\Theta}(k)$ at each sampling time.

Remark 4: To guarantee that the optimization is always feasible with the constraint (15), a strategy to determine λ can be suggested as follows.

1) Process the optimization (18) subject to (5), (10), and (17), but without the constraint (15).
2) Select an α satisfying $0 < \alpha < 2$. Calculate
$$\lambda_0 = \min\left[\frac{2-\alpha}{X(i|0)^T X(i|0)}, \frac{(2-\alpha)\bar{\Theta}}{\bar{r}_s}\right]$$
where $\bar{r}_s$ is the bound for the reference signal, and $\bar{\Theta}$ is the conservative bound given by Assumption 1.
3) Select a positive λ such that $(2-\alpha)/\lambda \gg (2-\alpha)/\lambda_0$.

Tracking performances of the estimated system (5) can be described by the following theorem.

Theorem 2: Consider the estimated system (5), with the system output defined by (10), and the adaptive updating law given by (6). The predictive horizon and the control horizon satisfy that $N_p = N_c = N$. Suppose that, the optimization (18) with constraints (15) and (17) is feasible at the initial time. Then, with the receding horizon control (19), tracking errors of the closed-loop estimated system are ultimately bounded. Moreover, if the reference signals are constant, the tracking errors can be stabilized asymptotically.

Proof: Suppose that, at sampling time k, the optimization (18) is feasible; or equivalently, there exists $\Delta U^*(k)$ satisfying (15) and (17), such that the cost function (13) reaches its optimal value J_y^*

$$J_y^*(k) = \sum_{i=0}^{N}\left(\|\hat{y}^*(k+i|k) - r_s(k+i)\|^2 + \bar{r}\,\|\Delta u^*(k+i|k)\|^2\right).$$

The above cost function can be used as the Lyapunov candidate for the estimated system, and it satisfies that

$$J_y(k)^* \geq \alpha_1\left(\|\hat{y}(k) - r_s(k)\|^2 + \bar{r}\,\|\Delta u(k)\|^2\right)$$
$$J_y(k)^* \leq \alpha_2\left(\|\hat{y}(k) - r_s(k)\|^2 + \bar{r}\,\|\Delta u(k)\|^2\right) \tag{20}$$

where existence of α_1 and α_2 can be proved by using the approach given in [19].

At sampling time $k+1$, a feasible control series satisfying constraints (15) and (17) can be selected by

$$\begin{aligned}\Delta U(k+1) &\triangleq [\Delta u(k+1|k+1),\ldots,\Delta u(k+N|k+1)]^T \\ &= [\Delta u^*(k+1|k),\ldots,\Delta u^*(k+N-1|k), \\ &\qquad \Delta u(k+N|k+1)]^T\end{aligned}$$

where appropriate $\Delta u(k+N|k+1)$ always exists for an unconstrained controllable linear system to satisfy

$$\hat{y}\,(k+N_c+1|k+1) = r_s(k+N_c+1) \tag{21}$$

indicating that the constraint (17) can be satisfied at time $k+1$. The constraint (15) can be guaranteed by the strategy introduced in Remark 4. Consequently, the optimization (18) is feasible at time $k+1$ if it is feasible at time k.

It follows that, at sampling time $k+1$:

$$\begin{aligned}\hat{y}(k+1|k+1) &= \hat{y}^*(k+1|k) + c_0\left(\hat{x}^*(k+1|k), \Delta\Theta(k+1)\right) \\ \hat{y}(k+2|k+1) &= \hat{y}^*(k+2|k) + c_1\left(\hat{x}^*(k+1|k), \Delta\Theta(k+1)\right) \\ &\vdots \\ \hat{y}(k+N|k+1) &= \hat{y}^*(k+N|k) + c_{N-1}(\hat{x}^*(k+1|k), \Delta\Theta(k+1)) \\ \hat{y}(k+1+N|k+1) &= r_s^T(k+1+N)\end{aligned}$$

where $\Delta\Theta(k+1) \triangleq \hat{\Theta}(k+1) - \hat{\Theta}(k)$. For simplicity, $c_i(\hat{x}^*(k+1|k), \Delta\Theta(k+1))$ is rewritten as $c_i(k+1)$ in the following derivations. The explicit expressions for c_i $(i = 1, 2, \ldots, N-1)$ can be calculated by

$$\begin{aligned}c_i(k+1) &= C\left(\hat{A}(k+1)^i - \hat{A}(k)^i\right)\hat{x}^*(k+1|k) \\ &\quad + C\sum_{j=1}^{i}\left(\hat{A}(k+1)^{j-1}\hat{B}(k+1) - \hat{A}(k)^{j-1}\hat{B}(k)\right)\Delta u^*(k+j|k)\end{aligned}$$

indicating that

$$\lim_{\Delta\Theta(k+1)\to 0} c_i\left(\hat{x}^*(k+1|k), \Delta\Theta(k+1)\right) = 0. \tag{22}$$

Consider the Lyapunov candidate at sampling time $k+1$

$$\begin{aligned}J_y(k+1) &= \sum_{i=0}^{N}\left(\|\hat{y}(k+1+i|k+1) - r_s(k+1+i)\|^2\right. \\ &\qquad \left. + \bar{r}\,\|\Delta u(k+i+1|k+1)\|^2\right) \\ &= \sum_{i=0}^{N-1}\left(\|\hat{y}(k+1+i|k+1) - r_s(k+1+i)\|^2\right. \\ &\qquad \left. + \bar{r}\,\|\Delta u(k+i+1|k+1)\|^2\right) \\ &\quad + \|\hat{y}(k+1+N|k+1) - r_s(k+1+N)\|^2 \\ &\quad + \bar{r}\,\|\Delta u(k+N+1|k+1)\|^2.\end{aligned}$$

It follows from (21) that:

$$\begin{aligned}
J_y(k+1) &= \sum_{i=0}^{N-1} \left(\|\hat{y}(k+1+i|k+1) - r_s(k+1+i)\|^2 \right.\\
&\qquad \left. + \bar{r}\,\|\Delta u(k+i+1|k+1)\|^2 \right)\\
&\quad + \bar{r}\,\|\Delta u(k+N+1|k+1)\|^2\\
&= \sum_{i=1}^{N} \left(\|\hat{y}^*(k+i|k) + c_{i-1}(k+1) - r_s(k+i)\|^2 \right.\\
&\qquad \left. + \bar{r}\,\|\Delta u(k+i|k)\|^2 \right)\\
&\quad + \bar{r}\,\|\Delta u(k+N+1|k+1)\|^2\\
&\le \sum_{i=0}^{N} \left(\|\hat{y}^*(k+i|k) - r_s(k+i)\|^2 \right.\\
&\qquad \left. + \bar{r}\,\|\Delta u^*(k+i|k)\|^2 \right)\\
&\quad + 2\sum_{i=1}^{N} c_{i-1}(k+1)\,\|\hat{y}^*(k+i|k) - r_s(k+i)\|\\
&\quad + \sum_{i=1}^{N} c_{i-1}(k+1)^2 - \|\hat{y}^*(k) - r_s(k)\|^2\\
&\quad - \bar{r}\,\|\Delta u(k)\|^2 + \bar{r}\,\|\Delta u(k+N+1|k+1)\|^2 . \qquad (23)
\end{aligned}$$

Notice that $\Delta\Theta(k)$ converges to zero asymptotically, because of (6) and Theorem 1. It follows from (22) that $c_i(k+1)$ converges to zero.

Since the system (4) is linear, and control $u(k)$ is bounded, there is no finite escaping time. Consequently, after finite time, $c_i(k+1)$ becomes extremely small, and it follows from (23) that:

$$\begin{aligned}
J_y^*(k+1) - J_y^*(k) &\le J_y(k+1) - J_y^*(k)\\
&\le -\|\hat{y}^*(k) - r_s(k)\|^2 - \bar{r}\,\|\Delta u(k)\|^2\\
&\quad + \bar{r}\,\|\Delta u(k+N+1|k+1)\|^2 . \qquad (24)
\end{aligned}$$

It can be implied from (20) and (24) that tracking errors of the estimated system are ultimately bounded.

Moreover, for constant reference tracking, it holds that $\Delta u(k+N+1|k+1) = 0$, indicating that tracking errors of the estimated system are asymptotically stable. ◊

C. Analysis on the Closed-Loop System

The proposed adaptive MPC can be summarized as follows.

Algorithm 1:

1. Run the algorithm given by Remark 4 to determine λ;
2. Calculate the constrained MPC $\Delta u(k)$ through (19), where ΔU is obtained by solving optimization problem (18) subject to (5), (10), (15) and (16);
3. Execute $\Delta u(k)$ to the system (4), and measure $x(k+1)$;
4. Update the estimated parameters $\hat{A}(k+1)$ and $\hat{B}(k+1)$ with the adaptive updating law (6), where $\Delta u(k)$ is obtained in Step 1, and $x(k+1)$ can be measured;
5. Let $k = k+1$, and go to Step 2.

The stability result of the closed-loop system can be given by the following theorem.

Theorem 3: Consider the discrete-time linear system (3) with uncertain constant matrices A_p and B_p. With the proposed model predictive control summarized in Algorithm 1, tracking errors of the closed-loop system are ultimately bounded. If the reference signals are constant, then tracking errors of the closed-loop system are asymptotically stable.

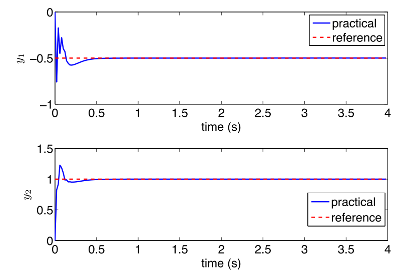

Fig. 1. Tracking constant reference signals: the tracking errors are asymptotically stable.

Proof: According to Theorem 1, with the adaptive updating law (6), the estimated parameters are bounded, and the estimated state error $\tilde{x}$ decreases exactly. It follows that the system state x tracks $\hat{x}$ asymptotically.

In another aspect, the MPC (19) guarantees bounded tracking of the estimated system (5), as can be shown by Theorem 2. As a result, with the proposed adaptive MPC given in Algorithm 1, the system (4) [or equivalently, the original system (3)] is capable of tracking its reference signals with bounded tracking errors. If the reference signals are constant, tracking errors of the closed-loop system are asymptotically stable. ◊

IV. Simulation Example

A simulation example is presented herein to illustrate the proposed theoretical results. The plant to be controlled is an MIMO linear system with two control inputs $u = [u_1, u_2]^T$ and two outputs $y = [y_1, y_2]^T$. Its uncertain system matrix and input matrix are given by

$$A_p = \begin{bmatrix} 0.8 & 0.4 & 1.1 \\ 0.6 & 1.5 & -0.1 \\ 0.1 & -0.2 & 1.8 \end{bmatrix}, \; B_p = \begin{bmatrix} 0.7 & 0 \\ 1.2 & 0 \\ -0.6 & 1.4 \end{bmatrix}.$$

The output matrix is known, and it is given by

$$C_p = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The sampling interval is given by $h = 0.02$ s.

Suppose that, though A_p and B_p are uncertain, their nominal values are known

$$A_{p0} = \begin{bmatrix} 1 & 0.5 & 1 \\ 0.5 & 1.5 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \; B_{p0} = \begin{bmatrix} 0.5 & 0 \\ 1 & 0 \\ -0.5 & 1.5 \end{bmatrix}$$

which are used as initial values of the estimated parameters.

In this example, the initial value of the system state x is given by $x(0) = [0.15, 0.1, -0.2, 0, 0]^T$, The initial value of the estimated state $\hat{x}$ is set to $\hat{x}(0) = [0, 0, 0, 0, 0]^T$. The predictive horizon and the control horizon are given by $N_p = N_c = 10$, respectively. The parameter λ is calculated by using the algorithm given in Remark 4. In this simulation case, $\lambda = 0.2$ and $\bar{R} = \mathrm{diag}(0.1)$ are assigned. The adaptive MPC is designed through Algorithm 1 in Section III-C.

The constant reference signals to be tracked are given by $r_s = [-0.5, 1]^T$. Simulation results of the closed-loop system are displayed in Figs. 1–4. As can be seen from Fig. 1, the closed-loop system is capable of tracking the constant reference signal asymptotically. The transient performance is satisfactory, though the overshoot and vibration are relatively large. The reason for the large overshoot and vibration might be that the estimated parameters were vibrating before reaching their steady values. It is shown in Fig. 2 that the estimated output errors defined by $\tilde{y} \triangleq y - \hat{y}$ converge to zeros, as are predicted

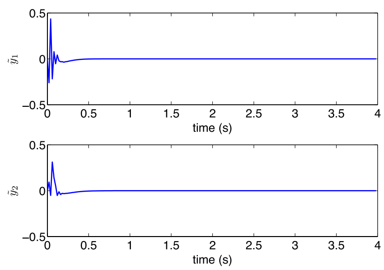

Fig. 2. Estimated output errors: they converge to zeros.

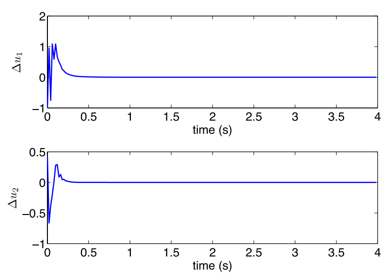

Fig. 3. Control inputs: they are within corresponding constraints.

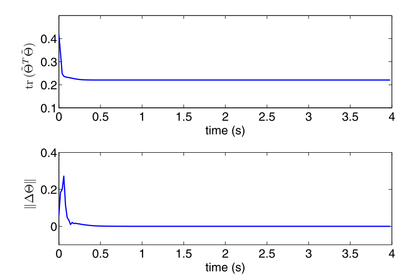

Fig. 4. Norm of estimation errors: it is bounded (upper). The variation of estimated parameters: it becomes zero (lower).

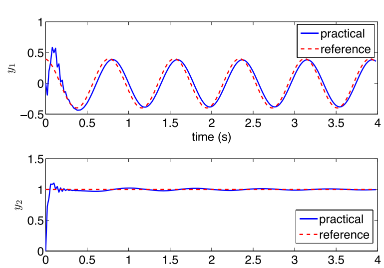

Fig. 5. Tracking bounded time-varying reference signals: the tracking errors are ultimately bounded.

by Theorem 1. The rates of control signals are displayed in Fig. 3, where they are within corresponding constraints. It can be seen from Fig. 4 that, the norm of estimation errors is ultimately bounded, and the variation of estimated parameters becomes zero.

To better illustrate the performances of the closed-loop system, the tracking result with respect to bounded time-varying reference signals is presented. The time-varying reference signals are given by $r_s(k) = [0.4\cos(8hk), 1]^T$. Control parameters are assigned as those in the constant tracking case. The simulation result is displayed in Fig. 5. As can be seen, with the proposed adaptive MPC, the tracking errors with respect to the time-varying reference signals are ultimately bounded. This result is in well accordance with Theorem 3.

V. Conclusion

In this technical note, an adaptive model predictive control is proposed for unconstrained discrete-time linear systems with parametric uncertainties. The control objective is reference tracking. Parametric uncertainties are estimated online by adaptive estimated parameters with a simple adaptive updating law, such that the prediction in MPC can be processed in case of parametric uncertainties. The proposed adaptive strategy transforms the MPC design for the unconstrained system with parametric uncertainties into a constrained MPC design for the estimated system. An MPC is then designed for the estimated system, such that bounded tracking of the closed-loop system can be guaranteed. Furthermore, if the reference signals are constant, tracking errors of the closed-loop system can be proved to be asymptotically stable. Performances of the closed-loop system are substantiated by both theoretical proofs and simulation results.

Future topics of this research may include: 1) some less conservative strategies to determine the updating rate, 2) finite-time approaching of the estimated system, 3) extension of this research to constrained adaptive MPC design, and 4) nonlinear adaptive MPC.

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