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Output consensus of multi-agent systems with delayed and sampled-data

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Abstract: This study considers the output consensus problem of high-order leader-following multi-agent systems with unknown non-linear dynamics, in which the delayed and sampled outputs of the system are the only available data. The unknown non-linear dynamics are assumed to satisfy the Lipschitz condition and the interconnected topologies are assumed to be undirected and connected. A distributed observer-based output feedback controller is proposed for the system to reach output consensus. Both of the bounds of the allowable delay and sampling period are also obtained. Stability analysis shows that the considered systems are globally exponentially stable under the output feedback controller. Finally, a simulation example is given to validate our theoretical results.

1 Introduction

The autonomy, distribution, and coordination render the multi-agent systems have strong robustness and reliability in solving practical problems. Its wide range of applications can be found in various areas including flocking, swarming, distributed sensor fusion, distributed coordination of mobile robots, congestion control in communication networks, synchronisation of dynamical networks, and so on. The distributed coordination problem also attracted the attention of scientists in control theory and many well-known works have been done in the context of control theory, for example [1–5], to name just a few.

In the past decades, the related topics on consensus problems have been comprehensively further studied in different situations, for example, consensus in networks with time-delays [6–8], finite time consensus [9–13], consensus in stochastic networks [14, 15], quantised consensus [16–18], and so on.

With the rapid development of intelligent instrument and digital measurement, the information of modern control systems tend to be sampled and sent periodically furthermore through the digital communication channels. Thus, the consensus with sampled-data and time-delay is a meaningful research topic. The articles studying sampled-data systems mainly include: discrete-time models [19–21], impulsive models [22–24], quantised models [16, 25], and time-delay systems [26–28]. Most of the works mentioned above are concerned with state consensus. In practical cases, most real-world systems are uncertain, and the full states of agents maybe unknown. Therefore, output consensus has attracted numerous papers' attention in recent years such as [26, 29–32]. Some representative works are summarised as follows. In [19], Chen *et al.* studied the multi-consensus problem and multi-tracking problem of second-order multi-agent systems by using only sampled current and past position data, a necessary and sufficient condition on gains and sampled period was given under directed topology. In [22], two kinds of impulsive distributed consensus algorithms which only utilise the sampled information were proposed to investigate the distributed consensus problem for second-order continuous-time multi-agent systems with sampled-data communication. In [27, 33], Yu *et al.* studied the state consensus in multi-agent dynamical systems with sampled-data, and distributed linear consensus protocols were proposed. In [33], both the current and some past sampled position data were utilised to design consensus protocols. While, using less information and saving energy, a protocol with more sampled-data and no current data was further designed in [27]. To the best of our knowledge, the studies on the output consensus problem of multi-agent systems with unknown non-linear dynamics and both delayed and sampled-data are rare.

In this paper, the output consensus problem of high-order leader-following multi-agent systems with unknown non-linear dynamics is considered. The only available data of the system outputs is assumed to be sampled and delayed. The unknown non-linear dynamics are assumed to satisfy the Lipschitz condition, and the interconnected graphs of the multi-agent systems are assumed to be undirected and connected. A distributed observer-based output feedback controller is proposed for the sampled-data multi-agent systems to reach output consensus. The stability analysis is conducted based on Lyapunov theory and algebraic graph theory, and shows that the error system is globally exponentially stable. Finally, the bounds of the allowable delay and sampling period are also given.

The contributions of this paper are mainly in three aspects. First, a novel distributed observer-based output feedback controller is presented for the considered sampled-data multi-agent systems to reach output consensus. Second, a sufficient condition is obtained to ensure the global exponential stability of the considered systems. Finally, we give the bounds of the allowable sampling period and time-delays.

This paper is organised as follows. In Section 2, some preliminary results and model formulation are presented, the related observer-based output feedback controller is also proposed in this section. In Section 3, the main results are given and proved. In Section 4, an example is given to illustrate the validity of the proposed design method. Finally, this paper is concluded in Section 5.

2 Problem statement

For the multi-agent system, the information exchange among N agents can be conveniently described by a simple and undirected graph $\mathscr{G}(\mathscr{V}, \mathscr{E})$. $\mathscr{V} = \{1, 2, \ldots, N\}$ is the node set, in which each node can be regarded as the N agents. $\mathscr{E} \subset \mathscr{V} \times \mathscr{V}$ is the edge set, a pair $(i, j) \in \mathscr{E}$ if and only if $(j, i) \in \mathscr{E}$, and an edge (i, j) in $\mathscr{G}$ means that agents i and j can obtain information from each other. The set of neighbours of node i is denoted by $\mathscr{N}_i = \{ j \in \mathscr{V} \mid (i, j) \in \mathscr{E}, j \neq i\}$. A path is a sequence of connected edges in a graph. If there is a path between any two nodes of a graph $\mathscr{G}$, then graph $\mathscr{G}$ is said to be connected; otherwise, disconnected. To model the interconnection relationship between N agents and the leader, we introduce another graph $\bar{\mathscr{G}}$ on nodes $0, 1, 2, \dots, N$, where 0 represents the leader agent. Obviously, $\mathscr{G}$ is a subgraph of $\bar{\mathscr{G}}$. Let $\mathscr{A} = [a_{ij}] \in R^{N\times N}$ be the adjacency matrix of graph $\mathscr{G}$, where $a_{ij} = 1$ if $(i, j) \in \mathscr{G}$; otherwise, $a_{ij} = 0$. The index number between the ith agent and the leader agent is denoted by b_i, where $b_i = 1$ if the leader agent is the neighbour of the ith agent; otherwise, $b_i = 0$. The degree matrix $\mathscr{D}$ of graph $\mathscr{G}$ is a diagonal matrix with the ith diagonal element being $|\mathscr{N}_i|$. The Laplacian of graph $\mathscr{G}$ is defined as $\mathscr{L} = \mathscr{D} - \mathscr{A}$, which is symmetric. Let $H = \mathscr{L} + \mathscr{B}$, and $\mathscr{B}$ be a diagonal matrix with diagonal elements $b_1, b_2, \dots, b_N$.

We consider a multi-agent system consisting of N agents and a leader. The dynamics of the ith agent, $i = 1, 2, \dots, N$, are described by

$$\begin{cases} \dot{x}^i(t) = A_0 x^i(t) + f^i(x^i(t)) + \bar{b}u^i(t), \\ y^i(t) = Cx^i(t), \end{cases} \tag{1}$$

where

$$A_0 = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \bar{b} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad C = (1, 0, \dots, 0),$$

$x^i(t) = (x^i_1(t), x^i_2(t), \dots, x^i_n(t))^{\mathrm{T}} \in R^n$ is the position state of the ith agent, $u^i(t) \in R$ is the control input of the ith agent, which will be designed later, $f^i(x^i(t)) = (f^i_1(x^i(t)), f^i_2(x^i(t)), \dots, f^i_n(x^i(t)))^{\mathrm{T}}$ is assumed to be unknown and non-linear with $f^i_m(x^i(t)) = f^i_m(x^i_1(t), x^i_2(t), \dots, x^i_m(t))$,, $m = 1, 2, \dots, n$, and $y^i(t)$ is the output of the system.

We assume that the leader agent moves in R^n and its underlying dynamics are described by

$$\begin{cases} \dot{x}^0(t) = A_0 x^0(t) + f^0(x^0(t)), \\ y^0(t) = Cx^0(t), \end{cases} \tag{2}$$

where $x^0(t) = (x^0_1(t), x^0_2(t), \dots, x^0_n(t))^{\mathrm{T}} \in R^n$ is the position state of the leader agent, and $f^0(x^0(t)) = (f^0_1(x^0(t)), f^0_2(x^0(t)), \dots, f^0_n(x^0(t)))^{\mathrm{T}}$ is a vector-function with $f^0_m(x^0(t)) = f^0_m(x^0_1(t), x^0_2(t), \dots, x^0_m(t))$, which is also assumed to be unknown and non-linear, and $y^0(t)$ is its output.

Remark 1: From the linear system theory, if the linear system $\dot{x} = Fx + gu$ is controllable, then it can be transformed into the Brunovsky controller form [34], that is, the linear part of the system (1). The system (1) is widely applied to some real-world systems such as two-link planar robots, aircraft wing rock control systems, and induction motor systems [35–38].

In this paper, the output $y^i(t)$ of system (1) and $y^0(t)$ of system (2) are assumed to be sampled at time instants t_k and are available at $t_k + \tau_k$, where $\{t_k\}, k = 1, 2, \dots, \infty$, is a strictly increasing sequence such that $\lim_{k\to\infty} t_k = \infty$ and $\tau_k \geqslant 0$, that is, the sampled-data $y^i(t_k)$ and $y^0(t_k)$ are available with a time-delay τ_k. The sampling interval $[t_{k-1}, t_k)$ satisfies $0 < T_{\min} \leqslant t_k - t_{k-1} = T_k \leqslant T_{\max}$ for all $k = 1, 2, \dots, \infty$, where T_k is the length of the kth sampling interval, $T_{\min} = \min\{T_k\}$ and $T_{\max} = \max\{T_k\}$. We assume that τ is one of the upper bound of τ_k, that is, $\tau_k \leqslant \tau$, and satisfy $\tau < T_{\min}$, which means that the sampled-data at time t_k can be used before next sampling time instant.

In this setting, the considered system (1) and (2) can be rewritten as

$$\begin{cases} \dot{x}^i(t) = A_0 x^i(t) + f^i(x^i(t)) + \bar{b}u^i(t), \\ y^i(t) = Cx^i(t_k), i = 1, 2, \dots, N, \\ t \in [t_k + \tau_k, t_{k+1} + \tau_{k+1}), k \geqslant 0, \\ x^i(t_{k+1} + \tau_{k+1}) = \lim_{t\to(t_{k+1}+\tau_{k+1})^-} x^i(t), \end{cases} \tag{3}$$

and

$$\begin{cases} \dot{x}^0(t) = A_0 x^0(t) + f^0(x^0(t)), \\ y^0(t) = Cx^0(t_k), \\ t \in [t_k + \tau_k, t_{k+1} + \tau_{k+1}), k \geqslant 0, \\ x^0(t_{k+1} + \tau_{k+1}) = \lim_{t\to(t_{k+1}+\tau_{k+1})^-} x^0(t). \end{cases} \tag{4}$$

Since the states of (3) and (4) are unmeasurable, an observer-based output feedback controller is proposed as follows:

$$\begin{cases} \dot{\hat{x}}^i(t) = A_0 \hat{x}^i(t) + Mae^i_1(t_k) \\ \qquad + f^i(\hat{x}^i(t)) + \bar{b}u^i(t), \\ \dot{\hat{x}}^0(t) = A_0 \hat{x}^0(t) + Mae^0_1(t_k) + f^0(\hat{x}^0(t)), \\ \hat{x}^i(t_{k+1} + \tau_{k+1}) = \lim_{t\to(t_{k+1}+\tau_{k+1})^-} \hat{x}^i(t), \\ \hat{x}^0(t_{k+1} + \tau_{k+1}) = \lim_{t\to(t_{k+1}+\tau_{k+1})^-} \hat{x}^0(t), \\ u^i(t) = -KW\Big[\sum_{j\in N^i} a_{ij}(\hat{x}^i(t) - \hat{x}^j(t)) \\ \qquad + b_i(\hat{x}^i(t) - \hat{x}^0(t))\Big], \\ t \in [t_k + \tau_k, t_{k+1} + \tau_{k+1}), k \geqslant 0, \end{cases} \tag{5}$$

where $\hat{x}^i(t) = (\hat{x}^i_1(t), \hat{x}^i_2(t), \dots, \hat{x}^i_n(t))^{\mathrm{T}}$, $e^i_1(t_k) = x^i_1(t_k) - \hat{x}^i_1(t_k), i = 1, 2, \dots, N$, $\hat{x}^0(t) = (\hat{x}^0_1(t), \hat{x}^0_2(t), \dots, \hat{x}^0_n(t))^{\mathrm{T}}$, $e^0_1(t_k) = x^0_1(t_k) - \hat{x}^0_1(t_k)$, $W = \mathrm{diag}(l^n, l^{n-1}, \dots, l)^{\mathrm{T}}$, with $l \geqslant 1$, $M = \mathrm{diag}(l, l^2, \dots, l^n)$, $f^i(\hat{x}^i(t)) = (f^i_1(\hat{x}^i(t)), f^i_2(\hat{x}^i(t)), \dots, f^i_n(\hat{x}^i(t)))^{\mathrm{T}}, i = 0, 1, \dots, N$, and $f^i_m(\hat{x}^i(t)) = f^i_m(\hat{x}^i_1(t), \hat{x}^i_2(t), \dots, \hat{x}^i_m(t))$, $m = 1, 2, \dots, n$. $K \in R^{1\times n}$ and $a = (a_1, a_2, \dots, a_n)^{\mathrm{T}}$ will be determined later.

Remark 2: From the literature, the consensus problem of multi-agent systems with sampled-data, time-delay, or sampled-data and time-delay has been extensively studied in the past years. Most of the existing literature are on state consensus problem. The papers on output consensus problem are rare. We consider the output consensus problem of multi-agent systems with delayed sampled-data and unknown non-linear dynamics in this paper. To the best of our knowledge, for the considered problem in this paper, no similar results appear in the existing literature. In this paper, a novel distributed controller is proposed for the considered system to reach output consensus globally exponentially. The essential difficulties to solve the problem are mainly in three aspects: first, the construction of the distributed controller. The distribution of the controller is very important in multi-agent systems, the design of distributed controller depends on the choice of the Lyapunov function; second, the construction of appropriate Lyapunov function and auxiliary integral function to deal with the delayed sampled-data and derive the global exponential stability; third, the derivation of the sufficiency conditions guaranteeing global exponential convergence. For this end, some inequality techniques are applied to solve the considered problem and give both the bounds of the allowable delay and sampling period.

Note that

$$\lim_{t \to (t_{k+1}+\tau_{k+1})^-} x^i(t) = \lim_{t \to (t_{k+1}+\tau_{k+1})^+} x^i(t),$$

and

$$\lim_{t \to (t_{k+1}+\tau_{k+1})^-} \hat{x}^i(t) = \lim_{t \to (t_{k+1}+\tau_{k+1})^+} \hat{x}^i(t).$$

Therefore, $x^i(t), \hat{x}^i(t)$ are continuous in all of the time intervals $[t_k + \tau_k, t_{k+1} + \tau_{k+1}), k = 0, 1, 2, \ldots, \infty$, that is, to say $x^i(t), \hat{x}^i(t)$ are continuous in the time interval $[t_0, \infty)$. Since $e_1^i(t_k) = x_1^i(t_k) - \hat{x}_1^i(t_k)$, we have the sequence $\{e_1^i(t)\}, i = 0, 1, \ldots, N$ is also continuous in the time interval $[t_0, \infty)$.

Definition 1 [39]: Let $x_1^i(t) = x_0^i$, $\hat{x}_1^i(t) = \hat{x}_0^i$, $i = 0, 1, \ldots, N$ for $t \in [t_0 - T_{\max} - \tau, t_0]$. We call that the system (3) and (4) are globally exponentially stabilisable, if there exists a system (5) such that the system (3) and (4) and (5) satisfy $\| \hat{x}_1^i(t) - \hat{x}_1^0(t) \| \leqslant e^{-\lambda(t-t_0)}\varphi(\| \hat{x}_0^i \|, \| \hat{x}_0^0 \|)$ and $\| \hat{x}_1^i(t) - x_1^i(t) \| \leqslant e^{-\lambda(t-t_0)}\varphi(\| \hat{x}_0^i \|, \| x_0^i \|)$ for any x_0^i, $\hat{x}_0^i$, where $\lambda > 0$ and $\varphi: R^+ \to R^+$ is a non-decreasing function. Or, we call that system (5) globally exponentially stabilises the system (3) and (4).

The control objective of this paper is to design distributed controllers $u^i(t), i = 1, 2, \ldots, N$, such that the system (3) and (4) reach output consensus, that is, $\lim_{t\to\infty} |y^i(t) - y^0(t)| = 0, i = 1, 2, \ldots, N$.

To proceed, some related assumptions and lemmas are given.

Assumption 1: Graph $\mathscr{G}$ is connected.

Lemma 1 [40]: Laplacian matrix $\mathscr{L}$ of graph $\mathscr{G}$ has at least one zero eigenvalue with $\mathbf{1}_N = (1, 1, \ldots, 1)^{\mathrm{T}} \in \mathscr{R}^N$ as its eigenvector, and all the non-zero eigenvalues of $\mathscr{L}$ are positive. Laplacian $\mathscr{L}$ has a simple zero eigenvalue if and only if graph $\mathscr{G}$ is connected.

Lemma 2 [41]: For any positive definite matrix $G \in R^{n\times n}$, scalar $\gamma > 0$, vector-function $\omega: [0, \gamma] \to R^n$ such that the integrations concerned are well defined, the following inequality holds:

$$\left[\int_0^\gamma \omega(s)\,\mathrm{d}s\right]^{\mathrm{T}} G \left[\int_0^\gamma \omega(s)\,\mathrm{d}s\right] \leqslant \gamma \left[\int_0^\gamma \omega(s)^{\mathrm{T}} G \omega(s)\,\mathrm{d}s\right]$$

.

Assumption 2: The non-linear functions $f^i(x_1(t), x_2(t), \ldots, x_m(t))$ and $f^i(y_1(t), y_2(t), \ldots, y_m(t)), m = 1, 2, \ldots, n$, are assumed to be continuous in the time interval $[t_0, \infty)$ and satisfying the following Lipschitz condition:

$$\begin{aligned} &\left| f^i(x_1(t), x_2(t), \ldots, x_m(t)) \right. \\ &\left. - f^i(y_1(t), y_2(t), \ldots, y_m(t)) \right| \\ \leqslant\ & l_m^i(|x_1(t) - y_1(t)| + |x_2(t) - y_2(t)| \\ &+ \cdots + |x_m(t) - y_m(t)|), \end{aligned}$$

where $l_m^i > 0$ is the Lipschitz constants.

3 Main results

In this section, we first derive the state estimation error system, then a sufficient condition is given to guarantee global exponential convergence of the consensus error. Since the systems are continuous in the time interval $[t_0, \infty)$, so we can only consider a sampling non-switching time interval $[t_k + \tau_k, t_{k+1} + \tau_{k+1}), k \geqslant 0$, in the following derivation.

From (3)–(5), we get the state estimation error system

$$\dot{e}^i(t) = A_0 e^i(t) - Mae_1^i(t_k) + \tilde{f}^i, \quad i = 1, 2, \ldots, N, \tag{6}$$

and

$$\dot{e}^0(t) = A_0 e^0(t) - Mae_1^0(t_k) + \tilde{f}^0, \tag{7}$$

where $e^i(t) = x^i(t) - \hat{x}^i(t)$, $\tilde{f}^i = f^i(x^i(t)) - f^i(\hat{x}^i(t)), i = 0, 1, \ldots, N$.

Let $\varepsilon^i(t) = (\varepsilon_1^i(t), \varepsilon_2^i(t), \ldots, \varepsilon_n^i(t))^{\mathrm{T}}$, $\varepsilon_m^i(t) = e_m^i(t)/l^{m-1}$, $i = 0, 1, \ldots, N, m = 1, 2, \ldots, n$, we have

$$\dot{\varepsilon}^i(t) = lA_0\varepsilon^i(t) - la\varepsilon_1^i(t_k) + \tilde{g}^i, \quad i = 1, 2, \ldots, N, \tag{8}$$

and

$$\dot{\varepsilon}^0(t) = lA_0\varepsilon^0(t) - la\varepsilon_1^0(t_k) + \tilde{g}^0, \tag{9}$$

where $\varepsilon_1^i(t_k) = x_1^i(t_k) - \hat{x}_1^i(t_k) = e_1^i(t_k)$, $\tilde{g}^i = (\tilde{g}_1^i, \tilde{g}_2^i, \ldots, \tilde{g}_n^i)^{\mathrm{T}}$, $\tilde{g}_m^i = \tilde{f}_m^i/l^{m-1}, i = 0, 1, \ldots, N$.

Let $z^i(t) = (z_1^i(t), z_2^i(t), \ldots, z_n^i(t))^{\mathrm{T}}$, $z_m^i(t) = (\hat{x}_m^i(t) - \hat{x}_m^0(t))/l^{m-1}$, $i = 1, 2, \ldots, N$, we get

$$\dot{z}^i(t) = lA_0 z^i(t) + la(\varepsilon_1^i(t_k) - \varepsilon_1^0(t_k)) + \bar{g}^i + \frac{\bar{b}}{l^{n-1}}u^i(t), \tag{10}$$

where $\bar{g}^i = (\bar{g}_1^i, \bar{g}_2^i, \ldots, \bar{g}_n^i)^{\mathrm{T}}$, $\bar{g}_m^i = \bar{f}_m^i/l^{m-1}$, $\bar{f}_m^i = f_m^i(\hat{x}^i(t)) - f_m^0(\hat{x}^0(t))$.

Let $\varepsilon(t) = ((\varepsilon^0(t))^{\mathrm{T}}, (\varepsilon^1(t))^{\mathrm{T}}, \ldots, (\varepsilon^N(t))^{\mathrm{T}})^{\mathrm{T}}$, $\varepsilon_1(t_k) = (\varepsilon_1^0(t_k), \varepsilon_1^1(t_k), \ldots, \varepsilon_1^N(t_k))^{\mathrm{T}}, i = 0, 1, \ldots, N$, we get the matrix form of equations (8) and (9):

$$\dot{\varepsilon}(t) = l(I_{N+1} \otimes A_0)\varepsilon(t) - l\varepsilon_1(t_k) \otimes a + \tilde{g}, \tag{11}$$

where $\tilde{g} = ((\tilde{g}^0)^{\mathrm{T}}, (\tilde{g}^1)^{\mathrm{T}}, \ldots, (\tilde{g}^N)^{\mathrm{T}})^{\mathrm{T}}$ and I_{N+1} is the $(N+1) \times (N+1)$ identity matrix.

Let $z(t) = ((z^1(t))^{\mathrm{T}}, (z^2(t))^{\mathrm{T}}, \ldots, (z^N(t))^{\mathrm{T}})^{\mathrm{T}}$, we get the matrix form of equation (10) as follows:

$$\begin{aligned} \dot{z}(t) = &\ l(I_N \otimes A_0)z(t) + l(\eta_1(t_k) - 1_N \otimes \varepsilon_1^0(t_k)) \otimes a \\ &+ \bar{g} + \frac{u \otimes \bar{b}}{l^{n-1}}, \end{aligned} \tag{12}$$

where $\eta_1(t_k) = (\varepsilon_1^1(t_k), \varepsilon_1^2(t_k), \ldots, \varepsilon_1^N(t_k))^{\mathrm{T}}$, $\bar{g} = ((\bar{g}^1)^{\mathrm{T}}, (\bar{g}^2)^{\mathrm{T}}, \ldots, (\bar{g}^N)^{\mathrm{T}})^{\mathrm{T}}$, $u = (u^1(t), u^2(t) \ldots, u^N(t))^{\mathrm{T}}$, and I_N is the $N \times N$ identity matrix.

Theorem 1: Under Assumptions 1 and 2, there exists an output feedback controller (5) such that the multi-agent system (3) and (4) are globally exponentially stable, if there exist symmetric positive definite matrices P and Q satisfying

$$D_1^{\mathrm{T}}P + PD_1 \leqslant -r_1 I, \tag{13}$$

$$A_0^{\mathrm{T}}Q + QA_0 - 2\underline{\lambda}_H K^{\mathrm{T}}K \leqslant -r_2 I, \quad K = \bar{b}^{\mathrm{T}}Q \tag{14}$$

and l satisfying

$$l > \max\left\{1, l_1, \frac{4n\sqrt{N}l_1\bar{\lambda}_P}{r_1}, \frac{4n\sqrt{N}l_1\bar{\lambda}_Q}{r_2}\right\}, \tag{15}$$

and

$$T_{\max} + \tau \leqslant \frac{1}{c_3 l}, T_{\min} - \tau > \left(\frac{c_2}{c_1 - c_2}\right)\frac{1}{c_1 l}, \tag{16}$$

where r_1, r_2 are two positive constants

$$c_1 = \min\left\{\frac{r_1}{8\bar{\lambda}_P}, \frac{r_2}{8\bar{\lambda}_Q}\right\}, \quad c_2 = \frac{\kappa_1}{c_3^2},$$

$$\kappa_1 = \frac{\bar{\lambda}_P^2}{r_1\underline{\lambda}_P}(12n\bar{a}_1 + 8n\bar{a}_1 c_3^2),$$

$$c_3 > \max\left\{4, \frac{12n\bar{\lambda}_P^2\bar{a}_1}{r_1} + c_1, \sqrt{\frac{\kappa_1 + 1}{c_1}}, \frac{\kappa_1 + \sqrt{\kappa_1^2 + 4c_1^3\kappa_1}}{2c_1^2}\right\},$$

$$\overline{a_1} = \max\{a_i^2\},\ l_1 = \max\{l_m^i\},$$

$$D_1 = \begin{pmatrix} -a_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & \cdots & 1 \\ -a_n & 0 & \cdots & 0 \end{pmatrix},$$

$\bar{\lambda}_P = \lambda_{\max}(P)$, $\underline{\lambda}_P = \lambda_{\min}(P)$, $\bar{\lambda}_Q = \lambda_{\max}(Q)$, $\underline{\lambda}_Q = \lambda_{\min}(Q)$, in which $\lambda_{\max}(\cdot)$, $\lambda_{\min}(\cdot)$ denote the maximum, minimum eigenvalues of a matrix, respectively.

Proof: For system (11) and (12), we consider two Lyapunov function candidates as follows:

$$V_P = \varepsilon(t)^{\mathrm{T}}(I_{N+1} \otimes P)\varepsilon(t),$$

and

$$V_Q = z(t)^{\mathrm{T}}(I_N \otimes Q)z(t).$$

Obviously, V_P and V_Q are continuously differentiable at any time, except for the switching time instants. At a non-switching time t, we obtain the time derivatives of the two Lyapunov functions

$$\begin{aligned} \dot{V}_P = {} & l\varepsilon(t)^{\mathrm{T}}[I_{N+1} \otimes (D_1^{\mathrm{T}}P + PD_1)]\varepsilon(t) \\ & +2\varepsilon(t)^{\mathrm{T}}(I_{N+1} \otimes P)\tilde{g} \\ & +2l\varepsilon(t)^{\mathrm{T}}(I_{N+1} \otimes P) \\ & \times [(\varepsilon_1(t) - \varepsilon_1(t_k)) \otimes a], \end{aligned} \tag{17}$$

and

$$\begin{aligned} \dot{V}_Q = {} & -lz(t)^{\mathrm{T}}[(H \otimes \bar{b}K)^{\mathrm{T}}(I_N \otimes Q) \\ & +(I_N \otimes Q)(H \otimes \bar{b}K)]z(t) \\ & +lz(t)^{\mathrm{T}}[(I_N \otimes A_0)^{\mathrm{T}}(I_N \otimes Q) \\ & +(I_N \otimes Q)(I_N \otimes A_0)]z(t) \\ & +2lz(t)^{\mathrm{T}}(I_N \otimes Q)[\eta_1(t_k) \\ & -1_N \otimes \varepsilon_1^0(t_k)] \otimes a \\ & +2z(t)^{\mathrm{T}}(I_N \otimes Q)\bar{g}, \end{aligned} \tag{18}$$

where $\varepsilon_1(t) = (\varepsilon_1^0(t), \varepsilon_1^1(t), \ldots, \varepsilon_1^N(t))^{\mathrm{T}}$.

Owing to the symmetricity of H, there exists an orthogonal matrix S such that

$$SHS^{\mathrm{T}} = \Lambda = \mathrm{diag}\{\lambda_H^1, \lambda_H^2, \ldots, \lambda_H^N\},$$

where $\lambda_H^1, \lambda_H^2, \ldots, \lambda_H^N$ are the N eigenvalues of H, let $\underline{\lambda}_H$ denote the minimum non-zero eigenvalues of the matrix H, and $K = \bar{b}^{\mathrm{T}}Q$. Then we have

$$\begin{aligned} & -lz(t)^{\mathrm{T}}[(H \otimes \bar{b}K)^{\mathrm{T}}(I_N \otimes Q) \\ & +(I_N \otimes Q)(H \otimes \bar{b}K)]z(t) \\ = {} & -lz(t)^{\mathrm{T}}[(S^{\mathrm{T}}\Lambda S \otimes \bar{b}K)^{\mathrm{T}}(I_N \otimes Q) \\ & +(I_N \otimes Q)(S^{\mathrm{T}}\Lambda S \otimes \bar{b}K)]z(t) \\ = {} & -lz(t)^{\mathrm{T}}(S^{\mathrm{T}} \otimes I_n)[(\Lambda \otimes \bar{b}K)^{\mathrm{T}}(I_N \otimes Q) \\ & +(I_N \otimes Q)(\Lambda \otimes \bar{b}K)](S \otimes I_n)z(t) \\ = {} & -lz(t)^{\mathrm{T}}(S^{\mathrm{T}} \otimes I_n)[(\Lambda \otimes \bar{b}\bar{b}^{\mathrm{T}}Q)^{\mathrm{T}}(I_N \otimes Q) \\ & +(I_N \otimes Q)(\Lambda \otimes \bar{b}\bar{b}^{\mathrm{T}}Q)](S \otimes I_n)z(t) \\ = {} & -2lz(t)^{\mathrm{T}}(S^{\mathrm{T}} \otimes I_n)(\Lambda \otimes Q^{\mathrm{T}}\bar{b}\bar{b}^{\mathrm{T}}Q)(S \otimes I_n)z(t) \\ \leqslant {} & -2lz(t)^{\mathrm{T}}(I_N \otimes \underline{\lambda}_H Q^{\mathrm{T}}\bar{b}\bar{b}^{\mathrm{T}}Q)z(t). \end{aligned} \tag{19}$$

From (18) and (19), we get

$$\begin{aligned} \dot{V}_Q \leqslant {} & lz(t)^{\mathrm{T}}\{I_N \otimes [A_0^{\mathrm{T}}Q + QA_0 \\ & -2\underline{\lambda}_H Q^{\mathrm{T}}\bar{b}\bar{b}^{\mathrm{T}}Q]\}z(t) \\ & +2lz(t)^{\mathrm{T}}(I_N \otimes Q) \\ & \times [\eta_1(t_k) - 1_N \otimes \varepsilon_1^0(t_k)] \otimes a \\ & +2z(t)^{\mathrm{T}}(I_N \otimes Q)\bar{g}. \end{aligned} \tag{20}$$

From (13) and (14), we have

$$l\varepsilon(t)^{\mathrm{T}}[I_{N+1} \otimes (D_1^{\mathrm{T}}P + PD_1)]\varepsilon(t) \leqslant -r_1 l\varepsilon(t)^{\mathrm{T}}\varepsilon(t), \tag{21}$$

and

$$\begin{aligned} & lz(t)^{\mathrm{T}}\{I_N \otimes [A_0^{\mathrm{T}}Q + QA_0 - 2\underline{\lambda}_H Q^{\mathrm{T}}\bar{b}\bar{b}^{\mathrm{T}}Q]\}z(t) \\ & \leqslant -r_2 lz(t)^{\mathrm{T}}z(t). \end{aligned} \tag{22}$$

According to the inequality $2a^{\mathrm{T}}b \leqslant a^{\mathrm{T}}Xa + b^{\mathrm{T}}X^{-1}b$, we obtain

$$\begin{aligned} & 2l\varepsilon(t)^{\mathrm{T}}(I_{N+1} \otimes P)[(\varepsilon_1(t) - \varepsilon_1(t_k)) \otimes a] \\ & \leqslant \frac{1}{4}r_1 l\varepsilon(t)^{\mathrm{T}}\varepsilon(t) + 4nl\bar{a}_1\bar{\lambda}_P^2(\varepsilon_1(t) - \varepsilon_1(t_k))^{\mathrm{T}} \\ & (\varepsilon_1(t) - \varepsilon_1(t_k))/r_1, \end{aligned} \tag{23}$$

and

$$\begin{aligned} & 2lz(t)^{\mathrm{T}}(I_N \otimes Q)(\bar{\varepsilon}_1(t_k) - 1_N \otimes \varepsilon_1^0(t_k)) \otimes a \\ & \leqslant \frac{1}{4}r_2 lz(t)^{\mathrm{T}}z(t) + 4nl\bar{a}_1\bar{\lambda}_Q^2(\bar{\varepsilon}_1(t_k) \\ & -1_N \otimes \varepsilon_1^0(t_k))^{\mathrm{T}}(\bar{\varepsilon}_1(t_k) - 1_N \otimes \varepsilon_1^0(t_k))/r_2 \\ & \leqslant \frac{1}{4}r_2 lz(t)^{\mathrm{T}}z(t) + 8nl\bar{a}_1\bar{\lambda}_Q^2\varepsilon(t_k)^{\mathrm{T}}\varepsilon(t_k)/r_2. \end{aligned} \tag{24}$$

From the Lipschitz condition, we get

$$2\varepsilon(t)^{\mathrm{T}}(I_{N+1} \otimes P)\tilde{g} \leqslant 2n\sqrt{N}l_1\bar{\lambda}_P\varepsilon(t)^{\mathrm{T}}\varepsilon(t), \tag{25}$$

and

$$2z(t)^{\mathrm{T}}(I_N \otimes Q)\bar{g} \leqslant 2n\sqrt{N}l_1\bar{\lambda}_Q z(t)^{\mathrm{T}}z(t). \tag{26}$$

On the basis of the Cauchy inequality and the Lemma 2, it follows that:

$$
\begin{aligned}
&|\varepsilon_1^i(t) - \varepsilon_1^i(t_k)|^2 \\
&\leqslant (t - t_k)\int_{t_k}^{t} |\dot{\varepsilon}_1^i(s)|^2 \,\mathrm{d}s \\
&= (t - t_k)\int_{t_k}^{t} \left| l\varepsilon_2^i(s) - l a_1 \varepsilon_1^i(t_k) + \frac{\tilde{f}_1^{\,i}}{l^0} \right|^2 \mathrm{d}s \\
&\leqslant 3l^2 (t - t_k)\int_{t_k}^{t} [\varepsilon_2^i(s)^2 + a_1^2 \varepsilon_1^i(t_k)^2 + \frac{l^2}{l^2}\varepsilon_1^i(s)^2] \,\mathrm{d}s \\
&\leqslant 3l^2 (t - t_k)^2 \bar{a}_1 \varepsilon_1^i(t_k)^2 \\
&\quad + 3l^2 (t - t_k)\int_{t_k}^{t} [\varepsilon_1^i(s)^2 + \varepsilon_2^i(s)^2] \,\mathrm{d}s, \\
&\quad i = 0, 1, \ldots, N,
\end{aligned}
$$

that is

$$
\begin{aligned}
&(\varepsilon_1(t) - \varepsilon_1(t_k))^{\mathrm{T}}(\varepsilon_1(t) - \varepsilon_1(t_k)) \\
&\leqslant 3l^2 (t - t_k)^2 \bar{a}_1 \varepsilon_1(t_k)^{\mathrm{T}} \varepsilon_1(t_k) \\
&\quad + 3l^2 (t - t_k)\int_{t_k}^{t} [\varepsilon_1(s)^{\mathrm{T}} \varepsilon_1(s) \\
&\quad + \varepsilon_2(s)^{\mathrm{T}} \varepsilon_2(s)] \,\mathrm{d}s .
\end{aligned} \tag{27}
$$

From (17)–(27), we have

$$
\begin{aligned}
\dot{V}_P &\leqslant \left(-\frac{3}{4} r_1 l + 2n\sqrt{N} l_1 \bar{\lambda}_P\right) \varepsilon(t)^{\mathrm{T}} \varepsilon(t) \\
&\quad + \frac{12n\bar{a}_1^2 l^3 \bar{\lambda}_P^2 (t - t_k)^2 \varepsilon_1(t_k)^{\mathrm{T}} \varepsilon_1(t_k)}{r_1} \\
&\quad + \frac{12n\bar{a}_1 l^3 \bar{\lambda}_P^2 (t - t_k)}{r_1} \int_{t_k}^{t} [\varepsilon_1(s)^{\mathrm{T}} \varepsilon_1(s) \\
&\quad + \varepsilon_2(s)^{\mathrm{T}} \varepsilon_2(s)] \,\mathrm{d}s, \\
\dot{V}_Q &\leqslant \left(-\frac{3}{4} r_2 l + 2n\sqrt{N} l_1 \bar{\lambda}_Q\right) z(t)^{\mathrm{T}} z(t) \\
&\quad + 8nl\bar{a}_1 \bar{\lambda}_Q^2 \varepsilon(t_k)^{\mathrm{T}} \varepsilon(t_k)/r_2 .
\end{aligned}
$$

Construct the following auxiliary integral function:

$$
V_R = \int_{t - T_{\max} - \tau}^{t} \int_{\rho}^{t} [\varepsilon(s)^{\mathrm{T}} \varepsilon(s) + z(s)^{\mathrm{T}} z(s)] \,\mathrm{d}s \,\mathrm{d}\rho .
$$

We have

$$
V_R \leqslant (T_{\max} + \tau) \int_{t - T_{\max} - \tau}^{t} [\varepsilon(s)^{\mathrm{T}} \varepsilon(s) + z(s)^{\mathrm{T}} z(s)] \,\mathrm{d}s .
$$

The derivative of V_R is

$$
\begin{aligned}
\dot{V}_R &= (T_{\max} + \tau)[\varepsilon(t)^{\mathrm{T}} \varepsilon(t) + z(t)^{\mathrm{T}} z(t)] \\
&\quad - \int_{t - T_{\max} - \tau}^{t} [\varepsilon(s)^{\mathrm{T}} \varepsilon(s) + z(s)^{\mathrm{T}} z(s)] \,\mathrm{d}s .
\end{aligned}
$$

Let $V(t) = V_P + V_Q + l^2 V_R$, we have

$$
\begin{aligned}
\dot{V}(t) &\leqslant \left(-\frac{3}{4} r_1 l + 2n\sqrt{N} l_1 \bar{\lambda}_P + (T_{\max} + \tau) l^2\right) \varepsilon(t)^{\mathrm{T}} \varepsilon(t) \\
&\quad + \left(-\frac{3}{4} r_2 l + 2n\sqrt{N} l_1 \bar{\lambda}_Q + (T_{\max} + \tau) l^2\right) z(t)^{\mathrm{T}} z(t) \\
&\quad + \left(\frac{12n\bar{a}_1^2 l^3 \bar{\lambda}_P^2 (t - t_k)^2}{r_1} + \frac{8n\bar{\lambda}_Q^2 \bar{a}_1 l}{r_2}\right) \varepsilon(t_k)^{\mathrm{T}} \varepsilon(t_k) \\
&\quad + \left[\frac{12n\bar{a}_1 l^3 \bar{\lambda}_P^2 (t - t_k)}{r_1} - l^2\right] \\
&\quad \times \int_{t - T_{\max} - \tau}^{t} [\varepsilon(s)^{\mathrm{T}} \varepsilon(s) + z(s)^{\mathrm{T}} z(s)] \,\mathrm{d}s, \\
&\leqslant -\left(\frac{1}{4} r_1 - (T_{\max} + \tau) l\right) l\varepsilon(t)^{\mathrm{T}} \varepsilon(t) \\
&\quad \times -\left(\frac{1}{4} r_2 - (T_{\max} + \tau) l\right) l z^{\mathrm{T}}(t) z(t) \\
&\quad + \left(\frac{12n\bar{a}_1^2 l^3 \bar{\lambda}_P^2 (t - t_k)^2}{r_1} + \frac{8n\bar{\lambda}_Q^2 \bar{a}_1 l}{r_2}\right) \varepsilon(t_k)^{\mathrm{T}} \varepsilon(t_k) \\
&\quad + \left[\frac{12n\bar{a}_1 l^3 \bar{\lambda}_P^2 (t - t_k)}{r_1} - l^2\right] \\
&\quad \times \int_{t - T_{\max} - \tau}^{t} [\varepsilon(s)^{\mathrm{T}} \varepsilon(s) + z(s)^{\mathrm{T}} z(s)] \,\mathrm{d}s, \\
&\leqslant -\frac{1}{\bar{\lambda}_P}\left(\frac{1}{4} r_1 - (T_{\max} + \tau) l\right) l V_P(t) \\
&\quad \times -\frac{1}{\bar{\lambda}_Q}\left(\frac{1}{4} r_2 - (T_{\max} + \tau) l\right) l V_Q(t) \\
&\quad + \frac{1}{\underline{\lambda}_P}\left(\frac{12n\bar{\lambda}_P^2 l^3 \bar{a}_1 (t - t_k)^2}{r_1} + \frac{8n\bar{\lambda}_Q^2 \bar{a}_1 l}{r_2}\right) V_P(t_k) \\
&\quad + \frac{(12n\bar{\lambda}_P^2 l^3 \bar{a}_1 (t - t_k)/r_1) - l^2}{T_{\max} + \tau} V_R(t) .
\end{aligned}
$$

Since

$$
l > \max\left\{1, l_1, \frac{4n\sqrt{N} l_1 \bar{\lambda}_P}{r_1}, \frac{4n\sqrt{N} l_1 \bar{\lambda}_Q}{r_2}\right\}, \quad T_{\max} + \tau < \frac{1}{c_3 l},
$$

we have

$$
\begin{aligned}
\dot{V}(t) &\leqslant -\frac{1}{\bar{\lambda}_P}\left(\frac{r_1}{4} - \frac{1}{c_3}\right) l V_P(t) - \frac{1}{\bar{\lambda}_Q}\left(\frac{r_2}{4} - \frac{1}{c_3}\right) l V_Q(t) \\
&\quad + \frac{1}{\underline{\lambda}_P c_3^2}\left(\frac{12n\bar{\lambda}_P^2 \bar{a}_1}{r_1} + \frac{8n\bar{\lambda}_Q^2 \bar{a}_1 c_3^2}{r_2}\right) l V_P(t_k) \\
&\quad + \left(\frac{12n\bar{\lambda}_P^2 \bar{a}_1}{r_1} - c_3\right) l^3 V_R(t) .
\end{aligned}
$$

Since

$$
\begin{aligned}
&c_3 > \max\left\{\frac{8}{r_1}, \frac{8}{r_2}, \frac{12n\bar{\lambda}_P^2 \bar{a}_1}{r_1} + c_1, \sqrt{\frac{\kappa_1 + 1}{c_1}}, \frac{\kappa_1 + \sqrt{\kappa_1^2 + 4c_1^3 \kappa_1}}{2c_1^2}\right\}, \\
&c_1 \leqslant \min\left\{\frac{r_1}{8\bar{\lambda}_P}, \frac{r_2}{8\bar{\lambda}_Q}\right\}, \quad c_2 = \frac{\kappa_1}{c_3^2},
\end{aligned}
$$

we have

$$
\begin{aligned}
\dot{V}(t) &\leqslant -c_1 l V_P(t) - c_1 l V_Q(t) - c_1 l^3 V_R(t) + c_2 l V_P(t_k) \\
&\leqslant -c_1 l V(t) + c_2 l V(t_k) .
\end{aligned} \tag{28}
$$

Multiplying $e^{c_1 l t}$ on both sides of (28), we have

$$e^{c_1lt}\frac{\mathrm{d}}{\mathrm{d}t}V(t)+e^{c_1lt}c_1ltV(t)\leqslant e^{c_1lt}c_2lV(t_k). \tag{29}$$

Integrating both sides of (29) from $t_k+\tau_k$ to t, we have

$$V(t)\leqslant e^{-c_1l(t-t_k-\tau_k)}V(t_k+\tau_k)+\frac{c_2}{c_1}V(t_k) -\frac{c_2}{c_1}e^{-c_1l(t-t_k-\tau_k)}V(t_k).$$

Note that $\varepsilon(t)$ and $z(t)$ are continuous in the interval $[t_0,\infty)$, we have

$$V(t_{k+1}+\tau_{k+1})\leqslant e^{-c_1l(T_{k+1}+\tau_{k+1}-\tau_k)}V(t_k+\tau_k)+\frac{c_2}{c_1}V(t_k) -\frac{c_2}{c_1}e^{-c_1l(T_{k+1}+\tau_{k+1}-\tau_k)}V(t_k), \tag{30}$$

and

$$V(t_{k+1})\leqslant e^{-c_1l(T_{k+1}-\tau_k)}V(t_k+\tau_k)+\frac{c_2}{c_1}V(t_k) -\frac{c_2}{c_1}e^{-c_1l(T_{k+1}-\tau_k)}V(t_k). \tag{31}$$

From (30) and (31), we have

$$\begin{aligned}&V(t_{k+1}+\tau_{k+1})+\rho_1V(t_{k+1})\\ &\leqslant(e^{-c_1l(T_{k+1}+\tau_{k+1}-\tau_k)}+\rho_1e^{-c_1l(T_{k+1}-\tau_k)})V(t_k+\tau_k)\\ &+\frac{c_2}{c_1}[1+\rho_1-e^{-c_1l(T_{k+1}-\tau_k)}(e^{-c_1l\tau_{k+1}}+\rho_1)]V(t_k),\end{aligned} \tag{32}$$

where ρ_1 is a positive constant number.

Since $c_2=\kappa_1/c_3^2$. Then, $c_3>\kappa_1/c_1$ implies that $c_1>\kappa_1/c_3=c_2c_3>8c_2$, i.e. $c_2/c_1<1/8$. From $T_{\min}-\tau>c_2/(c_1(c_1-c_2)l)>0$, we can choose some $\rho_1>0$ such that

$$\eta_1=\max_{k\geqslant 0}\left\{\frac{c_2}{c_1\rho_1}[1+\rho_1-e^{-c_1l(T_{k+1}-\tau_k)}(e^{-c_1l\tau_{k+1}}+\rho_1)]\right\}<1, \tag{33}$$

and

$$\eta_2=\max_{k\geqslant 0}\{\rho_1e^{-c_1l(T_{k+1}-\tau_k)}+e^{-c_1l(T_{k+1}+\tau_{k+1}-\tau_k)}\}<1. \tag{34}$$

Let $\eta=\max\{\eta_1,\eta_2\}$. From (33) and (34), we have $0<\eta<1$. Then, it follows that:

$$V(t_{k+1}+\tau_{k+1})+\rho_1V(t_{k+1})\leqslant\eta[V(t_k+\tau_k)+\rho_1V(t_k)]. \tag{35}$$

Through the iteration for (35), we have

$$V(t_k+\tau_k)+\rho_1V(t_k)\leqslant\eta^k[V(t_0+\tau_0)+\rho_1V(t_0)].$$

Since $0<\eta<1$, it is clear that the sequence $\{V(t_k+\tau_k)+\rho_1V(t_k)\}$ is convergent to zero, and

$$V(t)\leqslant V(t_k+\tau_k)+\rho_1V(t_k)\leqslant\eta^k[V(t_0+\tau_0)+\rho_1V(t_0)],\ t\in[t_k+\tau_k,t_{k+1}+\tau_{k+1}).$$

Since for any $t>t_0+\tau_0$, there exists $k\geqslant 0$ such that $t\in[t_k+\tau_k,t_{k+1}+\tau_{k+1})$. It follows that $((t-t_0-\tau_0)/(T_{\max}+\tau))-1\leqslant k$. Then

$$\begin{aligned}V(t)&\leqslant\eta^{((t-t_0-\tau_0)/(T_{\max}+\tau))-1}\\ &\times[V(t_0+\tau_0)+\rho_1V(t_0)]\\ &=e^{(t/(T_{\max}+\tau))\ln\eta}\eta^{((-t_0-\tau_0)/(T_{\max}+\tau))-1}\\ &\times[V(t_0+\tau_0)+\rho_1V(t_0)].\end{aligned}$$

From Definition 1, we get the system is globally exponentially stable. This completes the proof of the theorem. □

From Theorem 1, we have the following results which give the estimates of the allowable sampling interval and time-delays.

Corollary 1: There exists an output feedback control law in the form of (5), which globally exponentially stabilises the system (3) and (4), if there exist symmetric positive definite matrices P,Q and a constant l such that (13)–(15) and

$$\begin{aligned}\tau&<\frac{1}{\rho_2l}\left(\frac{1}{c_3}-\frac{1}{c_4}\right),\\ T_{\max}&<\frac{1}{c_3l}-\frac{1}{\rho_2l}\left(\frac{1}{c_3}-\frac{1}{c_4}\right),\\ T_{\min}&>\frac{1}{c_4l}+\frac{1}{\rho_2l}\left(\frac{1}{c_3}-\frac{1}{c_4}\right),\end{aligned} \tag{36}$$

are satisfied, where $\rho_2>2$, $c_4=(((c_1-c_2)c_1)/c_2)$, c_1,c_2,c_3 and κ_1 are given in Theorem 1.

Proof: From $c_3>((\kappa_1+\sqrt{\kappa_1^2+4c_1^3\kappa_1})/2c_1^2)$, we have $c_1^2c_3^2-c_3\kappa_1-c_1\kappa_1>0$. Then, $c_3<((c_1(c_1c_3^2-\kappa_1))/\kappa_1)=c_1((c_1/c_2)-1)=c_4$. Therefore, $(1/c_3-1/c_4)>0$. Since $c_2/c_1<1/8$ and $c_4>0$, it is easy to check that the conditions (36) imply that the conditions (16) are satisfied. The proof is completed. □

Remark 3: This corollary does not give the maximum sampling interval and time-delay. The maximum sampling interval and time-delay depend on the choices of a_i,P,Q.

On the basis of Theorem 1 and Corollary 1, an algorithm to set the design parameters is presented as follows:

Step 1: On the basis of the interconnected topology, we construct the matrix H and calculate the smallest eigenvalue of H which is denoted by $\underline{\lambda}_H$.

Step 2: We choose the appropriate values of a_i such that the inequality (13) holds and gets the matrix P, $\bar{\lambda}_P$, $\underline{\lambda}_P$, then by solving the inequality (14) gets the matrix Q, $\bar{\lambda}_Q$, $\underline{\lambda}_Q$, and let $K=\bar{b}^{\mathrm{T}}Q$.

Step 3: We select l such that the condition (15) is satisfied.

Step 4: Calculate c_1, select $\rho_2>2$ and c_3 such that

$$c_3>\max\left\{4,\frac{12n\bar{\lambda}_P^2\bar{a}_1}{r_1}+c_1,\sqrt{\frac{\kappa_1+1}{c_1}},\frac{\kappa_1+\sqrt{\kappa_1^2+4c_1^3\kappa_1}}{2c_1^2}\right\},$$

then calculate c_2 and c_4.

4 Simulations

In this section, we give an example to validate our theoretical results. In the example, we consider a multi-agent system consisting of four agents and a leader. The dynamics of the leader and the followers are described by:

$$\begin{cases}\dot{x}_1^0(t)=x_2^0(t)+l_0\cos(t)x_1^0(t),\\ \dot{x}_2^0(t)=x_3^0(t)+l_0\cos(t)x_2^0(t),\\ \dot{x}_3^0(t)=l_0(\sin(t)x_2^0(t)+2\cos(t)x_3^0(t)),\end{cases}$$

and

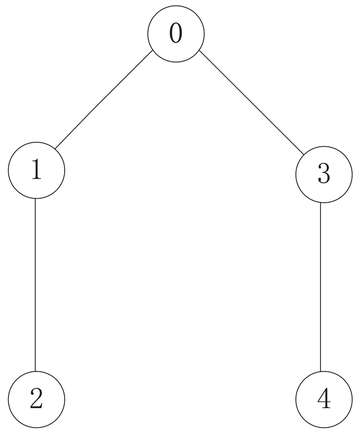

Fig. 1 *Connected graph*

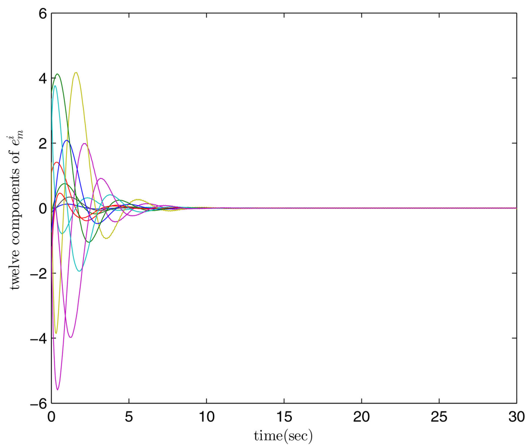

Fig. 2 *State estimation error*

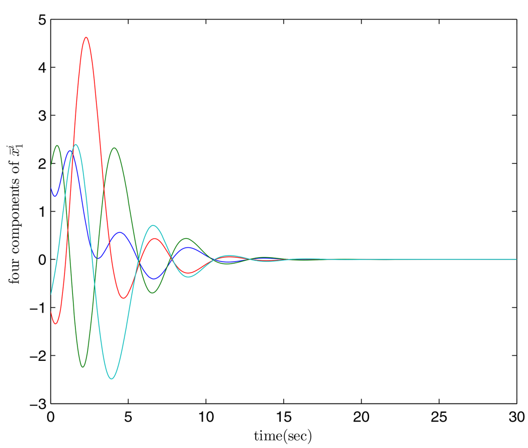

Fig. 3 *Output error*

$$\begin{cases} \dot{x}_1^i(t) = x_2^i(t) + l_0\cos(t)x_1^i(t), \\ \dot{x}_2^i(t) = x_3^i(t) + l_0\cos(t)x_2^i(t), \\ \dot{x}_3^i(t) = u^i + l_0(\sin(t)x_2^i(t) + 2\cos(t)x_3^i(t)), \end{cases}$$

respectively, where l is a constant.

An observer-based output feedback controller is proposed as follows:

$$\begin{cases} \dot{\hat{x}}_1^0(t) = \hat{x}_2^0(t) + la_1e_1^0(t_k) + l_0\cos(t)\hat{x}_1^0(t), \\ \dot{\hat{x}}_2^0(t) = \hat{x}_3^0(t) + l^2a_2e_1^0(t_k) + l_0\cos(t)\hat{x}_2^0(t), \\ \dot{\hat{x}}_3^0(t) = l^3a_3e_1^0(t_k) + l_0(\sin(t)\hat{x}_2^0(t) + 2\cos(t)\hat{x}_3^0(t)), \end{cases}$$

$$\begin{cases} \dot{\hat{x}}_1^i(t) = \hat{x}_2^i(t) + la_1e_1^i(t_k) + l_0\cos(t)\hat{x}_1^i(t), \\ \dot{\hat{x}}_2^i(t) = \hat{x}_3^i(t) + l^2a_2e_1^i(t_k) + l_0\cos(t)\hat{x}_2^i(t), \\ \dot{\hat{x}}_3^i(t) = u^i + l^3a_3e_1^i(t_k) + l_0(\sin(t)\hat{x}_2^i(t) + 2\cos(t)\hat{x}_3^i(t)), \end{cases}$$

where

$$u^i(t) = -KW\Big[\sum\nolimits_{j \in N^i} a_{ij}(\hat{x}^i(t) - \hat{x}^j(t)) + b_i(\hat{x}^i(t) - \hat{x}^0(t))\Big],$$

and $W = \mathrm{diag}(l^3, l^2, l)^{\mathrm{T}}$.

Suppose that the interconnected topology is described as in Fig. 1. We obtain that the largest and smallest non-zero eigenvalues of $H = \mathscr{L} + \mathscr{B}$ are $\bar{\lambda}_H = 2.618$ and $\underline{\lambda}_H = 0.382$. By choosing appropriate parameters of $a = (a_1, a_2, a_3) = (4, 4, 4)$, and solving the inequality (13), we get the solution $P > 0$, and $\bar{\lambda}_P = 7.0042, \underline{\lambda}_P = 0.2291$. By solving the inequality (14), we get the positive definite matrix

$$Q = \begin{pmatrix} 2.4841 & 2.5853 & 1.1440 \\ 2.5853 & 5.2780 & 2.8419 \\ 1.1440 & 2.8419 & 2.9577 \end{pmatrix},$$

then $K = \bar{b}^{\mathrm{T}}Q = (1.1440, 2.8419, 2.9577)$, $\bar{\lambda}_Q = 8.4858, \underline{\lambda}_Q = 0.6646$. Simulation is conducted in 30 s time, the initial states of the whole system (3) and (4) are

$$x_0 = [1.5, 2.6, 0.82, 3, 1.31, 3.27, 3.44, 3.74, 2.25, 0.42, 1.14, 4.57, 0.76, 4.12, 2.69]^{\mathrm{T}}$$

and

$$\hat{x}_0 = [3.13, 1.46, 2.16, 0.07, 4.92, 0.84, 0.53, 1.86, 0.99, 2.45, 1.69, 4.76]^{\mathrm{T}}.$$

Under the observer-based output feedback control law (5), selecting parameter $l_0 = 0.1, l = 1.5$. The sampling interval T_k and the time-delay τ_k are given as $T_k = 0.01s$ and $\tau_k = 0.001s$, respectively. The simulation results are shown in Figs. 2 and 3. Fig. 2 shows that the 12 components of the state estimation errors $e_m^i(t) = x_m^i(t) - \hat{x}_m^i(t), m = 1, 2, 3,\ i = 1, 2, 3, 4,$ converge to zero. Fig. 3 shows that the four components of the output error between the follower agents and the leader agents $\bar{x}_1^i(t) = x_1^i(t) - x_1^0(t), i = 1, 2, 3, 4,$ converge to zero.

Remark 4: So far, the consensus problem of multi-agent systems with the sampled-data [22, 33], sampled-data and time-delay [27, 28], and non-linear dynamics and time-delay [42–44] have been reported in the literatures. All of the above-mentioned literatures are on the state consensus. The literatures on output consensus problem are summarised as follows:

Case 1: The leader-following systems with unknown non-linear dynamics without considering sampled-data and time-delay [30, 45, 46].

Case 2: The leader-following systems with unknown non-linear dynamics and time-delay without considering sampled-data [47].

Case 3: The leaderless systems with unknown non-linear dynamics, time-delay, and sampled-data [48].

In this paper, the output consensus problem for leader-following multi-agent systems with unknown non-linear dynamics and delayed sampled-data is studied. The algorithm proposed in this paper is more general.

5 Conclusion

This paper considers the output consensus problem of high-order leader-following multi-agent systems with unknown non-linear dynamics, in which the output data of the system are delayed and sampled. A novel continuous observer-based output feedback controller is presented such that the outputs of the system reach consensus exponentially. The connectedness of interconnected graphs is required to guarantee the exponential convergence of the considered systems. Moreover, the estimates of the allowable sampling period and time-delay were also obtained.

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7 References

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