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A high-gain-based global finite-time nonlinear observer

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In this paper, a global finite-time observer is designed for a class of nonlinear systems with bounded rational powers imposed on the incremental nonlinearities. Compared with the previous global finite-time results, the new observer designed here is with a new gain update law. Moreover, an example is given to show that the proposed observer can reduce the time of the observation error convergence.

Keywords: global finite-time observer; high gain; nonlinear system; homogeneity

1. Introduction

Consider the problem of observer design for a nonlinear system described by

$$\begin{cases} \dot{x} = f(x, u), \\ y = h(x), \end{cases} \tag{1}$$

where $x \in \mathcal{R}^n$ is the state, $u \in \mathcal{R}^m$ is the input and $y \in \mathcal{R}^p$ is the output. Unlike in the case of linear system, the observability of nonlinear system depends on the inputs of the system (Gauthier & Bornard, 1981; Gauthier, Hammouri, & Othman, 1992; Shim & Seo, 2003). Perhaps for this reason, over the years, several papers have investigated the relationship between nonlinear observability and the existence of nonlinear observers (Fliess, 1982; Hermann & Krener, 1977). Since then, a lot of works have been done to try to design nonlinear observers through linearisation of nonlinear systems (Kotta, 1987; Krener & Isidori, 1983; Rugh, 1986). With the definition of uniform observability or observability for any input as proposed by Gauthier et al. (1992), thereafter, many existing results on nonlinear observer design are based on uniform observability. For example, Gauthier et al. (1992) proposed a simple nonlinear observer by a high gain method, then a nonlinear observer is designed in Hammouri, Targui, & Armanet (2002) for nonlinear systems with a triangular structure, and high gain observers in the presence of measurement noise (Ahrens & Khalil, 2009) are employed to output feedback control problem for a class of nonlinear systems through a switched-gain approach and so on. A common assumption for the observer design of nonlinear system is the Lipschitz condition in the nonlinear terms as discussed in the works (Chen & Chen, 2007; Pertew, Marquez, & Zhao, 2006; Rajamani, 1998) and references therein. Research on nonlinear observer design has also been done on some other kinds of nonlinear systems. Krishnamurthy, Khorrami, and Chandra (2003) give global high-gain-based observers for nonlinear systems with output-dependent upper diagonal terms, while global asymptotic high gain observers are studied in Praly (2003) for nonlinear systems with the nonlinear terms admitting an incremental rate of the measured output.

Based on the finite-time stability and homogeneity theory of nonlinear systems (Bhat & Bernstein, 2000, 2005), different kinds of finite-time observers for nonlinear systems are developed. For example, Perruquetti, Floquet, and Moulay (2008) introduced a finite-time observer with application to secure communication, where a homogeneous Lyapunov function is constructed. Then, based on this homogeneous Lyapunov function, semi-global finite-time and two different kinds of global finite-time observers are designed for single output triangular nonlinear systems, which are uniformly observable and globally Lipschitz (Ménard, Moulay, & Perruquetti, 2010; Shen & Huang, 2009; Shen & Xia, 2008). Global finite-time observers (Shen, Huang, & Gu, 2011) are proposed for a class of globally Lipschitz nonlinear systems with non-triangular structure where the interactions between all the states of the nonlinear terms are allowed. Then, in Burlion, Ahmed-Ali, & Lamnabhi-Lagarrigue (2011), a global finite-time observer with high gain is designed for a class of nonlinear systems where the nonlinear terms admit an incremental rate depending only on the output. Unfortunately, in all these papers, the derivative of the homogeneous Lyapunov function along the observation error system is not continuous. Then, Shen & Xia (2010) give a correct proof of the convergence of observation error and a semi-global finite-time observer is

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designed for the following nonlinear systems whose solutions exist for all positive time,

$$\begin{cases} \dot{x}_1 = x_2 + f_1(y, u), \\ \dot{x}_2 = x_3 + f_2(y, x_2, u), \\ \quad \vdots \\ \dot{x}_n = f_n(y, x_2, \dots, x_n, u), \\ y \ = x_1 = Cx,\ C = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}, \end{cases} \tag{2}$$

where $u \in \mathcal{R}^m$, $x \in \mathcal{R}^n$, $y \in \mathcal{R}$, with the nonlinear terms $f_i(\cdot)(i = 2, \dots, n)$ satisfying conditions

$$|f_i(y, x_2, \dots, x_i, u) - f_i(y, \hat{x}_2, \dots, \hat{x}_i, u)| \le \Gamma(u, y) \left(1 + \sum_{j=2}^{n} |\hat{x}_j|^{\upsilon_j}\right) \sum_{j=2}^{i} |x_j - \hat{x}_j| + l \sum_{j=2}^{i} |x_j - \hat{x}_j|^{\beta_{ij}}, \tag{3}$$

where $\Gamma(\cdot)$ is a continuous function, $l > 0$, $\upsilon_j \in [0, \frac{1}{j-1})(j = 2, \dots, n)$, the rational powers of the incremental terms satisfy $\frac{q-i}{q-j+1} < \beta_{ij} < \frac{i}{j-1}(2 \le j \le i \le n)$ (where $q > n$ is a positive real number). Asymptotic and finite-time stability are studied for a class of nonlinear homogeneous systems (Shen & Xia, 2011) where the best possible lower bound of homogeneity of degree is obtained. Then, motivated by Rosier (1992), a new kind of continuous homogeneous Lyapunov function and a global finite-time observer are constructed in Li, Shen, & Xia (2011) for a nonlinear system (2) under condition (3) with a better lower bound of the rational powers $\frac{n-i}{n-j+1} < \beta_{ij} < \frac{i}{j-1}(2 \le j \le i \le n)$.

In this paper, we restrict our attention to estimating the states only for those nonlinear systems (2) whose solutions globally exist and are unique for all positive time. The primary objective of this paper is to design a new global finite-time observer for nonlinear system (2) with condition (3). We will show that under the same rational powers $\frac{n-i}{n-j+1} < \beta_{ij} < \frac{i}{j-1}(2 \le j \le i \le n)$, global finite-time observers exist with a new gain update law where two new items are introduced compared with the dynamic high gain used in Li et al. (2011). Moreover, through an example, it will be shown that the observer proposed in this paper can render the observation error converging much more quickly than that in Li et al. (2011) although the amplitude of the observation error curve is a bit greater.

The rest of the paper is organised as follows. Some previous results are reviewed in Section 2. Then in Section 3, our main result, a global finite-time observer with a new gain update law is designed for system (2) under condition (3) with a detailed proof. An example is given in Section 4, highlighting the performance of the proposed observer and some comparisons are made with the results in Li et al. (2011). Then the paper is concluded in Section 5. Finally, the proofs of two useful lemmas are included in Appendix.

2. Previous results

Before we consider the global finite-time observer for system (2) with condition (3), let us recall some previous results for nonlinear system (2) with condition (3) where the rational powers satisfying $\frac{q-i}{q-j+1} \le \beta_{ij} < \frac{i}{j-1}(2 \le j \le i \le n)$ (where $q > n$ is a positive real number) in Shen & Xia (2010) and $\frac{n-i}{n-j+1} \le \beta_{ij} < \frac{i}{j-1}(2 \le j \le i \le n)$ in Li et al. (2011), respectively.

For nonlinear system (2), earlier Shen & Xia (2010) present a semi-global finite-time observer of the following form:

$$\begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 + La_1 \lceil e_1 \rfloor^{\alpha_1} + f_1(y, u), \\ \dot{\hat{x}}_2 = \hat{x}_3 + L^2 a_2 \lceil e_1 \rfloor^{\alpha_2} + f_2(y, \hat{x}_2, u), \\ \quad \vdots \\ \dot{\hat{x}}_n = L^n a_n \lceil e_1 \rfloor^{\alpha_n} + f_n(y, \hat{x}_2, \dots, \hat{x}_n, u), \end{cases} \tag{4}$$

with the observer gain L being dynamically updated by

$$\dot{L} = -L[\varphi_1(L^{1-\sigma} - \varphi_2) - \varphi_3 \Psi(u, y, \hat{x})],\ L(0) > \varphi_2, \tag{5}$$

where $\varphi_1, \varphi_2 \ge 1$ and φ_3 are three positive real numbers, $\Psi(u, y, \hat{x}) = \Gamma(u, y)(1 + \sum_{j=2}^{n} |\hat{x}_j|^{\upsilon_j})$ and $a_i > 0(i = 1, \dots, n)$ are the coefficients of the Hurwitz polynomial,

$$s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n \tag{6}$$

and

$$\alpha_i = i\alpha - (i - 1), \quad i = 1, \dots, n, \tag{7}$$

where $\alpha \in (1 - \frac{1}{n-1}, 1)$ and the rational power β_{ij} satisfy $\frac{q-i}{q-j+1} \le \beta_{ij} < \frac{i}{j-1}(2 \le j \le i \le n)$ (where $q > n$ is a positive real number).

Then, based on the same gain update law (5), a kind of global finite-time observers with two homogeneous terms (Li et al., 2011) with different degrees (one less than 1 and the other greater than 1) are constructed for nonlinear system (2) with condition (3) where the rational powers satisfying $\frac{n-i}{n-j+1} \le \beta_{ij} < \frac{i}{j-1}(2 \le j \le i \le n)$ as follows:

$$\begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 + La_1 \lceil e_1 \rfloor^{\alpha_1} + L^{1-(\beta_1-1)(1-\eta)\sigma} a_1 \lceil e_1 \rfloor^{\beta_1} \\ \qquad + f_1(y, u), \\ \dot{\hat{x}}_2 = \hat{x}_3 + L^2 a_2 \lceil e_1 \rfloor^{\alpha_2} + L^{2-(\beta_2-1)(1-\eta)\sigma} a_2 \lceil e_1 \rfloor^{\beta_2} \\ \qquad + f_2(y, \hat{x}_2, u), \\ \quad \vdots \\ \dot{\hat{x}}_n = L^n a_n \lceil e_1 \rfloor^{\alpha_n} + L^{n-(\beta_n-1)(1-\eta)\sigma} a_n \lceil e_1 \rfloor^{\beta_n} \\ \qquad + f_n(y, \hat{x}_2, \dots, \hat{x}_n, u), \end{cases} \tag{8}$$

where $\beta_i = i\beta - (i - 1), (i = 0, 1, \dots, n), \beta > \frac{1+\sigma}{\sigma}, 0 < \eta < 1 - \alpha < 1$.

3. Main result

The purpose of this paper is try to design a global finite-time observer with a new gain update law for the nonlinear system (2) with condition (3) where the rational powers satisfying $\frac{n-i}{n-j+1} \le \beta_{ij} < \frac{i}{j-1}(2 \le j \le i \le n)$. Before we give our result, let us introduce a useful lemma first.

The rational power $\beta_{ij}(2 \le j \le i \le n)$ in Equation (3) satisfies the following condition.

Lemma 3.1: *For $\beta_{ij}(2 \le j \le i \le n)$ given in Equation (3), $1-\frac{1}{n} < \alpha < 1$, if $\beta_{ij} > \frac{n-i}{n-j+1}$, we have $\alpha - 1 - \alpha_{j-1}\beta_{ij} + \alpha_{i-1} < 0$.*

Proof: The proof of Lemma 3.1 is given in Appendix. □

In the following, we will prove that the observer of the form (4) with the following dynamic gain,

$$\dot{L} = -L[\varphi_1(L^{1-\sigma} - \varphi_2) - \varphi_3\Psi(u, y, \hat{x}) - \varphi_4 L^{1-2\sigma}|y - \hat{x}_1|^m - \varphi_5\Psi(u, y, \hat{x})|y - \hat{x}_1|^m], \tag{9}$$

$L(0) > \varphi_2$ is a global finite-time observer for nonlinear system (2) with condition (3), where $\varphi_1, \varphi_2 > 1, \varphi_3, \varphi_4, \varphi_5$ are five positive numbers, m is a positive number satisfying

$$m \ge \max\{\alpha_{j-1}\beta_{ij} - \alpha_{i-1}, 1\}, \quad 2 \le j \le i \le n, \tag{10}$$

$\Psi(u, y, \hat{x})$ is the same as that in Equation (5).

For the gain update law $L(t)$ in Equation (9), we have the following result.

Lemma 3.2: *For the observer gain $L(t)$ in Equation (9), there exists $M > 0$ such that $L(t) < M$, $t \in [0, T], \forall T \in (0, \infty)$.*

Proof: The proof is simple, thus omitted here. □

The dynamics of the observation error $e = x - \hat{x}$ is given by

$$\begin{cases} \dot{e}_1 = e_2 - La_1\lceil e_1\rfloor^{\alpha_1}, \\ \dot{e}_2 = e_3 - L^2 a_2\lceil e_1\rfloor^{\alpha_2} + \tilde{f}_2, \\ \quad\vdots \\ \dot{e}_n = -L^n a_n\lceil e_1\rfloor^{\alpha_n} + \tilde{f}_n, \end{cases} \tag{11}$$

where $\tilde{f}_2 = f_2(y, x_2, u) - f_2(y, \hat{x}_2, u), \ldots, \tilde{f}_n = f_n(y, x_2, \ldots, x_n, u) - f_n(y, \hat{x}_2, \ldots, \hat{x}_n, u)$. Consider the change of coordinates

$$\varepsilon_i = \frac{e_i}{L^{i-1+\sigma}},$$

where $0 < \sigma < 1$ will be given later. Then Equation (11) can be expressed as

$$\begin{cases} \dot{\varepsilon}_1 = L\varepsilon_2 - L^{(\alpha_1-1)\sigma+1}a_1\lceil\varepsilon_1\rfloor^{\alpha_1} - \frac{\dot{L}}{L}\sigma\varepsilon_1, \\ \dot{\varepsilon}_2 = L\varepsilon_3 - L^{(\alpha_2-1)\sigma+1}a_2\lceil\varepsilon_1\rfloor^{\alpha_2} - \frac{\dot{L}}{L}(\sigma+1)\varepsilon_2 + \frac{\tilde{f}_2}{L^{1+\sigma}}, \\ \quad\vdots \\ \dot{\varepsilon}_n = -L^{(\alpha_n-1)\sigma+1}a_n\lceil\varepsilon_1\rfloor^{\alpha_n} - \frac{\dot{L}}{L}(n-1+\sigma)\varepsilon_n + \frac{\tilde{f}_n}{L^{n-1+\sigma}}. \end{cases} \tag{12}$$

Before we prove the global finite-time stability of the error system (12), let us investigate some properties of the following homogeneous nonlinear system,

$$\begin{cases} \dot{\varepsilon}_1 = L\varepsilon_2 - L^{(\alpha_1-1)\sigma+1}a_1\lceil\varepsilon_1\rfloor^{\alpha_1}, \\ \dot{\varepsilon}_2 = L\varepsilon_3 - L^{(\alpha_2-1)\sigma+1}a_2\lceil\varepsilon_1\rfloor^{\alpha_2}, \\ \quad\vdots \\ \dot{\varepsilon}_n = -L^{(\alpha_n-1)\sigma+1}a_n\lceil\varepsilon_1\rfloor^{\alpha_n}. \end{cases} \tag{13}$$

First, for system (13), suitably choose $a_i(1 \le i \le n)$ such that there exists $P^T = P > 0$ satisfying

$$A^T P + PA \le -I, \quad h_1 I \le D_1 P + PD_1 \le h_2 I, \tag{14}$$

where $h_1, h_2 > 0$ are real constants, $D_1 = \mathrm{diag}\{\sigma, 1+\sigma, \ldots, n-1+\sigma\}$, $A = \begin{bmatrix} -a_1 & 1\ldots 0 \\ \vdots & \ddots \\ -a_{n-1} & 0\ldots 1 \\ -a_n & 0\ldots 0 \end{bmatrix}$.

The following lemma gives a new homogeneous Lyapunov function. Under this Lyapunov function and condition (14), we will see that system (13) is finite-time stable.

Lemma 3.3: *For system (13), construct the following homogeneous function:*

$$V(\varepsilon) = \begin{cases} \int_0^\infty \frac{1}{v^{q+1}}(\chi \circ \bar{V})(v\varepsilon_1, v^{\alpha_1}\varepsilon_2, \ldots, v^{\alpha_{n-1}}\varepsilon_n)dv, & \varepsilon \in \mathcal{R}^n \setminus \{0\}, \\ 0, & \varepsilon = 0, \end{cases} \tag{15}$$

where $\bar{V}(\varepsilon) = \varepsilon^T P\varepsilon$, P, D_1 are given in Equation (14), $q > 0$ is an integer, $\chi(s) = \begin{cases} 0, & s \in (-\infty, 1] \\ 2(s-1)^2, & s \in (1, \frac{3}{2}) \\ 1-2(s-2)^2, & s \in [\frac{3}{2}, 2) \\ 1, & s \in [2, \infty) \end{cases}$, $\chi(s) \in C'(\mathcal{R}, \mathcal{R})$. Then

(i) $V(\varepsilon)$ is a positive definite function homogeneous of degree q with respect to the weights $\{\alpha_{i-1}\}_{1\le i\le n}$. $V(\varepsilon)$ is called a q h-Lyapunov function of $\bar{V}(\varepsilon)$ w.r.t. $\chi, L, (\alpha_0, \alpha_1, \ldots, \alpha_{n-1})$.

(ii) There exist $c_1, c_2 > 0$ such that

$$c_1 V(\varepsilon) \le \frac{\partial V(\varepsilon)}{\partial \varepsilon}^T D_1\varepsilon \le c_2 V(\varepsilon). \tag{16}$$

(iii) If $q > \max\{\alpha_i\}_{0\le i\le n-1}+1$, $\frac{dV(\varepsilon)}{dt}|_{(13)}$ is C^1 on $\mathcal{R}^n$, then there exists a $c_3 > 0$ such that

$$\left.\frac{dV(\varepsilon)}{dt}\right|_{(13)} \le -c_3 L^{1-\sigma} V(\varepsilon)^{\gamma}, \tag{17}$$

where $\gamma = \frac{q+\alpha-1}{q}$.

Proof: We give a direct and detailed proof of the lemma in Appendix. □

Based on Lemmas 3.1–3.3, our main result with explicit proof is given in the following.

Theorem 3.4: *If $\frac{n-i}{n-j+1} \le \beta_{ij} < \frac{i}{j-1} (2 \le j \le i \le n)$, then for any $1-\frac{1}{n} < \alpha < 1$, there exist $\varphi_i > 0\,(1 \le i \le 5)$ and $0 < \sigma < 1$ such that the system (4) with dynamic high gain (9) is a global finite-time observer for nonlinear system (2) with condition (3).*

Proof: Under the condition that $1-\frac{1}{n} < \alpha < 1$, $a_i(1 \le i \le n)$ satisfying Equation (14), $0 < \sigma < 1$ (which will be given later), we will use the homogeneous Lyapunov function $V(\varepsilon)$ as defined in Lemma 3.3 to derive the global finite-time stability.

For all $\varepsilon \in \mathcal{R}^n$, calculating the derivative of the Lyapunov function $V(\varepsilon)$ defined in Equation (15) along the solution of system (12), from Lemma 3.3, we have

$$\begin{aligned}\left.\frac{dV(\varepsilon)}{dt}\right|_{(12)} &\le -c_3L^{1-\sigma}V(\varepsilon)^{\gamma} + c_2\varphi_1(L^{1-\sigma}-\varphi_2)V(\varepsilon)\\ &-c_1\varphi_3\Psi(u,y,\hat{x})V(\varepsilon) - c_1\varphi_4L^{1+(m-2)\sigma}|\varepsilon_1|^mV(\varepsilon)\\ &-c_1\varphi_5L^{m\sigma}\Psi(u,y,\hat{x})|\varepsilon_1|^mV(\varepsilon) + \frac{\partial V(\varepsilon)}{\partial \varepsilon}^T\tilde{F},\end{aligned} \tag{18}$$

where $\tilde{F} = (0, \frac{\tilde{f}_2}{L^{1+\sigma}}, \ldots, \frac{\tilde{f}_n}{L^{n-1+\sigma}})^T$.

For $\frac{\partial V(\varepsilon)}{\partial\varepsilon}^T\tilde{F}$, we have

$$\begin{aligned}\left|\frac{\partial V(\varepsilon)}{\partial \varepsilon}^T\tilde{F}\right| &= \left|\sum_{i=2}^{n}\frac{\partial V(\varepsilon)}{\partial\varepsilon_i}\frac{\tilde{f}_i}{L^{i-1+\sigma}}\right|\\ &\le \sum_{i=2}^{n}\left|\frac{\partial V(\varepsilon)}{\partial\varepsilon_i}\right|\frac{1}{L^{i-1+\sigma}}\Bigg(\Psi(u,y,\hat{x})\\ &\quad\times\sum_{j=2}^{i}|x_j-\hat{x}_j| + l\sum_{j=2}^{i}|x_j-\hat{x}_j|^{\beta_{ij}}\Bigg)\\ &\le \sum_{i=2}^{n}\sum_{j=2}^{i}\Psi(u,y,\hat{x})\left|\frac{\partial V(\varepsilon)}{\partial\varepsilon_i}\right||\varepsilon_j|\\ &\quad + l\sum_{i=2}^{n}\sum_{j=2}^{i}\left|\frac{\partial V(\varepsilon)}{\partial\varepsilon_i}\right||\varepsilon_j|^{\beta_{ij}}L^{(j-1+\sigma)\beta_{ij}-(i-1+\sigma)}.\end{aligned}$$

If $\beta_{ij} < \frac{i}{j-1}$, there exists a $\sigma_1 > 0$ such that $\beta_{ij} < \frac{i-\sigma_1}{j-1+\sigma_1}$, $v_j < \frac{1-\sigma_1}{j-1+\sigma_1}$ and $(2 \le j \le i \le n)$. Choose $0 < \sigma < \sigma_1$, then we get

$$L^{(j-1+\sigma)\beta_{ij}-(i-1+\sigma)} < L^{1-2\sigma}.$$

Then, by Lemma 4.2 in Bhat & Bernstein (2005), we have

$$\begin{aligned}\left|\frac{\partial V(\varepsilon)}{\partial \varepsilon}^T\tilde{F}\right| &\le \sum_{i=2}^{n}\sum_{j=2}^{i}\Psi(u,y,\hat{x})\left|\frac{\partial V(\varepsilon)}{\partial\varepsilon_i}\right||\varepsilon_j|\\ &+lL^{1-2\sigma}\sum_{i=2}^{n}\sum_{j=2}^{i}\left|\frac{\partial V(\varepsilon)}{\partial\varepsilon_i}\right||\varepsilon_j|^{\beta_{ij}} \le k_1\Psi(u,y,\hat{x})\\ &\times\sum_{i=2}^{n}\sum_{j=2}^{i}V(\varepsilon)^{\frac{q-\alpha_{i-1}+\alpha_{j-1}}{q}}\\ &+k_2lL^{1-2\sigma}\sum_{i=2}^{n}\sum_{j=2}^{i}V(\varepsilon)^{\frac{q-\alpha_{i-1}+\alpha_{j-1}\beta_{ij}}{q}},\end{aligned} \tag{19}$$

where $k_1 = \max_{\{z:V(z)=1\}}|\frac{\partial V(z)}{\partial z_i}||z_j|$ and $k_2 = \max_{\{z:V(z)=1\}}|\frac{\partial V(z)}{\partial z_i}||z_j|^{\beta_{ij}}$.

Then, for $\delta > 0$, define $\overline{\mathcal{B}}_\delta \triangleq \{\varepsilon : V(\varepsilon) \le \delta\}$, $\mathcal{P}_\delta = \{\varepsilon : |\varepsilon_1| < \delta\}$. Let $\Omega = \{\varepsilon : (0, \varepsilon_2, \ldots, \varepsilon_n) \in \mathcal{R}^n\}$.

The proof is divided into two parts: $\varepsilon \in \mathcal{R}^n \setminus \Omega$ and $\varepsilon \in \Omega$, where part I consists of two small parts $\varepsilon \in \overline{\mathcal{B}}_1 \setminus \Omega$ and $\varepsilon \in (\mathcal{R}^n \setminus \overline{\mathcal{B}}_1) \setminus \Omega$, respectively. When $\varepsilon \in \overline{\mathcal{B}}_1 \setminus \Omega$, we can get $\frac{dV(\varepsilon)}{dt}|_{(12)} \le -\frac{1}{3}c_3L^{1-\sigma}V(\varepsilon)^{\gamma}$. Then we have $\frac{dV(\varepsilon)}{dt}|_{(12)} \le -c_3L^{1-\sigma}V(\varepsilon)^{\gamma}$ for $\varepsilon \in (\mathcal{R}^n \setminus \overline{\mathcal{B}}_1) \setminus \Omega$. Thus, we obtain $\frac{dV(\varepsilon)}{dt}|_{(12)} \le -\frac{1}{3}c_3L^{1-\sigma}V(\varepsilon)^{\gamma}$ for all $\varepsilon \in \mathcal{R}^n \setminus \Omega$. Then when $\varepsilon \in \Omega$, it can be verified that the non-trivial solution of system (12) can only pass through Ω finite times. Thus, from the combination of these two parts, we obtain the global finite-time stability of error system (12).

Part I:

1. When $\varepsilon \in \overline{\mathcal{B}}_1 \setminus \Omega$, from Equations (18) and (19), we have

$$\begin{aligned}\left.\frac{dV(\varepsilon)}{dt}\right|_{(12)} &\le -c_3L^{1-\sigma}V(\varepsilon)^{\gamma} + c_2\varphi_1(L^{1-\sigma}-\varphi_2)V(\varepsilon)\\ &\quad - c_1\varphi_3\Psi(u,y,\hat{x})V(\varepsilon)\\ &\quad - c_1\varphi_4L^{1+(m-2)\sigma}|\varepsilon_1|^mV(\varepsilon)\\ &\quad - c_1\varphi_5L^{m\sigma}\Psi(u,y,\hat{x})|\varepsilon_1|^mV(\varepsilon)\\ &\quad + k_1n^2\Psi(u,y,\hat{x})V(\varepsilon)\\ &\quad + k_2n^2lL^{1-2\sigma}V(\varepsilon)^{\frac{q+\underline{\beta}}{q}},\end{aligned} \tag{20}$$

where $\underline{\beta} = \min_{2\le j\le i\le n}\{\alpha_{j-1}\beta_{ij}-\alpha_{i-1}\}$. From Lemma 3.1, we can derive $\gamma < \frac{q+\underline{\beta}}{q}$, then there exist $d_{11}, d_{21}, d_{31} > 0$

such that when $\varphi_1 < d_{11}, \varphi_2 > d_{21}, \varphi_3 > d_{31}$ we have

$$\begin{aligned}\left.\frac{dV(\varepsilon)}{dt}\right|_{(12)} \le & -\frac{1}{3}c_3L^{1-\sigma}V(\varepsilon)^{\gamma} - c_2\varphi_1\varphi_2V(\varepsilon)\\ & - c_1\varphi_4L^{1+(m-2)\sigma}|\varepsilon_1|^mV(\varepsilon)\\ & - c_1\varphi_5L^{m\sigma}\Psi(u,y,\hat{x})|\varepsilon_1|^mV(\varepsilon)\\ \le & -\frac{1}{3}c_3L^{1-\sigma}V(\varepsilon)^{\gamma}, \end{aligned} \tag{21}$$

where $d_{11} = \frac{c_3}{3c_2}$, $d_{21} = (\frac{3k_2n^2l}{c_3})^{\frac{1}{\sigma}}$ and $d_{31} = \frac{k_1n^2}{c_1}$.

2. When $\varepsilon \in (\mathcal{R}^n \setminus \overline{\mathcal{B}}_1) \setminus \Omega$, from Equations (18) and (19), we can derive

$$\begin{aligned}\left.\frac{dV(\varepsilon)}{dt}\right|_{(12)} \le & -c_3L^{1-\sigma}V(\varepsilon)^{\gamma} + c_2\varphi_1(L^{1-\sigma} - \varphi_2)V(\varepsilon)\\ & - c_1\varphi_3\Psi(u,y,\hat{x})V(\varepsilon)\\ & - c_1\varphi_4L^{1+(m-2)\sigma}|\varepsilon_1|^mV(\varepsilon)\\ & - c_1\varphi_5L^{m\sigma}\Psi(u,y,\hat{x})|\varepsilon_1|^mV(\varepsilon)\\ & + k_1n^2\Psi(u,y,\hat{x})V(\varepsilon)^{\frac{q-\alpha_{n-1}+1}{q}}\\ & + k_2n^2lL^{1-2\sigma}V(\varepsilon)^{\frac{q+\bar{\beta}}{q}}, \end{aligned} \tag{22}$$

where $\bar{\beta} = \max_{2\le j\le i\le n}\{\alpha_{j-1}\beta_{ij} - \alpha_{i-1}\}$.

Let $\mathcal{G} = \{z : V(z) = 1\}$. For any $\varepsilon \in (\mathcal{R}^n \setminus \overline{\mathcal{B}}_1) \setminus \Omega$, there exist $\delta > 0$ and λ such that $\varepsilon = (\lambda\varepsilon_1^{\delta}, \lambda^{\alpha_1}\varepsilon_2^{\delta}, \ldots, \lambda^{\alpha_{n-1}}\varepsilon_n^{\delta})^T = \text{diag}\{\lambda, \lambda^{\alpha_1}, \ldots, \lambda^{\alpha_{n-1}}\}\varepsilon^{\delta}$, $\varepsilon^{\delta} = (\varepsilon_1^{\delta}, \ldots, \varepsilon_n^{\delta})^T \in \mathcal{G} \setminus \mathcal{P}_{\delta}$. Then we have

$$|\varepsilon_1|^mV(\varepsilon) = \lambda^{m+q}|\varepsilon_1^{\delta}|^mV(\varepsilon^{\delta}) = \lambda^{m+q}|\varepsilon_1^{\delta}|^m = V(\varepsilon)^{\frac{m+q}{q}}|\varepsilon_1^{\delta}|^m.$$

Because $|\varepsilon_1^{\delta}|^m \ge \min_{\varepsilon\in\mathcal{G}\setminus\mathcal{P}_{\delta}}|\varepsilon_1|^m = \delta^m$, then we can get the following inequality:

$$|\varepsilon_1|^mV(\varepsilon) \ge \delta^mV(\varepsilon)^{\frac{m+q}{q}}, \quad \varepsilon \in (\mathcal{R}^n \setminus \overline{\mathcal{B}}_1) \setminus \Omega. \tag{23}$$

Thus, from Equations (22) and (23), we obtain

$$\begin{aligned}\left.\frac{dV(\varepsilon)}{dt}\right|_{(12)} \le & -c_3L^{1-\sigma}V(\varepsilon)^{\gamma} + c_2\varphi_1(L^{1-\sigma} - \varphi_2)V(\varepsilon)\\ & - c_1\varphi_3\Psi(u,y,\hat{x})V(\varepsilon)\\ & - c_1\varphi_4L^{1+(m-2)\sigma}\delta^mV(\varepsilon)^{\frac{m+q}{q}}\\ & - c_1\varphi_5L^{m\sigma}\Psi(u,y,\hat{x})\delta^mV(\varepsilon)^{\frac{m+q}{q}}\\ & + k_1n^2\Psi(u,y,\hat{x})V(\varepsilon)^{\frac{q-\alpha_{n-1}+1}{q}}\\ & + k_2n^2lL^{1-2\sigma}V(\varepsilon)^{\frac{q+\bar{\beta}}{q}}. \end{aligned} \tag{24}$$

Because $m \ge \max\{\alpha_{j-1}\beta_{ij} - \alpha_{i-1}, 1\}(2 \le j \le i \le n)$, we can get $L^{1+(m-2)\sigma} \ge L^{1-\sigma}$. Then, there exist $d_{41}, d_{51} > 0$ such that $\varphi_4 > \frac{2c_2}{c_1\delta^m}\varphi_1$ holds when $\varphi_4 > d_{41}, \varphi_5 > d_{51}$. Thus, for $\varepsilon \in (\mathcal{R}^n \setminus \overline{\mathcal{B}}_1) \setminus \Omega$, we have

$$\begin{aligned}\left.\frac{dV(\varepsilon)}{dt}\right|_{(12)} \le & - c_3L^{1-\sigma}V(\varepsilon)^{\gamma} - c_2\varphi_1\varphi_2V(\varepsilon)\\ & - c_1\varphi_3\Psi(u,y,\hat{x})V(\varepsilon) \le -c_3L^{1-\sigma}V(\varepsilon)^{\gamma}, \end{aligned} \tag{25}$$

where $d_{41} = \max\{\frac{2k_2n^2l}{c_1\delta^m}, \frac{2c_3}{3c_1\delta^m}\}$ and $d_{51} = \frac{k_1n^2}{c_1\delta^m}$.

Finally, from Equations (21) and (25), by combining parts 1 and 2, we get that the following inequality,

$$\left.\frac{dV(\varepsilon)}{dt}\right|_{(12)} \le -\frac{1}{3}c_3L^{1-\sigma}V(\varepsilon)^{\gamma}, \tag{26}$$

holds for $\varepsilon \in \mathcal{R}^n \setminus \Omega$.

Part II:

When $\varepsilon \in \Omega$, let $\varepsilon(t, t_0, \varepsilon_0)$ denote a non-trivial solution of system (12). In the following, we will verify that there does not exist such $t_2 > t_1 \ge t_0$ that $\varepsilon(t, t_0, \varepsilon_0)$ stays on Ω in the interval (t_1, t_2). We will prove it using a contradiction argument. Suppose there exists such interval that $\varepsilon(t, t_0, \varepsilon_0)$ can stay on Ω. From the first equation of system (12), we can derive $\varepsilon_2 = 0$ on (t_1, t_2). Then, from the second equation, we can obtain $\varepsilon_3 = 0$ on (t_1, t_2). Then following the same steps, we have $\varepsilon_i = 0 (2 \le i \le n)$ on (t_1, t_2), which is a contradiction. Thus, $\varepsilon(t, t_0, \varepsilon_0)$ can only pass through Ω. Let t_k denote the time when $\varepsilon(t, t_0, \varepsilon_0)$ passes through Ω. From Equation (26), we have

$$\left.\frac{dV(\varepsilon)}{dt}\right|_{(12)} V(\varepsilon)^{-\gamma} \le -\frac{1}{3}c_3L^{1-\sigma} \le -\frac{1}{3}c_3\varphi_2^{1-\sigma}. \tag{27}$$

Integrate both sides of Equation (27), we have

$$\sum_{k=1}^{n}\int_{t_k}^{t_{k+1}} V(\varepsilon)^{-\gamma}dV(\varepsilon) \le -\frac{1}{3}c_3\varphi_2^{1-\sigma}\int_{t_k}^{t_{k+1}} dt,$$

i.e.

$$\begin{aligned}\frac{1}{1-\gamma}V(\varepsilon(t_{n+1}))^{1-\gamma} \le & \frac{1}{1-\gamma}V(\varepsilon(t_1))^{1-\gamma}\\ & - \frac{1}{3}c_3\varphi_2^{1-\sigma}(t_{n+1} - t_1). \end{aligned} \tag{28}$$

Here, we still use the contradiction argument to prove that $\{t_k\}$ is a finite sequence. If $\{t_k\}$ is not a finite sequence, then we have $t_n \longrightarrow +\infty$ as $n \longrightarrow +\infty$. And we can get that the left-hand side of Equation (28) approaches to zero while the right-hand side approaches to $-\infty$, which is a contradiction. Thus, $\{t_k\}$ is a finite sequence. Therefore, there exists a T_1 such that Equation (26) holds for all $\varepsilon \in \mathcal{R}^n (t > T_1)$.

Thus, from Theorem 4.2 in Bhat & Bernstein (2000) and by combining part I and part II, we get the global finite-time convergence of the observation error $\varepsilon_i (i = 1, \ldots, n)$. The

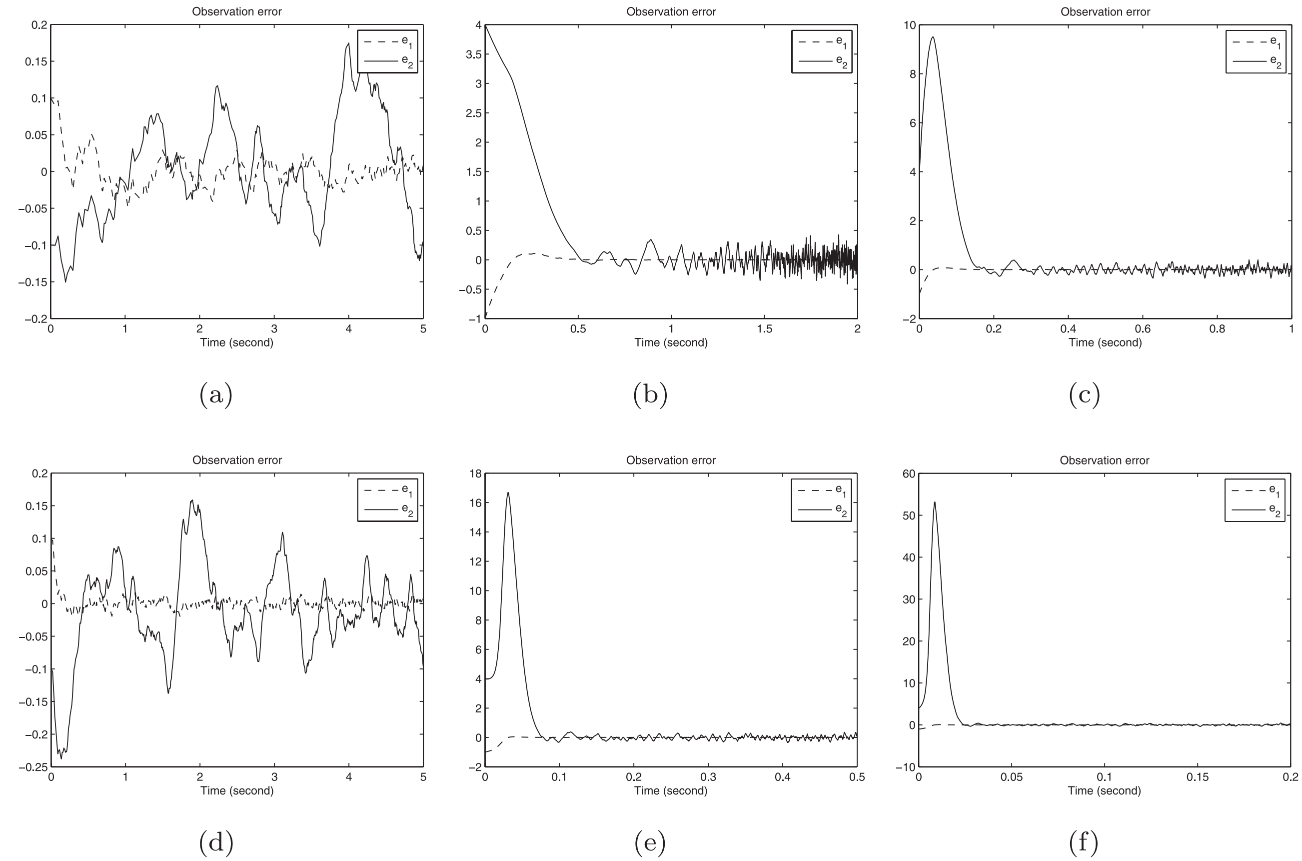

Figure 1. Observation errors of system (29) (shown in (a), (b) and (c)) and system (30) (shown in (d), (e) and (f)) under condition I, condition II and condition III.

settling time $T(\varepsilon^0)$ is $T(\varepsilon^0) \le \frac{3}{c_3 \varphi_2{}^{1-\sigma}(1-\gamma)} V(\varepsilon^0)^{1-\gamma} + T_1$, where t_0 is the initial time and $\varepsilon^0 = (e_1^0, \frac{e_2^0}{\varphi_2{}^{\sigma}}, \ldots, \frac{e_n^0}{\varphi_2{}^{n-1+\sigma}})^T$ is the initial state. Then from Lemma 3.2, we get $\frac{e_i}{M^{i-1+\sigma}} < \frac{e_i}{L^{i-1+\sigma}} = \varepsilon_i = 0$ when $t > T(\varepsilon^0) + T_1$ $(1 \le i \le n)$, i.e. the system (4) with update gain (9) is a global finite-time observer for system (2) with condition (3).

This completes the proof. □

4. Example

Example 1: Consider the same nonlinear system as in Li et al. (2011),

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -1.5x_2 - x_2^{1.4} - x_1, \\ y \ = x_1, \end{cases}$$

where the following nonlinear condition holds: $|(-1.5x_2 - x_2^{1.4} - x_1) - (-1.5\hat{x}_2 - \hat{x}_2^{1.4} - x_1)| \le (1.5 + 1.4|\hat{x}_2|^{0.4})$ $|x_2 - \hat{x}_2| + |x_2 - \hat{x}_2|^{1.4}$. Following the results in Li et al. (2011), a global finite-time observer is designed as

$$\begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 + 4L\lceil y - \hat{x}_1 \rfloor^{\alpha} + 4L^{1-(\beta-1)(1-\eta)\sigma} \lceil y - \hat{x}_1 \rfloor^{\beta}, \\ \dot{\hat{x}}_2 = 3L^2 \lceil y - \hat{x}_1 \rfloor^{2\alpha-1} + 3L^{2-2(\beta-1)(1-\eta)\sigma} \lceil y - \hat{x}_1 \rfloor^{2\beta-1} \\ \qquad - 1.5\hat{x}_2 - \hat{x}_2^{1.4} - y, \\ \dot{L} \ = -L[\varphi_1(L^{1-\sigma} - \varphi_2) - \varphi_3(1.5 + 1.4|\hat{x}_2|^{0.4})], \end{cases} \tag{29}$$

while the global finite-time observer designed in this paper is as follows:

$$\begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 + 4L\lceil y - \hat{x}_1 \rfloor^{\alpha}, \\ \dot{\hat{x}}_2 = 3L^2 \lceil y - \hat{x}_1 \rfloor^{2\alpha-1} - 1.5\hat{x}_2 - \hat{x}_2^{1.4} - y, \\ \dot{L} \ = -L[\varphi_1(L^{1-\sigma} - \varphi_2) - \varphi_3(1.5 + 1.4|\hat{x}_2|^{0.4}) \\ \qquad - \varphi_4 L^{1-2\sigma}|x_1 - \hat{x}_1|^2 \\ \qquad - \varphi_5(1.5 + 1.4|\hat{x}_2|^{0.4})|x_1 - \hat{x}_1|^2]. \end{cases} \tag{30}$$

To illustrate the performance of systems (29) and (30) more clearly, several figures are given under the following three different initial conditions and parameters.

Condition I

Parameters: $\alpha = 0.95$, $\beta = 10^5$, $\sigma = 0.01$, $\eta = 0.01$, $\varphi_1 = 0.1$, $\varphi_2 = 1.2$, $\varphi_3 = 0.2$, $\varphi_4 = 500$, $\varphi_5 = 400$.

The initial values: $x_1(0) = 0.2, x_2(0) = 0.3, \hat{x}_1(0) = 0.1, \hat{x}_2(0) = 0.4, L(0) = 1.5$.

Condition II

Parameters: $\alpha = 0.8, \beta = 10^4, \sigma = 0.001, \eta = 0.1, \varphi_1 = 0.01, \varphi_2 = 1, \varphi_3 = 1, \varphi_4 = 20, \varphi_5 = 30$. The initial values: $x_1(0) = 2, x_2(0) = 5, \hat{x}_1(0) = 3, \hat{x}_2(0) = 1, L(0) = 1.5$.

Condition III

Parameters: $\alpha = 0.8, \beta = 10^4, \sigma = 0.001, \eta = 0.1, \varphi_1 = 0.01, \varphi_2 = 1, \varphi_3 = 1, \varphi_4 = 20, \varphi_5 = 30$. The initial values: $x_1(0) = 2, x_2(0) = 5, \hat{x}_1(0) = 3, \hat{x}_2(0) = 1, L(0) = 10$.

From the simulations (with uniform random number noise added to the observers) as shown in Figure 1, we can see that the change of different parameters as well as the initial values of the states and the high gain L do have some effect on the convergence of the observation error system. However, it is very clear that no matter under which case, the new global finite-time observer (30) proposed by this paper can render the error systems converge more quickly while it is a bit more noise-sensitive than the one (29) designed previously.

5. Conclusion

A global finite-time observer was designed for a class of nonlinear systems with rational powers imposed on the incremental nonlinear terms. Compared with the previous global finite-time results, the observer was given with a new gain update law where the term $|y - \hat{x}_1|^m$ is introduced. Through an example, we showed that the observer proposed in this paper can reduce the convergence time of the observation error.

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Appendix: Proofs of Lemma 3.1 and Lemma 3.3

A.1. *Proof of Lemma 3.1*

Proof: To prove $\alpha - 1 - \alpha_{j-1}\beta_{ij} + \alpha_{i-1} < 0$ is equivalent to prove $\beta_{ij} > \frac{\alpha_i}{\alpha_{j-1}}$ for $2 \le j \le i \le n$. For $1 - \frac{1}{n} < \alpha < 1$, we have $\frac{\alpha_i}{\alpha_{j-1}} = \frac{i\alpha-(i-1)}{(j-1)\alpha-(j-2)}$, which is strictly increasing with respect to α. Because $\alpha < 1$, $\beta_{ij} > \frac{n-i}{n-j+1}$, there exists a $\epsilon > 0$ such that $\alpha < \frac{n-1+\epsilon}{n}$ and $\beta_{ij} > \frac{n-i+i\epsilon}{n-j+1+j\epsilon-\epsilon}$. Then we get $\frac{\alpha_i}{\alpha_{j-1}} < \frac{i\frac{n-1+\epsilon}{n}-(i-1)}{(j-1)\frac{n-1+\epsilon}{n}-(j-2)} = \frac{n-i+i\epsilon}{n-j+1+j\epsilon-\epsilon} < \beta_{ij}$. □

Thus, the proof is completed.

A.2. *Proof of Lemma 3.3*

Proof: First, for $\pi > 0, 0 < \sigma < 1$, define $\mathcal{F}_\pi \triangleq \{\varepsilon : |\varepsilon_1| = \pi\}$, $\overline{\mathcal{B}}_{1,\pi} \triangleq \{\varepsilon : \varepsilon^T\varepsilon \le \pi\}$, $\mathcal{B}_{1,\pi} \triangleq \{\varepsilon : \varepsilon^T\varepsilon < \pi\}$, $\overline{\mathcal{B}}_{2,\pi} \triangleq \{(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)^T : \sum_{i=2}^n \varepsilon_i^2 \le \pi^2\}$, $\mathcal{B}_{2,\pi} \triangleq \{(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)^T : \sum_{i=2}^n \varepsilon_i^2 < \pi^2\}$, $\overline{\mathcal{B}}_{3,\pi,i} \triangleq \{(\varepsilon_1, L^{-i\sigma\alpha_1}\varepsilon_2, \ldots, L^{-i\sigma\alpha_{n-1}}\varepsilon_n)^T : \sum_{i=2}^n \varepsilon_i^2 \le \pi^2\}$, $\mathcal{B}_{3,\pi,i} \triangleq \{(\varepsilon_1, L^{-i\sigma\alpha_1}\varepsilon_2, \ldots, L^{-i\sigma\alpha_{n-1}}\varepsilon_n)^T : \sum_{i=2}^n \varepsilon_i^2 < \pi^2\}$, $\overline{\mathcal{P}}_\pi \triangleq \{\varepsilon : |\varepsilon_1| \le \pi\}$, $\mathcal{P}_\pi \triangleq \{\varepsilon : |\varepsilon_1| < \pi\}$ and $\mathcal{S}_\pi \triangleq \{\varepsilon : \varepsilon^T\varepsilon = \pi\}$.

The proofs of (i) and (ii) are quite easy. For (i), by change of integration, it can be verified that $V(\varepsilon)$ is homogeneous of degree q with respect to the weights $\{\alpha_i\}_{0\le i\le n-1}$. From condition (14), it is also not difficult to derive the inequality (16) in (ii).

The proof of (iii) is a bit complicated. We will see that the proof is divided into two parts. The first part is to construct a compact set $\overline{\mathcal{A}}$ (where $\overline{\mathcal{A}}$ will be given later) encircling the origin where some inequalities are obtained. The compact set is derived by combination of four sets. In the second part, for any $\varepsilon \in \mathcal{R}^n \setminus \{0\}$, the inequality (17) in (iii) is derived through establishing the relationship between $\frac{dV(\varepsilon)}{dt}|_{(13)}$ and $\frac{dV(\varepsilon_0)}{dt}|_{(13)}$, $\varepsilon_0 \in \overline{\mathcal{A}}$ by use of the homogeneity theory.

Part I: This part is divided into six parts. In the first four parts, we will show that $\frac{dV(\varepsilon)}{dt}|_{(13)}$ satisfies some inequalities on the following sets $\mathcal{S}_1 \cap \overline{\mathcal{P}}_{L^{-\sigma}}$, $(\overline{\mathcal{P}}_{(1+\pi_1)L^{-\sigma}} \setminus \mathcal{P}_{(1-\pi_1)L^{-\sigma}}) \cap \overline{\mathcal{B}}_{3,\pi_1,2}$, $\mathcal{F}_{L^{-h\sigma}} \cap (\overline{\mathcal{B}}_{1,1} \setminus \mathcal{B}_{3,\pi_1,2})$ and $(\overline{\mathcal{P}}_{L^{-\sigma}} \setminus \mathcal{P}_{L^{-h\sigma}}) \cap (\overline{\mathcal{B}}_{3,\pi_1,2} \setminus \mathcal{B}_{3,\pi_1,2})$ separately, where $\pi_1 > 0$, $h > 2$ will be given later. Then in the fifth part, $V(\varepsilon)$ admits some inequalities for ε belonging to each of these four sets. Finally, in the sixth part, the compact set $\overline{\mathcal{A}}$ is derived from the combination of these four sets.

(1) Let l_1 be the largest $l > 0$ such that $\max_{\{v\le l\}} \max_{\{\varepsilon\in\overline{\mathcal{B}}_{1,2}\setminus\mathcal{B}_{1,\frac{1}{2}}\}} \bar{V}(v\varepsilon_1, \ldots, v^{\alpha_{n-1}}\varepsilon_n) \le 1$.
Let l_2 be the smallest $l > 0$ such that $\min_{\{v\ge l\}} \min_{\{\varepsilon\in\overline{\mathcal{B}}_{1,2}\setminus\mathcal{B}_{1,\frac{1}{2}}\}} \bar{V}(v\varepsilon_1, \ldots, v^{\alpha_{n-1}}\varepsilon_n) \ge 2$. Then we have $V(\varepsilon) = \int_{l_1}^{l_2} \frac{1}{v^{q+1}}(\chi \circ \bar{V}(v\varepsilon_1, \ldots, v^{\alpha_{n-1}}\varepsilon_n))dv + \frac{1}{ql_2^q}$, $\varepsilon \in \overline{\mathcal{B}}_{1,2} \setminus \mathcal{B}_{1,\frac{1}{2}}$. And

$$\frac{dV(\varepsilon)}{dt}\bigg|_{(13)} = 2L\int_{l_1}^{l_2} \frac{\chi'(\bar{V}(v\varepsilon_1, \ldots, v^{\alpha_{n-1}}\varepsilon_n))}{v^{q+\alpha}} K(v, \varepsilon_1, \ldots, \varepsilon_n) \times dv, \varepsilon \in \overline{\mathcal{B}}_{1,2} \setminus \mathcal{B}_{1,\frac{1}{2}}, \tag{A.1}$$

where

$$K(v, \varepsilon_1, \ldots, \varepsilon_n) = \begin{bmatrix} 0 \\ v^{\alpha_1}\varepsilon_2 \\ \vdots \\ v^{\alpha_{n-1}}\varepsilon_n \end{bmatrix}^T P \begin{bmatrix} v^{\alpha_1}\varepsilon_2 \\ \vdots \\ v^{\alpha_{n-1}}\varepsilon_n \\ 0 \end{bmatrix} + \begin{bmatrix} v\varepsilon_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}^T \times P \begin{bmatrix} v^{\alpha_1}\varepsilon_2 - a_1 L^{(\alpha_1-1)\sigma}\lceil v\varepsilon_1\rfloor^{\alpha_1} \\ \vdots \\ -a_n L^{(\alpha_n-1)\sigma}\lceil v\varepsilon_1\rfloor^{\alpha_n} \end{bmatrix} + \begin{bmatrix} 0 \\ v^{\alpha_1}\varepsilon_2 \\ \vdots \\ v^{\alpha_{n-1}}\varepsilon_n \end{bmatrix}^T P \begin{bmatrix} -a_1 L^{(\alpha_1-1)\sigma}\lceil v\varepsilon_1\rfloor^{\alpha_1} \\ \vdots \\ -a_n L^{(\alpha_n-1)\sigma}\lceil v\varepsilon_1\rfloor^{\alpha_n} \end{bmatrix}. \tag{A.2}$$

When $\varepsilon \in \mathcal{S}_1 \cap \overline{\mathcal{P}}_{L^{-\sigma}}$, from Equations (A.1) and (A.2), there exists $L_1 > 2$ such that when $L > L_1$, we have $\frac{dV(\varepsilon)}{dt}\Big|_{(13)} < -\frac{L}{2}\int_{l_1}^{l_2} \frac{1}{v^{q+\alpha}} \sum_{i=2}^n v^{2\alpha_{i-1}}\varepsilon_i^2 \chi'(\bar{V}(v\varepsilon_1, \ldots, v^{\alpha_{n-1}}\varepsilon_n))dv$, $\varepsilon \in \mathcal{S}_1 \cap \overline{\mathcal{P}}_{L^{-\sigma}}$, where $a^* = \max_{\{1\le i\le n\}} a_i$, $\bar{p} = \max_{\{1\le i,j\le n\}} |P_{ij}|$.

And clearly, we have $(\mathcal{S}_1 \cap \overline{\mathcal{P}}_0) \subset (\mathcal{S}_1 \cap \overline{\mathcal{P}}_{L^{-\sigma}}) \subset (\mathcal{S}_1 \cap \overline{\mathcal{P}}_{2^{-\sigma}})$. Let l_3 be the largest $l > 0$ such that $\max_{\{v\le l\}} \max_{\{\varepsilon\in\mathcal{S}_1\cap\overline{\mathcal{P}}_0\}} \bar{V}(v\varepsilon, \ldots, v^{\alpha_{n-1}}\varepsilon_n) \le 1$. Let l_4 be the smallest $l > 0$ such that $\min_{\{v\ge l\}} \min_{\{\varepsilon\in\mathcal{S}_1\cap\overline{\mathcal{P}}_0\}} \bar{V}(v\varepsilon, \ldots, v^{\alpha_{n-1}}\varepsilon_n) \ge 2$. It is not difficult to get $l_3 \ge l_1$, $l_4 \le l_2$. Then we have

$$\frac{dV(\varepsilon)}{dt}\bigg|_{(13)} < -Ld_1, \quad \varepsilon \in \mathcal{S}_1 \cap \overline{\mathcal{P}}_{L^{-\sigma}}, \tag{A.3}$$

where $d_1 = \frac{1}{2}\min_{\{\varepsilon\in\mathcal{S}_1\cap\overline{\mathcal{P}}_{2^{-\sigma}}\}} \int_{l_3}^{l_4} \frac{1}{v^{q+\alpha}} \sum_{i=2}^n v^{2\alpha_{i-1}}\varepsilon_i^2 \chi'(\bar{V}(v\varepsilon_1, \ldots, v^{\alpha_{n-1}}\varepsilon_n))dv$.

(2) Because $a_1P_{11} > 0$, from Equations (A.1) and (A.2), there exist $\pi_1 \in (0, 1)$ such that for $\varepsilon \in (\overline{\mathcal{P}}_{1+\pi_1} \setminus \mathcal{P}_{1-\pi_1}) \cap \overline{\mathcal{B}}_{3,\pi_1,1}$, we have $\frac{dV(\varepsilon)}{dt}|_{(13)} < -L^{1-\sigma}\int_{l_1}^{l_2} \frac{a_1P_{11}v^{1+\alpha_1}}{v^{q+\alpha}} \chi'(\bar{V}(\pm v, 0, \ldots, 0))dv$.

Because $\frac{dV(\varepsilon)}{dt}|_{(13)}$ is homogeneous of degree $q + \alpha - 1$ with respect to the weights $\{\alpha_i\}_{0\le i\le n-1}$, we get

$$\frac{dV(\varepsilon)}{dt}\bigg|_{(13)} < -d_2L^{1-(q+\alpha)\sigma}, \quad \varepsilon \in (\overline{\mathcal{P}}_{(1+\pi_1)L^{-\sigma}} \setminus \mathcal{P}_{(1-\pi_1)L^{-\sigma}}) \times \cap \overline{\mathcal{B}}_{3,\pi_1,2}, \tag{A.4}$$

where $d_2 = \int_{l_1}^{l_2} \frac{a_1P_{11}v^{1+\alpha_1}}{v^{q+\alpha}} \chi'(\bar{V}(\pm v, 0, \ldots, 0))dv$.

(3) Let l_5 be the largest $l > 0$ such that $\max_{\{v\le l\}} \max_{\{\varepsilon\in\overline{\mathcal{P}}_{(1+\pi_1)L^{-\sigma}}\cap(\overline{\mathcal{B}}_{1,1}\setminus\mathcal{B}_{3,\pi_1,2})\}} \bar{V}(v\varepsilon_1, \ldots, v^{\alpha_{n-1}}\varepsilon_n) \le 1$. And let l_6 be the smallest $l > 0$ such that $\min_{\{v\ge l\}} \min_{\{\varepsilon\in\overline{\mathcal{P}}_{(1+\pi_1)L^{-\sigma}}\cap(\overline{\mathcal{B}}_{1,1}\setminus\mathcal{B}_{3,\pi_1,2})\}} \bar{V}(v\varepsilon_1, \ldots, v^{\alpha_{n-1}}\varepsilon_n) \ge 2$. Then, for $\varepsilon \in \overline{\mathcal{P}}_{(1+\pi_1)L^{-\sigma}} \cap (\overline{\mathcal{B}}_{1,1} \setminus \mathcal{B}_{3,\pi_1,2})$, we have

$V(\varepsilon)=\int_{l_5}^{l_6}\frac{1}{v^{q+\alpha}}(\chi\circ\bar{V}(v\varepsilon_1,\ldots,v^{\alpha_{n-1}}\varepsilon_n))dv+\frac{1}{ql_6^q}$ and $\left.\frac{dV(\varepsilon)}{dt}\right|_{(13)}=2L\int_{l_5}^{l_6}\frac{1}{v^{q+\alpha}}\chi'(\bar{V}(v\varepsilon_1,\ldots,v^{\alpha_{n-1}}\varepsilon_n))K(v,\varepsilon_1,\ldots,\varepsilon_n)dv$.

And for any $\varepsilon\in\overline{\mathcal{P}}_{(1+\pi_1)L^{-\sigma}}\cap(\overline{\mathcal{B}}_{1,1}\setminus\mathcal{B}_{3,\pi_1,2})$, there exists $\tilde{L}\geq 1$ such that $\varepsilon=(\tilde{L}^{\sigma}(\tilde{L}^{-\sigma}L^{-\sigma}\varepsilon_1),\tilde{L}^{\alpha_1\sigma}L^{-2\alpha_1\sigma}\varepsilon_2,\ldots,\tilde{L}^{\alpha_{n-1}\sigma}L^{-2\alpha_{n-1}\sigma}\varepsilon_n)^T$, $|\varepsilon_1|\leq 1+\pi_1$, $\sum_{i=2}^n\varepsilon_i^2=\pi_1^2$.

Then, for any $\varepsilon\in\mathcal{F}_{L^{-h\sigma}}\cap(\overline{\mathcal{B}}_{1,1}\setminus\mathcal{B}_{3,\pi_1,2})$, there exists $\bar{h}_1>2$ such that when $h\geq\bar{h}_1$, we have $\left.\frac{dV(\varepsilon)}{dt}\right|_{(13)}<-\frac{L}{2}\int_{l_5}^{l_6}\frac{\chi'(\bar{V}(vL^{-h\sigma},\ldots,v^{\alpha_{n-1}}\tilde{L}^{\alpha_{n-1}\sigma}L^{-2\alpha_{n-1}\sigma}\varepsilon_n))}{v^{q+\alpha}}\sum_{i=2}^n\tilde{L}^{2\alpha_{i-1}\sigma}L^{-4\alpha_{i-1}\sigma}v^{2\alpha_{i-1}}\varepsilon_i^2dv$.

Moreover, for any $\varepsilon\in\mathcal{F}_{L^{-h\sigma}}\cap(\overline{\mathcal{B}}_{1,1}\setminus\mathcal{B}_{3,\pi_1,2})$, let $l_7(\varepsilon)$ and $l_8(\varepsilon)$ be such that $\frac{5}{4}\leq\bar{V}(v\varepsilon_1,\ldots,v^{\alpha_{n-1}}\varepsilon_n)\leq\frac{7}{4}$ when $l_7(\varepsilon)\leq l\leq l_8(\varepsilon)$ (without loss of generality, it is assumed that $0\leq l_7(\varepsilon)\leq l_8(\varepsilon)$). Note that from the definition of $\chi(s)$, $1\leq\chi'(s)\leq 2$ for $\frac{5}{4}\leq s\leq\frac{7}{4}$. Then, there exists $\bar{h}_2>2$ such that when $h>\bar{h}_2$ we can get $\left.\frac{dV(\varepsilon)}{dt}\right|_{(13)}<-\frac{L}{2}\int_{l_7(\varepsilon)}^{l_8(\varepsilon)}\frac{\sum_{i=2}^n\tilde{L}^{2\alpha_{i-1}\sigma}L^{-4\alpha_{i-1}\sigma}v^{2\alpha_{i-1}}\varepsilon_i^2}{v^{q+\alpha}}dv<-\frac{5L}{16\bar{\lambda}(q+\alpha-1)}\frac{l_8(\varepsilon)^{q+\alpha-1}-l_7(\varepsilon)^{q+\alpha-1}}{l_7(\varepsilon)^{q+\alpha-1}l_8(\varepsilon)^{q+\alpha-1}}$, where $\bar{\lambda}=\lambda_{\max}(P)$.

It is clear that $\{z:z^TPz=\frac{5}{4}\}\cap\{z:z^TPz=\frac{7}{4}\}=\emptyset$, thus, we can derive $M_1<\sum_{i=1}^n(z_i^{1\,\frac{q+\alpha-1}{\alpha_{i-1}}}-z_i^{2\,\frac{q+\alpha-1}{\alpha_{i-1}}})^2$, where $M_1>0$ is a positive real number, $z^1=(z_1^1,\ldots,z_n^1)^T\in\{z:z^TPz=\frac{7}{4}\}$ and $z^2=(z_1^2,\ldots,z_n^2)^T\in\{z:z^TPz=\frac{5}{4}\}$. Because $(l_8(\varepsilon)\tilde{L}^{\sigma}(\tilde{L}^{-\sigma}L^{-h\sigma}\varepsilon_1),l_8(\varepsilon)^{\alpha_1}\tilde{L}^{\alpha_1\sigma}L^{-2\alpha_1\sigma}\varepsilon_2,\ldots,l_8(\varepsilon)^{\alpha_{n-1}}\tilde{L}^{\alpha_{n-1}\sigma}L^{-2\alpha_{n-1}\sigma}\varepsilon_n)^T\in\{z:z^TPz=\frac{7}{4}\}$ and $(l_7(\varepsilon)\tilde{L}^{\sigma}(\tilde{L}^{-\sigma}L^{-h\sigma}\varepsilon_1),l_7(\varepsilon)^{\alpha_1}\tilde{L}^{\alpha_1\sigma}L^{-2\alpha_1\sigma}\varepsilon_2,\ldots,l_7(\varepsilon)^{\alpha_{n-1}}\tilde{L}^{\alpha_{n-1}\sigma}L^{-2\alpha_{n-1}\sigma}\varepsilon_n)^T\in\{z:z^TPz=\frac{5}{4}\}$, we can get $M_1\leq\tilde{L}^{2(q+\alpha-1)\sigma}L^{-4(q+\alpha-1)\sigma}(l_8(\varepsilon)^{q+\alpha-1}-l_7(\varepsilon)^{q+\alpha-1})^2(\sum_{i=2}^n\varepsilon_i^{\frac{2(q+\alpha-1)}{\alpha_{i-1}}}+1)$, $\sum_{i=2}^n\varepsilon_i^2=\pi_1^2$.

Because $\{z:1\leq z^TPz\leq 2\}$ is a bounded compact set, then there exist $M_2,M_3>0$ such that $M_2<\sum_{i=2}^n z_i^{\frac{2(q+\alpha-1)}{\alpha_{i-1}}}<M_3$, $z\in\{z:1\leq z^TPz\leq 2\}$. And it is not difficult to see that there exist $\varepsilon^j\in\overline{\mathcal{P}}_{(1+\pi_1)L^{-\sigma}}\cap(\overline{\mathcal{B}}_{1,1}\setminus\mathcal{B}_{3,\pi_1,2})$ such that $(l_j(\varepsilon)\tilde{L}^{\sigma}(\tilde{L}^{-\sigma}L^{-h\sigma}\varepsilon_1^j),l_j(\varepsilon)^{\alpha_1}\tilde{L}^{\alpha_1\sigma}L^{-2\alpha_1\sigma}\varepsilon_2^j,\ldots,l_j(\varepsilon)^{\alpha_{n-1}}\tilde{L}^{\alpha_{n-1}\sigma}L^{-2\alpha_{n-1}\sigma}\varepsilon_n^j)^T\in\{z:1\leq z^TPz\leq 2\}$, $j=7,8$. And $M_3>\tilde{L}^{2(q+\alpha-1)\sigma}L^{-4(q+\alpha-1)\sigma}l_j(\varepsilon)^{2(q+\alpha-1)}\sum_{i=2}^n\varepsilon_i^{j\,\frac{2(q+\alpha-1)}{\alpha_{i-1}}}$, $j=7,8$, $\sum_{i=2}^n\varepsilon_i^{j\,2}=\pi_1^2$.

Then we can get $l_8(\varepsilon)^{q+\alpha-1}-l_7(\varepsilon)^{q+\alpha-1}>\min_{\{\varepsilon:\sum_{i=2}^n\varepsilon_i^2=\pi_1^2\}}\sqrt{\frac{L^{4(q+\alpha-1)\sigma}M_1}{\tilde{L}^{2(q+\alpha-1)\sigma}(\sum_{i=2}^n\varepsilon_i^{\frac{2(q+\alpha-1)}{\alpha_{i-1}}}+1)}}$ and $\frac{1}{l_j(\varepsilon)^{q+\alpha-1}}>\min_{\{\varepsilon:\sum_{i=2}^n\varepsilon_i^2=\pi_1^2\}}\sqrt{\frac{\tilde{L}^{2(q+\alpha-1)\sigma}\sum_{i=2}^n\varepsilon_i^{\frac{2(q+\alpha-1)}{\alpha_{i-1}}}}{L^{4(q+\alpha-1)\sigma}M_3}}$, $j=7,8$. Therefore, we have

$$\left.\frac{dV(\varepsilon)}{dt}\right|_{(13)}<-L^{1-2(q+\alpha-1)\sigma}\tilde{L}^{(q+\alpha-1)\sigma}d_3,\ \varepsilon\in\mathcal{F}_{L^{-h\sigma}}\cap(\overline{\mathcal{B}}_{1,1}\setminus\mathcal{B}_{3,\pi_1,2}),\tag{A.5}$$

where $d_3=\min_{\{\varepsilon:\sum_{i=2}^n\varepsilon_i^2=\pi_1^2\}}\frac{5\sqrt{M_1}\sum_{i=2}^n\varepsilon_i^{\frac{2(q+\alpha-1)}{\alpha_{i-1}}}}{16\bar{\lambda}(q+\alpha-1)M_3\sqrt{\sum_{i=2}^n\varepsilon_i^{\frac{2(q+\alpha-1)}{\alpha_{i-1}}}+1}}$.

(4) When $\varepsilon\in(\overline{\mathcal{P}}_{L^{-\sigma}}\setminus\mathcal{P}_{L^{-h\sigma}})\cap(\overline{\mathcal{B}}_{3,\pi_1,2}\setminus\mathcal{B}_{3,\pi_1,2})$, because for any $\varepsilon^1=(\varepsilon_1^1,\varepsilon_2^1,\ldots,\varepsilon_n^1)^T\in(\overline{\mathcal{P}}_{L^{-\sigma}}\setminus\mathcal{P}_{L^{-h\sigma}})\cap(\overline{\mathcal{B}}_{3,\pi_1,2}\setminus\mathcal{B}_{3,\pi_1,2})$ and any $\varepsilon^2=(\pm L^{-\sigma},\varepsilon_2^1,\ldots,\varepsilon_n^1)^T\in\mathcal{F}_{L^{-\sigma}}\cap(\overline{\mathcal{B}}_{3,\pi_1,2}\setminus\mathcal{B}_{3,\pi_1,2})$, we have $\|\varepsilon^1-\varepsilon^2\|_2^2\leq 4L^{-2\sigma}$.

Because of the continuity of $\left.\frac{dV(\varepsilon)}{dt}\right|_{(13)}$ on $\varepsilon\in\mathcal{R}^n$, we derive

$$\left.\frac{dV(\varepsilon)}{dt}\right|_{(13)}<-\frac{d_2}{2}L^{1-(q+\alpha)\sigma}<0,\quad\varepsilon\in(\overline{\mathcal{P}}_{L^{-\sigma}}\setminus\mathcal{P}_{L^{-h\sigma}})\cap(\overline{\mathcal{B}}_{3,\pi_1,2}\setminus\mathcal{B}_{3,\pi_1,2}).\tag{A.6}$$

(5) From Equation (A.3), we can select $L>\max_{\{1\leq i\leq 2\}}\{2,L_i\}$ such that

$$V(\varepsilon)^{-\gamma}\geq d_4^{-\gamma},\quad\varepsilon\in\mathcal{S}_1\cap\mathcal{P}_{L^{-\sigma}},\tag{A.7}$$

where $d_4=\max_{\sum_{i=2}^n\varepsilon_i^2=1}V(\varepsilon)$.

When $\varepsilon\in\mathcal{F}_{L^{-\sigma}}\cap\overline{\mathcal{B}}_{3,\pi_1,2}$, we can have $V(\pm L^{-\sigma},L^{-2\alpha_1\sigma}\varepsilon_2,\ldots,L^{-2\alpha_{n-1}\sigma}\varepsilon_n)=L^{-q\sigma}V(\pm 1,L^{-\alpha_1\sigma}\varepsilon_2,\ldots,L^{-\alpha_{n-1}\sigma}\varepsilon_n)\leq d_5L^{-q\sigma}$, where $d_5=\max_{\sum_{i=2}^n\varepsilon_i^2\leq\pi_1^2}V(\pm 1,\varepsilon_2,\ldots,\varepsilon_n)$. Then, we get

$$V(\varepsilon)^{-\gamma}>d_5^{-\gamma}L^{\sigma(q+\alpha-1)},\quad\varepsilon\in\mathcal{F}_{L^{-\sigma}}\cap\overline{\mathcal{B}}_{3,\pi_1,2}.\tag{A.8}$$

When $\varepsilon\in\mathcal{F}_{L^{-h\sigma}}\cap(\overline{\mathcal{B}}_{1,1}\setminus\mathcal{B}_{3,\pi_1,2})$, by use of homogeneity property, we have $V(\pm\tilde{L}^{\sigma}\tilde{L}^{-\sigma}L^{-h\sigma},\tilde{L}^{\alpha_1\sigma}L^{-2\alpha_1\sigma}\varepsilon_2,\ldots,\tilde{L}^{\alpha_{n-1}\sigma}L^{-2\alpha_{n-1}\sigma}\varepsilon_n)=\tilde{L}^{q\sigma}L^{-q\sigma}V(\pm\tilde{L}^{-\sigma}L^{-(h-2)\sigma},\varepsilon_2,\ldots,\varepsilon_n)\leq d_6\tilde{L}^{q\sigma}L^{-2q\sigma}$, where $d_6=\max_{|\varepsilon_1|\leq 1,\sum_{i=2}^n\varepsilon_i^2\leq\pi_1^2}V(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n)$. Then the following inequality holds:

$$V(\varepsilon)^{-\gamma}>d_6^{-\gamma}L^{2\sigma(q+\alpha-1)}\tilde{L}^{-\sigma(q+\alpha-1)},\quad\varepsilon\in\mathcal{F}_{L^{-h\sigma}}\cap(\overline{\mathcal{B}}_{1,1}\setminus\mathcal{B}_{3,\pi_1,2}).\tag{A.9}$$

When $\varepsilon\in(\overline{\mathcal{P}}_{L^{-\sigma}}\setminus\mathcal{P}_{L^{-h\sigma}})\cap(\overline{\mathcal{B}}_{3,\pi_1,2}\setminus\mathcal{B}_{3,\pi_1,2})$, we can get $V(\pm L^{-(1+(h-1)s)\sigma},L^{-2\alpha_1\sigma}\varepsilon_2,\ldots,L^{-2\alpha_{n-1}\sigma}\varepsilon_n)=L^{-q\sigma}V(\pm L^{-(h-1)s\sigma},L^{-\alpha_1\sigma}\varepsilon_2,\ldots,L^{-\alpha_{n-1}\sigma}\varepsilon_n)\leq d_6L^{-q\sigma}$, where $0<s<1$. Therefore, we get

$$V(\varepsilon)^{-\gamma}>d_6^{-\gamma}L^{(q+\alpha-1)\sigma},\ \varepsilon\in(\overline{\mathcal{P}}_{L^{-\sigma}}\setminus\mathcal{P}_{L^{-h\sigma}})\cap(\overline{\mathcal{B}}_{3,\pi_1,2}\setminus\mathcal{B}_{3,\pi_1,2}).\tag{A.10}$$

(6) Thus, from the above inequalities (A.3), (A.7); (A.4), (A.8); (A.5), (A.9) and (A.6), (A.10), we can obtain a compact set, which encircles the origin and is shown in the following:

$$\overline{\mathcal{A}}\triangleq(\mathcal{S}_1\cap\overline{\mathcal{P}}_{L^{-h\sigma}})\cup(\mathcal{F}_{L^{-\sigma}}\cap\overline{\mathcal{B}}_{3,\pi_1,2})\cup(\mathcal{F}_{L^{-h\sigma}}\cap(\overline{\mathcal{B}}_{1,1}\setminus\mathcal{B}_{3,\pi_1,2}))\cup((\overline{\mathcal{P}}_{L^{-\sigma}}\setminus\mathcal{P}_{L^{-h\sigma}})\cap(\overline{\mathcal{B}}_{3,\pi_1,2}\setminus\mathcal{B}_{3,\pi_1,2})),$$

and

$$\left.\frac{dV(\varepsilon)}{dt}\right|_{(13)}V(\varepsilon)^{-\gamma}\leq -c_3L^{1-\sigma},\quad\varepsilon\in\mathcal{A},\tag{A.11}$$

where $c_3=\min\{d_1d_4^{-\gamma},d_2d_5^{-\gamma},d_3d_6^{-\gamma},\frac{d_2d_6^{-\gamma}}{2}\}>0$.

Part II: Because $V(\varepsilon)$ and $\left.\frac{dV(\varepsilon)}{dt}\right|_{(13)}$ are homogeneous of degrees q and $q+\alpha-1$ with respect to the weights $\{\alpha_i\}_{0\leq i\leq n-1}$, for any $\varepsilon\in\mathcal{R}^n\setminus\{0\}$, there exist $v_0>0$ and $\varepsilon^0\in\mathcal{A}$ such that $\varepsilon=(\varepsilon_1,\ldots,\varepsilon_n)^T=(v_0\varepsilon_1^0,\ldots,v_0^{\alpha_{n-1}}\varepsilon_n^0)^T$. Moreover, we have $\frac{dV(\varepsilon)}{dt}|_{(13)}=v_0^{q+\alpha-1}\frac{dV(\varepsilon^0)}{dt}|_{(13)}$ and $V(\varepsilon)=v_0^qV(\varepsilon^0)$. Then, from Equation (A.11), we derive

$$\left.\frac{dV(\varepsilon)}{dt}\right|_{(13)}=V(\varepsilon)^{-\gamma}\left.\frac{dV(\varepsilon^0)}{dt}\right|_{(13)}V(\varepsilon^0)^{-\gamma}\leq -c_3L^{1-\sigma}V(\varepsilon)^{-\gamma},\quad\varepsilon\in\mathcal{R}^n\setminus\{0\}.\tag{A.12}$$

This completes the proof. □