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Distributed optimization of multi-agent systems with delayed sampled-data

Junxiu Yan[a,b], Hui Yu[a,b,*], Xiaohua Xia[c]

[a] College of Science, China Three Gorges University, Yichang 443002, China
[b] Three Gorges Mathematical Research Center, China Three Gorges University, Yichang 443002, China
[c] Centre of New Energy Systems, Department of Electrical, Electronics and Computer Engineering, University of Pretoria, Pretoria 0002, South Africa

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ABSTRACT

In this paper, we study the distributed optimization problem of multi-agent systems with delayed sampled-data, where the interconnected topology is directed, weighted-balanced and strongly connected, and also local cost functions are strongly convex with globally Lipschitz gradients. Based on synchronous and asynchronous sampled-data, we construct two respective algorithms. Our main results, sufficient conditions for the convergence to an optimal solution, are obtained under assumption that all design parameters are chosen properly. We also present one example to validate our theoretical results.

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1. Introduction

Over the past years, distributed optimization problem has been a hot topic. As a result, there are an increasing number of studies conducted on distributed optimization in the context of control theory. Its wide range of applications can be found in various fields, such as statistical machine learning [1], smart grid [2,3], sensor networks [4], and so on. Based on multi-agent environments, objective of a distributed optimization problem is to solve an optimization problem cooperatively in a distributed way, where the objective function formed by a sum of local objective functions, and each agent can access to one local objective function only. The ultimate goal is to make states of all agents converge to the optimal solution of the optimization problem via a local computation and information exchange with its neighbors. Compared with the consensus problem of multi-agent systems, which makes all agents achieve a common state [5–12], the optimization problem of consensus does not only make all agents achieve the same state but also minimizes the optimization problem.

It is common that time-delay exists in practical systems [13–15] because of the finite speeds of information transmission and spreading as well as traffic congestions, and time-delay may result in undesirable dynamics such that the system runs out. Therefore, it is important to analyze the robustness against time-delay of a system and take time-delay into account in the algorithm design of multi-agent systems [16–18]. Meanwhile, in the real situation, agents in systems usually communicate with each other in some certain time intervals. Due to implementation of digital sensors, filters, and controllers, it is desirable that sampled-data takes place only at the discrete sampling instants but not the entire continuous process [19–21]. It is well known that effective methods to deal with sampled-data consensus problems is one of the input delay approach [22–24]. The consensus of multi-agent systems with both sampling data and time-delay is considered in [25]. The simultaneous stability problem of a finite number of linear subsystems is studied in [26] under asynchronous and aperiodic sampling, time-varying delays, and measurement errors. Furthermore, [27,28] provided overviews of recent advances in distributed sampled-data cooperative control and event-triggered consensus of multi-agent systems, respectively. Thus, the consensus problem with sampled-data and time-delay is a meaningful research topic.

In distributed optimization problems of multi-agent systems, most algorithms in earlier works were time-varying, consensus-based dynamics implemented in discrete time [29–31]. In the context of time-varying network topology, discrete time subgradient algorithms are proposed for unconstrained, separable, convex optimization problems in [29,30]. Recent works have introduced continuous-time methods whose convergence properties can be

* Corresponding author at: College of Science, China Three Gorges University, Yichang 443002, China.

E-mail addresses: Yanjunx_2017@163.com (J. Yan), yuhui@ctgu.edu.cn (H. Yu), xxia@up.ac.za (X. Xia).

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analyzed via classical stability theory. Based on the gradient algorithm and integral feedback, auxiliary-variables are introduced in stability analysis of continuous-time dynamical systems [32–34]. From the control system viewpoint, a continuous-time multi-agent system is proposed with strongly connected and weight-balanced directed communication topology in [32]. A modified system is proposed in [33] with auxiliary-variables no longer need to exchange information, where centralized synchronous and distributed asynchronous event-triggered communication schemes are also considered to reduced communication bandwidth. In [34], time-delays are considered in continuous-time multi-agent systems for distributed optimization and a sampled-data communication scheme is formulated based on the results of delay systems, where conditions are derived in form of Linear Matrix Inequality(LMI). In order to avoid using auxiliary-variables, a family of Zero-Gradient-Sum algorithms are proposed over fixed communication topology in [35]. In [36,37], the continuous time Zero-Gradient-Sum algorithm, sampled-data, event-triggered communication for distributed convex optimization problem are considered over directed networks and undirected, connected networks, respectively. In [38], a periodic event-triggered consensus of first-order time-delayed multi-agent systems under switching topologies can be achieved with appropriate choices of the event-triggering parameters, sampling period, and time-delay. Moreover, output consensus problem of delayed sampled-data is considered in [39], where the data of the system is sampled at a sampling instant but can be available with a time-delay. However, so far, studies on the distributed optimization problem of multi-agent systems with delayed sampled-data are rare.

In this paper, the distributed consensus optimization problem of multi-agent systems with delayed sampled-data is considered. The interconnected graph is assumed to be directed, strongly connected and weight balanced. Only available data of the system is assumed to be sampled and delayed. Local costs are strongly convex with global Lipschitz gradients. Two control algorithms under synchronous and asynchronous sampled-data are proposed for the sampled-data multi-agent systems to reach the consensus and optimal state, respectively. A stability analysis is conducted based on Lyapunov theory and algebraic graph theory. Finally, sufficient conditions are obtained such that optimization problems can be solved in the consensus state.

The main contributions of this paper are listed as follows: Firstly, two control algorithms using sampled-data with time-delay under synchronous and asynchronous sampling are presented for the considered multi-agent systems, respectively. Secondly, sufficient conditions are obtained to guarantee the convergence to the optimal solution. In general, the multi-agent system with sampled-data is transformed into time-delay system, and then LMI conditions can be obtained such as in [34]. Other works related to this issue are based on event-triggered scheme due to the advantages of reducing communication resources such as in [37]. The main differences between this paper and previously mentioned works are that the sampled-data becomes available with a time-delay, and then some inequalities conditions are obtained such that the parameters can be chosen properly. In other words, we consider a distributed optimization problem of multi-agent systems with delayed sampled-data in this paper. To the best of our knowledge, no similar results appear in the existing literatures.

This paper is organized as follows. Some preliminaries on algebraic graph theory, useful lemmas and model formulation are presented in Section 2. The convergence results of the proposed algorithm are established and proved under a given communication condition on network topology in Section 3. An example is provided to illustrate results in this paper in Section 4. Finally, this paper concludes in Section 5.

Notations: $\mathcal{R}$ and $\mathcal{R}^n$ represent the set of real numbers and the set of $n \times 1$ real vectors, respectively; $I_n \in \mathcal{R}^{n\times n}$ is the $n \times n$ identity matrix; $\mathbf{1}_n$ (or $\mathbf{0}_n$) denotes an n dimensional column vector whose all entries being 1 (or 0); A^T represents the transpose of a matrix A; for vectors $x_1, x_2, \ldots, x_n$, $\text{col}(x_1, x_2, \ldots, x_n) = [x_1^T, x_2^T, \ldots, x_n^T]^T$; for a vector w, $\|w\| = \sqrt{w^T w}$ represents the standard Euclidean norm; for a matrix P, $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ denote the smallest and largest eigenvalue.

2. Preliminaries and problem statement

2.1. Preliminaries

For a multi-agent system, the information exchange among N agents can be modeled by a weighted digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ with the finite set of nodes $\mathcal{V} = \{1, 2, \ldots, N\}$ and edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. An edge starts from i and ends on j, which means that agent j can obtain information from agent i. The weighted adjacency matrix $\mathcal{A} = [a_{ij}] \in R^{N\times N}$ with $a_{ij} > 0$ if $(j, i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. If $\sum_{j=1}^N a_{ij} = \sum_{j=1}^N a_{ji}$ for all $i \in \mathcal{V}$, the digraph $\mathcal{G}$ is called weighted-balanced. A path is a sequence of connected edges in a graph. If for every pair of nodes there is a directed path connecting them, the digraph $\mathcal{G}$ is said to be strongly connected, otherwise disconnected. The Laplacian $L = [\ell_{ij}] \in R^{N\times N}$ of graph $\mathcal{G}$ is defined by

$$\ell_{ij} = \begin{cases} \sum_{k=1,k\neq i}^N a_{ik} & j = i \\ -a_{ij} & j \neq i \end{cases}.$$

The next lemmas related to the important properties of Laplace L and provide useful mathematical tools.

Lemma 1 [40]**.** *Laplace matrix L has least one zero eigenvalue with* $\mathbf{1}_N = [1, 1, \ldots, 1] \in \mathcal{R}^N$ *as its eigenvector, and all the non-zero eigenvalues of L have positive real parts. Laplacian L has a simple zero eigenvalue if and only if* $\mathcal{G}$ *is strongly connected.*

Lemma 2. *For matrices A, B, C and D with appropriate dimensions, the Kronecker product ⊗ satisfies* $(1)(A\otimes B)(C\otimes D) = (AC)\otimes(BD)$; $(2)(A\otimes B)^T = A^T \otimes B^T$; $(3)(A\otimes B)^{-1} = A^{-1}\otimes B^{-1}$.

Lemma 3 [41]**.** *For a given* $n \times n$*−matrix* $G > 0$ *and for all continuous functions* ω *in* $[a, b] \to \mathcal{R}^n$*, the following inequality holds:*

$$\left[\int_a^b \omega(s)ds\right]^T G\left[\int_a^b \omega(s)ds\right] \le (b-a)\int_a^b \omega^T(s)G\omega(s)ds.$$

2.2. Problem statement

We consider a multi-agent system consisting of N agents. The dynamics of the ith agent, $i \in \mathcal{V}$, are described by

$$\dot{x}_i(t) = u_i(t), \tag{1}$$

where $x_i \in \mathcal{R}^m$ denotes the state of agent i, $u_i \in \mathcal{R}^m$ is the control input.

Consider the multi-agent optimization problem, in which the goal is to minimize the sum of local cost functions associated to the individual agent. More specially, it can be expressed as

$$\text{minimize} \quad f(x) = \sum_{i=1}^N f_i(x), x \in \mathcal{R}^m. \tag{2}$$

Let $\boldsymbol{x} = \text{col}(x_1, x_2, \ldots, x_N) \in \mathcal{R}^{Nm}$. Next, we provide an alternative formulation of (2), i.e.,

$$\begin{aligned} &\text{minimize} \quad f(\boldsymbol{x}) = \sum_{i=1}^N f_i(x_i), x_i \in \mathcal{R}^m, \\ &\text{subject to} \quad (L\otimes I_m)\boldsymbol{x} = 0_{Nm}. \end{aligned} \tag{3}$$

We can see problem (2) on $\mathcal{R}^m$ is equivalent to problem (3) on $\mathcal{R}^{Nm}$.

In this paper, our goal is to design a distributed controller for each agent such that the states of all agents converge to a optimal solution of the optimization problem (2) via local communication.

Before proceed, we give the following assumption on the local cost function f_i based on the convex analysis [42].

Assumption 1. (a) For each $i \in \mathcal{V}$, f_i is differentiable and its gradient is Lipschitz with constant $\rho_i > 0$ in $\mathcal{R}^m$:

$$\|\nabla f_i(x) - \nabla f_i(y)\| \le \rho_i \|x - y\|, \quad \forall x, y \in \mathcal{R}^m. \tag{4}$$

(b) For $i \in \mathcal{V}$, f_i is m_i-strongly convex with constant $m_i > 0$:

$$(x-y)^T(\nabla f_i(x) - \nabla f_i(y)) \ge m_i \|x-y\|^2, \quad \forall x, y \in \mathcal{R}^m. \tag{5}$$

Remark 1. Under Assumption 1(b), we can note that f is strictly convex, then the optimization problem (3) has an unique optimal solution.

Assumption 2. The directed graph $\mathcal{G}$ is weighted-balanced and strongly connected.

Remark 2. From Assumption 2, zero is a simple eigenvalue of matrix L and $1_N^T L = 0$. Moreover, there exists a matrix $Q \in \mathcal{R}^{N\times(N-1)}$ with

$$\mathbf{1}_N^T Q = 0, \quad Q^T Q = I_{N-1}, \quad QQ^T = I_N - \frac{1}{N}\mathbf{1}_N \mathbf{1}_N^T, \tag{6}$$

such that the matrix $Q^T L Q = H$, where the real parts of all eigenvalues of H are positive, and $H + H^T$ is a positive definite matrix.

3. Main results

3.1. Synchronous sampling

The state $x_i(t)$ of system (1) and (3) is assumed to be sampled at time instants t_k and available at $t_k + \tau_k$. $\{t_k\}$, $k = 0, 1, \ldots, \infty$, is a strictly increasing sequence such that $\lim_{k\to\infty} t_k = \infty$ and $\tau_k \ge 0$, that is the sampled-data $x_i(t_k)$ is available with a time-delay τ_k. The sampling interval $[t_{k-1}, t_k)$ satisfy $0 < T_{\min} \le t_k - t_{k-1} = T_k \le T_{\max}$ for all $k = 0, 1, \ldots, \infty$, where T_k is the length of the kth sampling interval, $T_{\min} = \min\{T_k\}$ and $T_{\max} = \max\{T_k\}$, and when $t \in [t_k + \tau_k, t_{k+1} + \tau_{k+1})$, $x_i(t_{k+1} + \tau_{k+1}) = \lim_{t\to(t_{k+1}+\tau_{k+1})^-} x_i(t)$. We assume that τ is the upper bound of τ_k, that is $\tau_k \le \tau$, and satisfy $\tau < T_{\min}$, which means that the sampled-data at time t_k can be used before next sampling time instant.

We use the following sampled-data based control algorithm to achieve consensus and optimum:

$$\begin{aligned} u_i(t) &= -k\sum_{j=1}^N a_{ij}[x_i(t_k) - x_j(t_k)] - w_i(t) - \gamma \nabla f_i(x_i(t)), \\ \dot{w}_i(t) &= \alpha \sum_{j=1}^N a_{ij}[x_i(t_k) - x_j(t_k)], \\ w_i(0) &= 0, \quad t \in [t_k + \tau_k, t_{k+1} + \tau_{k+1}), \quad k \ge 0, \end{aligned} \tag{7}$$

where $w_i(t)$ is an auxiliary state of agent i and k, α, γ are the scalar tuning positive parameter. It can be seen from (7) that each agent only uses the information at time t_k when $t \in [t_k + \tau_k, t_{k+1} + \tau_{k+1})$, $k \ge 0$.

Noticed that $x_i(t_k)(i \in \mathcal{V})$ are constant in all of the time intervals $[t_k + \tau_k, t_{k+1} + \tau_{k+1})$, $k \ge 0$. From the second equation of (7), we can know that $w_i(t)$ is continuous in $[t_k + \tau_k, t_{k+1} + \tau_{k+1})$. From Assumption 1, $\nabla f_i(x_i(t))$ is Lipschitz and then continuous in $[t_k + \tau_k, t_{k+1} + \tau_{k+1})$. Therefore, $\dot{x}_i(t)$ and then $x_i(t)$ is continuous in time intervals $[t_k + \tau_k, t_{k+1} + \tau_{k+1})$, $k \ge 0$. According to the definition $x_i(t_{k+1} + \tau_{k+1}) = \lim_{t\to(t_{k+1}+\tau_{k+1})^-} x_i(t)$ in the beginning of this section, we have $\lim_{t\to(t_{k+1}+\tau_{k+1})^-} x_i(t) = \lim_{t\to(t_{k+1}+\tau_{k+1})^+} x_i(t) = x_i(t_{k+1} + \tau_{k+1})$, which means that $x_i(t)$ is continuous in the time instant $t_{k+1} + \tau_{k+1}$. Thus, $x_i(t)(i \in \mathcal{V})$ are continuous in the time interval $[t_0, \infty)$.

Let

$$\boldsymbol{w}(t) = \mathrm{col}(w_1(t), w_2(t), \ldots, w_N(t)),$$

and

$$\nabla \bar{f}(x(t)) = \mathrm{col}(\nabla f_1(x_1(t)), \nabla f_2(x_2(t)), \ldots, \nabla f_N(x_N(t))).$$

Then the closed-loop systems of (1) and (7) can be expressed as a compact form:

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= -k(L\otimes I_m)\boldsymbol{x}(t_k) - \boldsymbol{w}(t) - \gamma \nabla \bar{f}(\boldsymbol{x}(t)), \\ \dot{\boldsymbol{w}}(t) &= \alpha(L\otimes I_m)\boldsymbol{x}(t_k). \end{aligned} \tag{8}$$

Let the right-side of the closed-loop system (8) equal to 0, then we can get a equilibrium point $(\boldsymbol{x}^*, \boldsymbol{w}^*)$, i.e.

$$\begin{aligned} -k(L\otimes I_m)\boldsymbol{x}^* - \boldsymbol{w}^* - \gamma \nabla \bar{f}(\boldsymbol{x}^*) &= 0, \\ \alpha(L\otimes I_m)\boldsymbol{x}^* &= 0. \end{aligned} \tag{9}$$

According to the properties of Laplacian matrix, and from (9), one can obtain

$$\begin{aligned} \boldsymbol{x}^* &= \mathbf{1}_N \otimes \pi, \pi \in \mathcal{R}^m, \\ \boldsymbol{w}^* &= -\gamma \nabla \bar{f}(\boldsymbol{x}^*). \end{aligned} \tag{10}$$

Under Assumption 2, we have $\mathbf{1}_N^T L = 0$. Left multiplying the second equation of (8) by $\mathbf{1}_N^T \otimes I_m$, we obtain $\sum_{j=1}^N \dot{w}_j(t) = 0$, and using initial condition $w_i(0) = 0$, then

$$\sum_{j=1}^N w_j(t) = \sum_{j=1}^N w_j(0) = 0, \quad \forall t \ge 0. \tag{11}$$

Using $\mathbf{1}_N^T \otimes I_m$ left multiply the second equation of (10) again results in

$$0 = \sum_{j=1}^N w_j^* = -\gamma(\mathbf{1}_N^T \otimes I_m)\nabla \bar{f}(\boldsymbol{x}^*) = -\gamma \sum_{j=1}^N \nabla f_i(\pi) = -\gamma \nabla f(\boldsymbol{x}^*).$$

Thus, the optimal condition $\nabla f(\boldsymbol{x}^*) = 0$ is satisfied, which means that $\boldsymbol{x}^* = \mathbf{1}_N \otimes x^*, x^* \in \mathcal{R}^m$ is the optimal solution of the optimization problem (3).

Using the transformation

$$\bar{\boldsymbol{x}}(t) = \boldsymbol{x}(t) - \boldsymbol{x}^*, \bar{\boldsymbol{w}}(t) = \boldsymbol{w}(t) - \boldsymbol{w}^*, \tag{12}$$

one can shift the equilibrium point into the origin, then system (8) can be transformed into the following form:

$$\begin{aligned} \dot{\bar{\boldsymbol{x}}}(t) &= -k(L\otimes I_m)\bar{\boldsymbol{x}}(t_k) - \bar{\boldsymbol{w}}(t) - \gamma \Psi(\bar{\boldsymbol{x}}(t)), \\ \dot{\bar{\boldsymbol{w}}}(t) &= \alpha(L\otimes I_m)\bar{\boldsymbol{x}}(t_k), \end{aligned} \tag{13}$$

where $\Psi(\bar{\boldsymbol{x}}(t)) = \nabla \bar{f}(\boldsymbol{x}(t)) - \nabla \bar{f}(\boldsymbol{x}^*)$.

Let

$$e(t) = (T^T \otimes I_m)\bar{\boldsymbol{x}}(t), \quad \vartheta(t) = (T^T \otimes I_m)\bar{\boldsymbol{w}}(t), \quad T = \begin{bmatrix} \frac{\mathbf{1}_N}{\sqrt{N}} & Q \end{bmatrix}.$$

Denote $e = \mathrm{col}(e_1, e_2)$, and $\vartheta = \mathrm{col}(\vartheta_1, \vartheta_2)$ with $e_1, \vartheta_1 \in \mathcal{R}^m$, and $e_2, \vartheta_2 \in \mathcal{R}^{m(N-1)}$. By the structure of T and (6), we can know T is an orthogonal matrix. Then system (13) can be rewritten as:

$$\begin{aligned} \dot{e_1}(t) &= -\gamma\left(\frac{\mathbf{1}_N^T}{\sqrt{N}} \otimes I_m\right)\Psi(\bar{\boldsymbol{x}}(t)), \\ \dot{e_2}(t) &= -k(H\otimes I_m)e_2(t_k) - \vartheta_2(t) - \gamma(Q^T\otimes I_m)\Psi(\bar{\boldsymbol{x}}(t)), \\ \dot{\vartheta}_1(t) &= 0, \\ \dot{\vartheta}_2(t) &= \alpha(H\otimes I_m)e_2(t_k). \end{aligned} \tag{14}$$

Let $\varepsilon(t) = \mathrm{col}(e_2(t), \vartheta_2(t))$, then

$$\dot{\varepsilon}(t) = C\varepsilon(t) - E\int_{t_k}^{t} \dot{\varepsilon}(s)ds + F(t), \tag{15}$$

with

$$C = \begin{pmatrix} -kH & -I_{N-1} \\ \alpha H & 0 \end{pmatrix} \otimes I_m, \quad E = \begin{pmatrix} -kH & 0 \\ \alpha H & 0 \end{pmatrix} \otimes I_m,$$

and

$$F(t) = \begin{pmatrix} -\gamma(Q^T \otimes I_m)\Psi(\bar{\boldsymbol{x}}(t)) \\ 0 \end{pmatrix}.$$

Then the main results can be obtained as follows.

Theorem 1. *Suppose Assumptions 1 and 2 hold, the optimization problem (3) for multi-agent system (1) can be solved by the optimization control (7), if the following conditions are satisfied:*

$$\underline{m} - (3m+1)\gamma\bar{\rho}^2 > 0, \tag{16}$$

$$\underline{\lambda}_1 - 6m[(\alpha^2 + k^2)\bar{\lambda}_2 + 1] - \frac{1}{2} > 0, \tag{17}$$

and

$$T_{\max} + \tau < \sqrt{\frac{m}{\bar{\lambda}_2[(k-\alpha)^2 + 3m(\alpha^2+k^2)]}}, \tag{18}$$

where $m > 0$ is a constant, $\underline{\lambda}_1 = \lambda_{\min}(R)$, $\bar{\lambda}_2 = \lambda_{\max}(H^T H)$, and $R = \begin{pmatrix} (k-\alpha)(H+H^T) & I_{N-1} \\ I_{N-1} & 2I_{N-1} \end{pmatrix} \otimes I_m$.

Proof. Consider the following Lyapunov function:

$$V_1(t) = \frac{1}{2}e_1^T(t)e_1(t) + \frac{1}{2}\boldsymbol{\varepsilon}^T(t)\Omega\boldsymbol{\varepsilon}(t),$$

where $\Omega = \begin{pmatrix} I_{N-1} & I_{N-1} \\ I_{N-1} & \frac{k}{\alpha}I_{N-1} \end{pmatrix} \otimes I_m$ is positive definite for $k > \alpha$, the condition $k > \alpha$ will be proved later. The derivation of V_1 along the first equality of (14) and - system (15) yields:

$$\begin{aligned} \dot{V}_1 &= e_1^T(t)\dot{e}_1(t) + \frac{1}{2}\varepsilon^T(t)(\Omega C + C^T\Omega)\varepsilon(t) \\ &\quad - \varepsilon^T(t)\Omega E\int_{t_k}^{t}\dot{\varepsilon}(s)ds + \varepsilon^T(t)\Omega F(t). \end{aligned} \tag{19}$$

Due to $e(t) = (T^T \otimes I_m)\bar{\boldsymbol{x}}(t)$ and from Assumption 1, we have

$$\begin{aligned} & e_1^T(t)\dot{e}_1(t) + \varepsilon^T(t)\Omega F(t) \\ &= -\gamma e_1^T(t)\left(\frac{\mathbf{1}_N^T}{\sqrt{N}} \otimes I_m\right)\Psi(\bar{\boldsymbol{x}}(t)) + \varepsilon^T(t)\Omega F(t) \\ &= -\gamma\bar{\boldsymbol{x}}^T(t)\Psi(\bar{\boldsymbol{x}}(t)) + \gamma e_2^T(t)(Q^T \otimes I_m)\Psi(\bar{\boldsymbol{x}}(t)) \\ &\quad - \gamma e_2^T(t)(Q^T \otimes I_m)\Psi(\bar{\boldsymbol{x}}(t)) - \gamma\vartheta_2^T(t)(Q^T \otimes I_m)\Psi(\bar{\boldsymbol{x}}(t)) \\ &= -\gamma\bar{\boldsymbol{x}}^T(t)\Psi(\bar{\boldsymbol{x}}(t)) - \gamma\vartheta_2^T(t)(Q^T \otimes I_m)\Psi(\bar{\boldsymbol{x}}(t)) \\ &\le -\gamma\underline{m}\bar{\boldsymbol{x}}^T(t)\bar{\boldsymbol{x}}(t) + \frac{1}{4}\vartheta_2^T(t)\vartheta_2(t) + \gamma^2\|(Q^T \otimes I_m)\Psi(\bar{\boldsymbol{x}}(t))\|^2 \\ &\le -\gamma\underline{m}\bar{\boldsymbol{x}}^T(t)\bar{\boldsymbol{x}}(t) + \frac{1}{4}\vartheta_2^T(t)\vartheta_2(t) + \gamma^2\bar{\rho}^2\bar{\boldsymbol{x}}^T(t)\bar{\boldsymbol{x}}(t), \end{aligned} \tag{20}$$

where $\underline{m} = \min\{m_1, m_2, \ldots, m_N\}$, $\bar{\rho} = \max\{\rho_1, \rho_2, \ldots, \rho_N\}$.

Let $R = -(\Omega C + C^T\Omega)$, according to condition (17), we can know R is positive definite, and due to $H + H^T$ is positive definite, then $k > \alpha$. Thus

$$\begin{aligned} & -\varepsilon^T(t)\Omega E\int_{t_k}^{t}\dot{\varepsilon}(s)ds \\ &= (k-\alpha)e_2^T(t)(H \otimes I_m)\int_{t_k}^{t}\dot{e}_2(s)ds \\ &\le \frac{1}{4}e_2^T(t)e_2(t) + (k-\alpha)^2\bar{\lambda}_2\left(\int_{t_k}^{t}\dot{e}_2(s)ds\right)^T\left(\int_{t_k}^{t}\dot{e}_2(s)ds\right) \\ &\le \frac{1}{4}e_2^T(t)e_2(t) + (k-\alpha)^2\bar{\lambda}_2\left(\int_{t_k}^{t}\dot{\varepsilon}(s)ds\right)^T\left(\int_{t_k}^{t}\dot{\varepsilon}(s)ds\right). \end{aligned} \tag{21}$$

From (19)–(21) and $\boldsymbol{\varepsilon}^T(t)R\boldsymbol{\varepsilon}(t) \ge \underline{\lambda}_1\boldsymbol{\varepsilon}^T(t)\boldsymbol{\varepsilon}(t)$, we have

$$\begin{aligned} \dot{V}_1(t) &\le -\gamma\underline{m}\bar{\boldsymbol{x}}^T(t)\bar{\boldsymbol{x}}(t) + \frac{1}{4}\vartheta_2^T(t)\vartheta_2(t) + \gamma^2\bar{\rho}^2\bar{\boldsymbol{x}}^T(t)\bar{\boldsymbol{x}}(t) \\ &\quad - \frac{1}{2}\varepsilon^T(t)R\varepsilon(t) \\ &\quad + \frac{1}{4}e_2^T(t)e_2(t) + (k-\alpha)^2\bar{\lambda}_2\left(\int_{t_k}^{t}\dot{\varepsilon}(s)ds\right)^T\left(\int_{t_k}^{t}\dot{\varepsilon}(s)ds\right) \\ &\le -\gamma\underline{m}\bar{\boldsymbol{x}}^T(t)\bar{\boldsymbol{x}}(t) + \frac{1}{4}\varepsilon^T(t)\varepsilon(t) + \gamma^2\bar{\rho}^2\bar{\boldsymbol{x}}^T(t)\bar{\boldsymbol{x}}(t) \\ &\quad - \frac{1}{2}\underline{\lambda}_1\varepsilon^T(t)\varepsilon(t) \\ &\quad + (k-\alpha)^2\bar{\lambda}_2\left(\int_{t_k}^{t}\dot{\varepsilon}(s)ds\right)^T\left(\int_{t_k}^{t}\dot{\varepsilon}(s)ds\right). \end{aligned} \tag{22}$$

Construct the following auxiliary integral function

$$V_2(t) = \int_{t-T_{\max}-\tau}^{t}\int_{\theta}^{t}\dot{\varepsilon}^T(s)\dot{\varepsilon}(s)dsd\sigma,$$

we can obtain

$$\dot{V}_2(t) = (T_{\max}+\tau)\dot{\varepsilon}^T(t)\dot{\varepsilon}(t) - \int_{t-T_{\max}-\tau}^{t}\dot{\varepsilon}^T(s)\dot{\varepsilon}(s)ds.$$

By calculation, we have

$$\begin{aligned} \dot{\varepsilon}^T(t)\dot{\varepsilon}(t) &= \varepsilon^T(t)C^TC\varepsilon(t) + \left(\int_{t_k}^{t}\dot{\varepsilon}(s)ds\right)^T E^TE\left(\int_{t_k}^{t}\dot{\varepsilon}(s)ds\right) \\ &\quad + F^T(t)F(t) \\ &\quad - 2\varepsilon^T(t)C^TE\left(\int_{t_k}^{t}\dot{\varepsilon}(s)ds\right) - 2\left(\int_{t_k}^{t}\dot{\varepsilon}(s)ds\right)^T E^TF(t) \\ &\quad + 2\varepsilon^T(t)C^TF(t). \end{aligned} \tag{23}$$

Due to $2a^Tb \le a^TXa + b^TX^{-1}b$, we have

$$\begin{aligned} -2\varepsilon^T(t)C^TE\left(\int_{t_k}^{t}\dot{\varepsilon}(s)ds\right) &\le \varepsilon^T(t)C^TC\varepsilon(t) + \left(\int_{t_k}^{t}\dot{\varepsilon}(s)ds\right)^T \\ &\quad \times E^TE\left(\int_{t_k}^{t}\dot{\varepsilon}(s)ds\right), \end{aligned} \tag{24}$$

$$\begin{aligned} -2\left(\int_{t_k}^{t}\dot{\varepsilon}(s)ds\right)^T E^TF(t) &\le \left(\int_{t_k}^{t}\dot{\varepsilon}(s)ds\right)^T E^TE\left(\int_{t_k}^{t}\dot{\varepsilon}(s)ds\right) \\ &\quad + F^T(t)F(t), \end{aligned} \tag{25}$$

and

$$2\varepsilon^T(t)C^TF(t) \le \varepsilon^T(t)C^TC\varepsilon(t) + F^T(t)F(t). \tag{26}$$

Based on Assumption 1(a), the following result can be obtained:

$$F^T(t)F(t) = \gamma^2\|(Q^T \otimes I_m)\Psi(\bar{\boldsymbol{x}}(t))\|^2 \le \gamma^2\bar{\rho}^2\bar{\boldsymbol{x}}^T(t)\bar{\boldsymbol{x}}(t). \tag{27}$$

From (23)–(27), we have

$$\begin{aligned} \dot{\varepsilon}^T(t)\dot{\varepsilon}(t) &\le 3[\varepsilon^T(t)C^TC\varepsilon(t) + \left(\int_{t_k}^{t}\dot{\varepsilon}(s)ds\right)^T E^TE\left(\int_{t_k}^{t}\dot{\varepsilon}(s)ds\right) \\ &\quad + F^T(t)F(t)] \\ &\le 3[(\alpha^2+k^2)\bar{\lambda}_2 + 1]\varepsilon^T(t)\varepsilon(t) + 3\gamma^2\bar{\rho}^2\bar{\boldsymbol{x}}^T(t)\bar{\boldsymbol{x}}(t) \end{aligned}$$

$$+3(\alpha^2+k^2)\overline{\lambda}_2\left(\int_{t_k}^{t}\dot{\varepsilon}(s)ds\right)^T\left(\int_{t_k}^{t}\dot{\varepsilon}(s)ds\right). \tag{28}$$

Let $V(t)=V_1(t)+\frac{m}{T_{\max}+\tau}V_2(t)$, we have

$$\begin{aligned}\dot{V}(t)\le& -\gamma\underline{m}\overline{\boldsymbol{x}}^T(t)\overline{\boldsymbol{x}}(t)+\frac{1}{4}\varepsilon^T(t)\varepsilon(t)+\gamma^2\overline{\rho}^2\overline{\boldsymbol{x}}^T(t)\overline{\boldsymbol{x}}(t)\\&-\frac{1}{2}\underline{\lambda}_1\varepsilon^T(t)\varepsilon(t)\\&+(k-\alpha)^2\overline{\lambda}_2\left(\int_{t_k}^{t}\dot{\varepsilon}(s)ds\right)^T\left(\int_{t_k}^{t}\dot{\varepsilon}(s)ds\right)\\&+3m[(\alpha^2+k^2)\overline{\lambda}_2+1]\varepsilon^T(t)\varepsilon(t)+3m\gamma^2\overline{\rho}^2\overline{\boldsymbol{x}}^T(t)\overline{\boldsymbol{x}}(t)\\&+3m(\alpha^2+k^2)\overline{\lambda}_2\left(\int_{t_k}^{t}\dot{\varepsilon}(s)ds\right)^T\left(\int_{t_k}^{t}\dot{\varepsilon}(s)ds\right)\\&-\frac{m}{T_{\max}+\tau}\int_{t-T_{\max}-\tau}^{t}\dot{\varepsilon}^T(s)\dot{\varepsilon}(s)ds.\end{aligned} \tag{29}$$

From Lemma 3, we have

$$\left(\int_{t_k}^{t}\dot{\varepsilon}(s)ds\right)^T\left(\int_{t_k}^{t}\dot{\varepsilon}(s)ds\right)\le(t-t_k)\int_{t_k}^{t}\dot{\varepsilon}^T(s)\dot{\varepsilon}(s)ds,$$

where $t\in[t_k+\tau_k,t_{k+1}+\tau_{k+1})$, $t-t_k\le T_{\max}+\tau$, that is $t-T_{\max}-\tau\le t_k$. Then, we have

$$\begin{aligned}\dot{V}(t)\le& -\gamma\underline{m}\overline{\boldsymbol{x}}^T(t)\overline{\boldsymbol{x}}(t)+\frac{1}{4}\varepsilon^T(t)\varepsilon(t)+\gamma^2\overline{\rho}^2\overline{\boldsymbol{x}}^T(t)\overline{\boldsymbol{x}}(t)\\&-\frac{1}{2}\underline{\lambda}_1\varepsilon^T(t)\varepsilon(t)\\&+(k-\alpha)^2\overline{\lambda}_2(t-t_k)\int_{t_k}^{t}\dot{\varepsilon}^T(s)\dot{\varepsilon}(s)ds\\&+3m[(\alpha^2+k^2)\overline{\lambda}_2+1]\varepsilon^T(t)\varepsilon(t)+3m\gamma^2\overline{\rho}^2\overline{\boldsymbol{x}}^T(t)\overline{\boldsymbol{x}}(t)\\&+3m(\alpha^2+k^2)\overline{\lambda}_2(t-t_k)\int_{t_k}^{t}\dot{\varepsilon}^T(s)\dot{\varepsilon}(s)ds\\&-\frac{m}{T_{\max}+\tau}\int_{t-T_{\max}-\tau}^{t}\dot{\varepsilon}^T(s)\dot{\varepsilon}(s)ds.\end{aligned} \tag{30}$$

Note that

$$(t-t_k)\int_{t_k}^{t}\dot{\varepsilon}^T(s)\dot{\varepsilon}(s)ds\le(T_{\max}+\tau)\int_{t-T_{\max}-\tau}^{t}\dot{\varepsilon}^T(s)\dot{\varepsilon}(s)ds,$$

we have

$$\begin{aligned}\dot{V}(t)\le& -\gamma\underline{m}\overline{\boldsymbol{x}}^T(t)\overline{\boldsymbol{x}}(t)+\frac{1}{4}\varepsilon^T(t)\varepsilon(t)+\gamma^2\overline{\rho}^2\overline{\boldsymbol{x}}^T(t)\overline{\boldsymbol{x}}(t)\\&-\frac{1}{2}\underline{\lambda}_1\varepsilon^T(t)\varepsilon(t)\\&+(k-\alpha)^2\overline{\lambda}_2(T_{\max}+\tau)\int_{t-T_{\max}-\tau}^{t}\dot{\varepsilon}^T(s)\dot{\varepsilon}(s)ds\\&+3m[(\alpha^2+k^2)\overline{\lambda}_2+1]\varepsilon^T(t)\varepsilon(t)+3m\gamma^2\overline{\rho}^2\overline{\boldsymbol{x}}^T(t)\overline{\boldsymbol{x}}(t)\\&+3m(\alpha^2+k^2)\overline{\lambda}_2(T_{\max}+\tau)\int_{t-T_{\max}-\tau}^{t}\dot{\varepsilon}^T(s)\dot{\varepsilon}(s)ds\\&-\frac{m}{T_{\max}+\tau}\int_{t-T_{\max}-\tau}^{t}\dot{\varepsilon}^T(s)\dot{\varepsilon}(s)ds,\end{aligned} \tag{31}$$

and then

$$\begin{aligned}\dot{V}(t)\le& -[\gamma\underline{m}-(3m+1)\gamma^2\overline{\rho}^2]\overline{\boldsymbol{x}}^T(t)\overline{\boldsymbol{x}}(t)\\&-\left\{\frac{1}{2}\underline{\lambda}_1-3m[(\alpha^2+k^2)\overline{\lambda}_2+1]-\frac{1}{4}\right\}\varepsilon^T(t)\varepsilon(t)\\&-\left[\frac{m}{T_{\max}+\tau}-(k-\alpha)^2\overline{\lambda}_2(T_{\max}+\tau)\right.\\&\left.-3m(\alpha^2+k^2)\overline{\lambda}_2(T_{\max}+\tau)\right]\int_{t-T_{\max}-\tau}^{t}\dot{\varepsilon}^T(s)\dot{\varepsilon}(s)ds.\end{aligned} \tag{32}$$

Hence, conditions (16)–(18) guarantee that $\dot{V}(t)<0$. Based on Lyapunov stability theory, we can conclude that $e_1(t)\to 0$ and $\boldsymbol{\varepsilon}(t)\to 0$, that is $e(t)\to 0_{mN}$, $\vartheta(t)\to 0_{mN}$ as $t\to\infty$.

With the transformation $\overline{\boldsymbol{x}}(t)=(T\otimes I_m)e(t)$ and $\overline{\boldsymbol{w}}(t)=(T\otimes I_m)\vartheta(t)$ and T is a orthogonal matrix, we can obtain $\overline{\boldsymbol{x}}(t)\to 0_{mN}$, $\overline{\boldsymbol{w}}(t)\to 0_{mN}$, which means $\boldsymbol{x}(t)\to\boldsymbol{x}^*$, $\boldsymbol{w}(t)\to\boldsymbol{w}^*$ as $t\to\infty$. As a result, this proof is completed. □

3.2. Asynchronous sampling

Based on the dynamics (1) and the optimization problem (3), we assume that each agent i independently samples its own state at sampling instant t_k^i and the sampled-data is available at $t_k^i+\tau_k^i$, $i\in\mathcal{V}, k=0,1,\ldots,\infty$, which is determined by its own clock. $\{t_k^i\}$ is a strictly increasing sequence such that $\lim_{k\to\infty}t_k^i=\infty$. The sampling interval $[t_k^i,t_{k+1}^i)$ satisfies $0<T_{\min}\le t_{k+1}^i-t_k^i\le T_{\max}$ for all $k\ge 0$. $\tau_k^i>0$ denote the transmission delay with an upper bound $\tau\ge\max\{\tau_k^i\}$, and satisfy $\tau<T_{\min}$, which means that the sampled-data at time t_k^i can be used before next sampling time instant. When $t\in[t_k^i+\tau_k^i,t_{k+1}^i+\tau_{k+1}^i)$, $x_i(t_{k+1}^i+\tau_{k+1}^i)=\lim_{t\to(t_{k+1}^i+\tau_{k+1}^i)^-}x_i(t)$.

The following asynchronous sampled-data control algorithm is proposed:

$$\begin{aligned}u_i(t)&=-k\sum_{j=1}^{N}a_{ij}[x_i(t_k^i)-x_j(t_k^j)]-w_i(t)-\gamma\nabla f_i(x_i(t)),\\\dot{w}_i(t)&=\alpha\sum_{j=1}^{N}a_{ij}[x_i(t_k^i)-x_j(t_k^j)],\\w_i(0)&=0,\quad t\in[t_k^i+\tau_k^i,t_{k+1}^i+\tau_{k+1}^i),\quad k\ge 0.\end{aligned} \tag{33}$$

Similar mechanism of asynchronous sampling for consensus problem of multi-agent systems can be found in [26].

By the similar discussion as in the Section 3.1, we can also conclude that $\lim_{t\to(t_{k+1}^i+\tau_{k+1}^i)^-}x_i(t)=\lim_{t\to(t_{k+1}^i+\tau_{k+1}^i)^+}x_i(t)=x_i(t_{k+1}^i+\tau_{k+1}^i)$, which means that $x_i(t)$ is continuous in time instant $t_{k+1}^i+\tau_{k+1}^i$. Thus, $x_i(t)$ is continuous in the time interval $[t_0,\infty)$.

According to the definition of Laplacian matrix L, (1) and (33) can be rewritten as

$$\begin{aligned}\dot{x}_i(t)&=-k\sum_{j=1}^{N}\ell_{ij}x_j(t_k^j)-w_i(t)-\gamma\nabla f_i(x_i(t)),\\\dot{w}_i(t)&=\alpha\sum_{j=1}^{N}\ell_{ij}x_j(t_k^j),\quad w_i(0)=0,\\&\quad t\in[t_k^i+\tau_k^i,t_{k+1}^i+\tau_{k+1}^i),\quad k\ge 0.\end{aligned} \tag{34}$$

Let $\hat{\boldsymbol{x}}(t)=\mathrm{col}(x_1(t_k^1),x_2(t_k^2),\ldots,x_N(t_k^N))$, then system (34) can be expressed as the following compact form:

$$\begin{aligned}\dot{\boldsymbol{x}}(t)&=-k(L\otimes I_m)\hat{\boldsymbol{x}}(t)-\boldsymbol{w}(t)-\gamma\nabla\overline{f}(\boldsymbol{x}(t)),\\\dot{\boldsymbol{w}}(t)&=\alpha(L\otimes I_m)\hat{\boldsymbol{x}}(t).\end{aligned} \tag{35}$$

Similarly, we can obtain the equilibrium point $\boldsymbol{x}^*=\mathbf{1}_N\otimes x^*$, $\boldsymbol{w}^*=-\gamma\nabla\overline{f}(\boldsymbol{x}^*)$, where $x^*\in\mathcal{R}^m$ is the optimal solution of the optimization problem (3).

Using transformation (12) and $\hat{\overline{\boldsymbol{x}}}(t)=\hat{\boldsymbol{x}}(t)-\boldsymbol{x}^*$, one can shift the equilibrium point into the origin, then system (35) can be transformed into the following form:

$$\dot{\overline{\boldsymbol{x}}}(t) = -k(L\otimes I_m)\hat{\overline{\boldsymbol{x}}}(t) - \overline{\boldsymbol{w}}(t) - \gamma\Psi(\overline{\boldsymbol{x}}(t)),$$
$$\dot{\overline{\boldsymbol{w}}}(t) = \alpha(L\otimes I_m)\hat{\overline{\boldsymbol{x}}}(t), \tag{36}$$

Let

$$e(t) = (T^T\otimes I_m)\overline{\boldsymbol{x}}(t), \quad \hat{e}(t) = (T^T\otimes I_m)\hat{\overline{\boldsymbol{x}}}(t),$$

$$\vartheta(t) = (T^T\otimes I_m)\overline{\boldsymbol{w}}(t), \quad T = \left[\frac{\mathbf{1}_N}{\sqrt{N}} \quad Q\right].$$

Denote $e=\mathrm{col}(e_1,e_2)$, $\hat{e}=\mathrm{col}(\hat{e}_1,\hat{e}_2)$ and $\vartheta=\mathrm{col}(\vartheta_1,\vartheta_2)$ with $e_1,\hat{e}_1,\vartheta_1\in\mathcal{R}^m$, and $e_2,\hat{e}_2,\vartheta_2\in\mathcal{R}^{m(N-1)}$. By the structure of T and (6), we can know T is an orthogonal matrix. Then system (36) can be rewritten as:

$$\dot{e}_1(t) = -\gamma\left(\frac{\mathbf{1}_N^T}{\sqrt{N}}\otimes I_m\right)\Psi(\overline{\boldsymbol{x}}(t)),$$
$$\dot{e}_2(t) = -k(H\otimes I_m)\hat{e}_2(t) - \vartheta_2(t) - \gamma(Q^T\otimes I_m)\Psi(\overline{\boldsymbol{x}}(t)),$$
$$\dot{\vartheta}_1(t) = 0,$$
$$\dot{\vartheta}_2(t) = \alpha(H\otimes I_m)\hat{e}_2(t). \tag{37}$$

Let $\varepsilon(t)=\mathrm{col}(e_2(t),\vartheta_2(t)), \hat{\varepsilon}(t)=\mathrm{col}(\hat{e}_2(t),\hat{\vartheta}_2(t))$, and $\tilde{\varepsilon}(t)=\varepsilon(t)-\hat{\varepsilon}(t)$, then

$$\dot{\varepsilon}(t) = C\varepsilon(t) - E\tilde{\varepsilon}(t) + F(t), \tag{38}$$

with

$$C = \begin{pmatrix} -kH & -I_{N-1} \\ \alpha H & 0\end{pmatrix}\otimes I_m, \quad E = \begin{pmatrix} -kH & 0 \\ \alpha H & 0\end{pmatrix}\otimes I_m,$$

and

$$F(t) = \begin{pmatrix} -\gamma(Q^T\otimes I_m)\Psi(\overline{\boldsymbol{x}}(t)) \\ 0\end{pmatrix}.$$

Theorem 2. *Suppose Assumptions 1 and 2 hold, the optimization problem (3) for multi-agent system (1) can be solved by the optimization control (33), if the following conditions are satisfied:*

$$\underline{m} - (4m+1)\gamma\overline{\rho}^2 > 0, \tag{39}$$

$$\underline{\lambda}_1 - 6m(1+2k^2\overline{\lambda}_2) - \frac{1}{2} > 0, \tag{40}$$

and

$$T_{\max}+\tau < \sqrt{\frac{m}{\overline{\lambda}_2[(k-\alpha)^2+6mk^2]}}, \tag{41}$$

where $m>0$ is a constant, $\underline{\lambda}_1=\lambda_{\min}(R)$, $\overline{\lambda}_2=\lambda_{\max}(H^TH)$, and $R=\begin{pmatrix}(k-\alpha)(H^T+H) & I_{N-1}\\ I_{N-1} & 2I_{N-1}\end{pmatrix}\otimes I_m$.

Proof. Consider Lyapunov function $V_1(t)$ given in Theorem 1, the derivation of V_1 along the first equality of (37) and system (38) yields:

$$\dot{V}_1 = e_1^T(t)\dot{e}_1(t) + \frac{1}{2}\varepsilon^T(t)(\Omega C + C^T\Omega)\varepsilon(t)$$
$$- \varepsilon^T(t)\Omega E\tilde{\varepsilon}(t) + \varepsilon^T(t)\Omega F(t), \tag{42}$$

Let $R=-(\Omega C+C^T\Omega)$, we obtain that R is positive definite and $k>\alpha$. Then, we have

$$-\varepsilon^T(t)\Omega E\tilde{\varepsilon}(t) = (k-\alpha)e_2^T(t)(H\otimes I_m)\tilde{e}_2(t)$$
$$\leq \frac{1}{4}e_2^T(t)e_2(t) + (k-\alpha)^2\tilde{e}_2^T(t)(H^TH\otimes I_m)\tilde{e}_2(t)$$
$$\leq \frac{1}{4}e_2^T(t)e_2(t) + (k-\alpha)^2\overline{\lambda}_2\tilde{e}_2^T(t)\tilde{e}_2(t), \tag{43}$$

where $\tilde{e}_2(t)=e_2(t)-\hat{e}_2(t)=(Q^T\otimes I_m)(\overline{\boldsymbol{x}}(t)-\hat{\overline{\boldsymbol{x}}}(t))$, and

$$\overline{\boldsymbol{x}}(t)-\hat{\overline{\boldsymbol{x}}}(t) = \mathrm{col}[\overline{x}_1(t)-\overline{x}_1(t_k^1), \overline{x}_2(t)-\overline{x}_2(t_k^2),\ldots,\overline{x}_N(t)-\overline{x}_N(t_k^N)]$$
$$= \mathrm{col}\left[\int_{t_k^1}^t \dot{\overline{x}}_1(s)ds, \int_{t_k^2}^t \dot{\overline{x}}_2(s)ds,\ldots,\int_{t_k^N}^t \dot{\overline{x}}_N(s)ds\right]. \tag{44}$$

Combining (20), (42) and (43), we have

$$\dot{V}_1(t) \leq -\gamma\underline{m}\overline{\boldsymbol{x}}^T(t)\overline{\boldsymbol{x}}(t) + \frac{1}{4}\vartheta_2^T(t)\vartheta_2(t) + \gamma^2\overline{\rho}^2\overline{\boldsymbol{x}}^T(t)\overline{\boldsymbol{x}}(t)$$
$$-\frac{1}{2}\varepsilon^T(t)R\varepsilon(t)$$
$$+\frac{1}{4}e_2^T(t)e_2(t) + (k-\alpha)^2\overline{\lambda}_2\tilde{e}_2^T(t)\tilde{e}_2(t)$$
$$\leq -\gamma\underline{m}\overline{\boldsymbol{x}}^T(t)\overline{\boldsymbol{x}}(t) + \frac{1}{4}\varepsilon^T(t)\varepsilon(t) + \gamma^2\overline{\rho}^2\overline{\boldsymbol{x}}^T(t)\overline{\boldsymbol{x}}(t)$$
$$-\frac{1}{2}\underline{\lambda}_1\varepsilon^T(t)\varepsilon(t) + (k-\alpha)^2\overline{\lambda}_2\tilde{e}_2^T(t)\tilde{e}_2(t). \tag{45}$$

Construct the following auxiliary integral function

$$V_2(t) = \int_{t-T_{\max}-\tau}^t \int_\theta^t \dot{e}^T(s)\dot{e}(s)dsd\sigma,$$

we can obtain

$$\dot{V}_2(t) = (T_{\max}+\tau)\dot{e}^T(t)\dot{e}(t) - \int_{t-T_{\max}-\tau}^t \dot{e}^T(s)\dot{e}(s)ds,$$

where

$$\dot{e}^T(t)\dot{e}(t) = \dot{e}_1^T(t)\dot{e}_1(t) + \dot{e}_2^T(t)\dot{e}_2(t),$$

with

$$\dot{e}_1^T(t)\dot{e}_1(t) = \gamma^2\left\|\left(\frac{\mathbf{1}_N^T}{\sqrt{N}}\otimes I_m\right)\Psi(\overline{\boldsymbol{x}}(t))\right\|^2 \leq \gamma^2\overline{\rho}^2\overline{\boldsymbol{x}}^T(t)\overline{\boldsymbol{x}}(t),$$

and

$$\dot{e}_2^T(t)\dot{e}_2(t) = k^2\hat{e}_2^T(t)(H^TH\otimes I_m)\hat{e}_2(t) + \vartheta_2^T(t)\vartheta_2(t)$$
$$+\gamma^2\|(Q^T\otimes I_m)\Psi(\overline{\boldsymbol{x}}(t))\|^2$$
$$+2k\hat{e}_2^T(t)(H^T\otimes I_m)\vartheta_2(t)$$
$$+2k\gamma\hat{e}_2^T(t)(H^T\otimes I_m)(Q^T\otimes I_m)\Psi(\overline{\boldsymbol{x}}(t))$$
$$+2\gamma\vartheta_2^T(t)(Q^T\otimes I_m)\Psi(\overline{\boldsymbol{x}}(t)). \tag{46}$$

Similarly to (24)–(27), and due to the fact that

$$\hat{e}_2^T(t)\hat{e}_2(t) = (e_2(t)-\tilde{e}_2(t))^T(e_2(t)-\tilde{e}_2(t)) \leq 2(e_2^T(t)e_2(t) + \tilde{e}_2^T(t)\tilde{e}_2(t)),$$

we have

$$\dot{e}_2^T(t)\dot{e}_2(t) \leq 3[k^2\hat{e}_2^T(t)(H^TH\otimes I_m)\hat{e}_2(t) + \vartheta_2^T(t)\vartheta_2(t) + \gamma^2\|(Q^T\otimes I_m)\Psi(\overline{\boldsymbol{x}}(t))\|^2]$$
$$\leq 3[k^2\overline{\lambda}_2\hat{e}_2^T(t)\hat{e}_2(t) + \vartheta_2^T(t)\vartheta_2(t) + \gamma^2\overline{\rho}^2\overline{\boldsymbol{x}}^T(t)\overline{\boldsymbol{x}}(t)]$$
$$\leq 6k^2\overline{\lambda}_2\tilde{e}_2^T(t)\tilde{e}_2(t) + 6k^2\overline{\lambda}_2 e_2^T(t)e_2(t) + 3\vartheta_2^T(t)\vartheta_2(t) + 3\gamma^2\overline{\rho}^2\overline{\boldsymbol{x}}^T(t)\overline{\boldsymbol{x}}(t)$$
$$\leq 6k^2\overline{\lambda}_2\tilde{e}_2^T(t)\tilde{e}_2(t) + 6k^2\overline{\lambda}_2\varepsilon^T(t)\varepsilon(t) + 3\varepsilon^T(t)\varepsilon(t) + 3\gamma^2\overline{\rho}^2\overline{\boldsymbol{x}}^T(t)\overline{\boldsymbol{x}}(t). \tag{47}$$

Let $V(t)=V_1(t)+\frac{m}{T_{\max}+\tau}V_2(t)$, then

$$\dot{V}(t) \leq -\gamma\underline{m}\overline{\boldsymbol{x}}^T(t)\overline{\boldsymbol{x}}(t) + \frac{1}{4}\varepsilon^T(t)\varepsilon(t) + \gamma^2\overline{\rho}^2\overline{\boldsymbol{x}}^T(t)\overline{\boldsymbol{x}}(t) - \frac{1}{2}\underline{\lambda}_1\varepsilon^T(t)\varepsilon(t)$$
$$+[(k-\alpha)^2+6mk^2]\overline{\lambda}_2\tilde{e}_2^T(t)\tilde{e}_2(t) + 6mk^2\overline{\lambda}_2\varepsilon^T(t)\varepsilon(t)$$
$$+3m\varepsilon^T(t)\varepsilon(t) + 4m\gamma^2\overline{\rho}^2\overline{\boldsymbol{x}}^T(t)\overline{\boldsymbol{x}}(t)$$
$$-\frac{m}{T_{\max}+\tau}\int_{t-T_{\max}-\tau}^t \dot{e}^T(s)\dot{e}(s)ds. \tag{48}$$

Recalling that $\tilde{e}_2(t) = (Q^T \otimes I_m)(\bar{\boldsymbol{x}}(t) - \hat{\bar{\boldsymbol{x}}}(t))$ and (44), it follows that

$$\begin{aligned}\tilde{e}_2^T(t)\tilde{e}_2(t) &= \|(Q^T \otimes I_m)(\bar{\boldsymbol{x}}(t) - \hat{\bar{\boldsymbol{x}}}(t))\|^2 \\ &\le \|\bar{\boldsymbol{x}}(t) - \hat{\bar{\boldsymbol{x}}}(t)\|^2 \\ &= \sum_{i=1}^{N}\left(\int_{t_k^i}^{t}\dot{\bar{x}}_i(s)ds\right)^T\left(\int_{t_k^i}^{t}\dot{\bar{x}}_i(s)ds\right). \end{aligned} \tag{49}$$

From Lemma 3, we have

$$\left(\int_{t_k^i}^{t}\dot{\bar{x}}_i(s)ds\right)^T\left(\int_{t_k^i}^{t}\dot{\bar{x}}_i(s)ds\right) \le (t - t_k^i)\int_{t_k^i}^{t}\dot{\bar{x}}_i^T(s)\dot{\bar{x}}_i(s)ds,$$

where $t \in [t_k^i + \tau_k^i, t_{k+1}^i + \tau_{k+1}^i)$, $t - t_k^i \le T_{\max} + \tau$, that is $t - T_{\max} - \tau \le t_k^i$, and $\dot{\bar{x}}^T(t)\dot{\bar{x}}(t) = \dot{e}^T(t)\dot{e}(t)$.

Then, we have

$$\begin{aligned}\tilde{e}_2^T(t)\tilde{e}_2(t) &\le \sum_{i=1}^{N}(t - t_k^i)\int_{t_k^i}^{t}\dot{\bar{x}}_i^T(s)\dot{\bar{x}}_i(s)ds \\ &\le \sum_{i=1}^{N}(T_{\max} + \tau)\int_{t-T_{\max}-\tau}^{t}\dot{\bar{x}}_i^T(s)\dot{\bar{x}}_i(s)ds \\ &= (T_{\max} + \tau)\int_{t-T_{\max}-\tau}^{t}\sum_{i=1}^{N}\dot{\bar{x}}_i^T(s)\dot{\bar{x}}_i(s)ds \\ &= (T_{\max} + \tau)\int_{t-T_{\max}-\tau}^{t}\dot{\bar{x}}^T(s)\dot{\bar{x}}(s)ds \\ &= (T_{\max} + \tau)\int_{t-T_{\max}-\tau}^{t}\dot{e}^T(s)\dot{e}(s)ds. \end{aligned} \tag{50}$$

From (48) and (50), we have

$$\begin{aligned}\dot{V}(t) \le& -\gamma\underline{m}\bar{\boldsymbol{x}}^T(t)\bar{\boldsymbol{x}}(t) + \frac{1}{4}\varepsilon^T(t)\varepsilon(t) + \gamma^2\overline{\rho}^2\bar{\boldsymbol{x}}^T(t)\bar{\boldsymbol{x}}(t) \\ & - \frac{1}{2}\underline{\lambda}_1\varepsilon^T(t)\varepsilon(t) \\ & + [(k-\alpha)^2 + 6mk^2]\overline{\lambda}_2(T_{\max}+\tau)\int_{t-T_{\max}-\tau}^{t}\dot{e}^T(s)\dot{e}(s)ds \\ & + 6mk^2\overline{\lambda}_2\varepsilon^T(t)\varepsilon(t) + 3m\varepsilon^T(t)\varepsilon(t) + 4m\gamma^2\overline{\rho}^2\bar{\boldsymbol{x}}^T(t)\bar{\boldsymbol{x}}(t) \\ & - \frac{m}{T_{\max}+\tau}\int_{t-T_{\max}-\tau}^{t}\dot{e}^T(s)\dot{e}(s)ds, \end{aligned} \tag{51}$$

and then

$$\begin{aligned}\dot{V}(t) \le& -[\gamma\underline{m} - (4m+1)\gamma^2\overline{\rho}^2]\bar{\boldsymbol{x}}^T(t)\bar{\boldsymbol{x}}(t) \\ & - \left[\frac{1}{2}\underline{\lambda}_1 - 3m(1 + 2k^2\overline{\lambda}_2) - \frac{1}{4}\right]\varepsilon^T(t)\varepsilon(t) \\ & - \left[\frac{m}{T_{\max}+\tau} - ((k-\alpha)^2 + 6mk^2)\overline{\lambda}_2(T_{\max}+\tau)\right] \\ & \times \int_{t-T_{\max}-\tau}^{t}\dot{e}^T(s)\dot{e}(s)ds. \end{aligned} \tag{52}$$

By the similar analysis as the proof of Theorem 1, we can conclude that $\dot{V}(t) < 0$, which completes the proof. □

4. Simulations

In this section, we give an example to validate our theoretical results. In this example, we consider a multi-agent system consisting of five agents. Supposes that the interconnected topology is described as in Fig. 1. The weight of every edge is 1.

Consider the following optimization problem

$$\text{minimize} \quad f(x) = \sum_{i=1}^{N} f_i(x), x \in \mathcal{R},$$

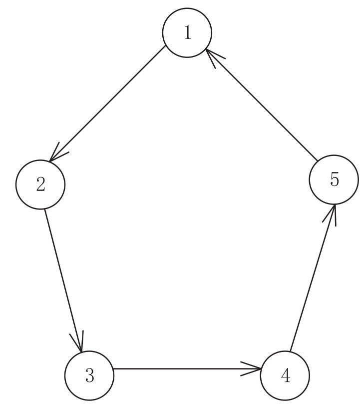

Fig. 1. Connected graph.

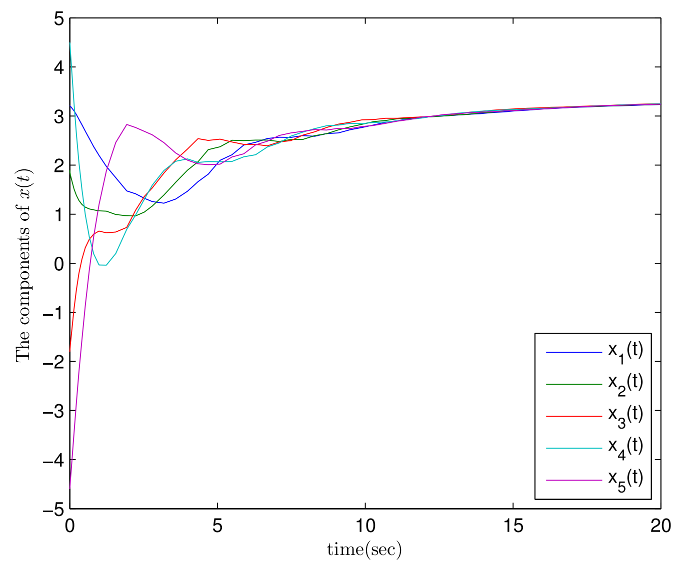

Fig. 2. The trajectories of $x_i(t)$ with $T_k = 0.01$, $\tau_k = 0.07$.

where the local objective function is given as following

$$\begin{aligned} f_1(x) &= 0.7(x-6)^2, \\ f_2(x) &= (x-4)^2, \\ f_3(x) &= \frac{x^2}{\ln(x^2+2)}, \\ f_4(x) &= \sin\frac{x}{2} + \frac{x^2}{4}, \\ f_5(x) &= \frac{x^2}{\sqrt{x^2+1}} + 0.2x^2. \end{aligned} \tag{53}$$

Obviously, for $i = 1, 2, \ldots, 5$, f_i is differentiable and satisfies Assumption 1. Choosing $\alpha = 0.6, k = 1.0, \gamma = 0.2$, we can obtain $\overline{\rho} = 2, \underline{m} = 1, \underline{\lambda}_1 = 0.6861, \overline{\lambda}_2 = 3.6180$. Let the initial values

$$x(0) = [x_1(0), x_2(0), x_3(0), x_4(0), x_5(0)]^T = [-3.2, 1.9, -1.8, 4.5, -4.6]^T,$$

and

$$w(0) = [w_1(0), w_2(0), w_3(0), w_4(0), w_5(0)]^T = [0, 0, 0, 0, 0]^T.$$

(1) synchronous sampling: The sampling interval T_k and the time-delay τ_k are given as $T_k = 0.01s$, $\tau_k = 0.07$. The simulation results are shown in Figs. 2 and 3.

(2) asynchronous sampling: The sampling interval is given as $T = 0.01s$, time-delay $\tau_k^i (i = 1, 2, \ldots, 5)$ are simulated by random numbers in the interval $[0, 0.3T]$. The simulation results are shown in Figs. 4 and 5.

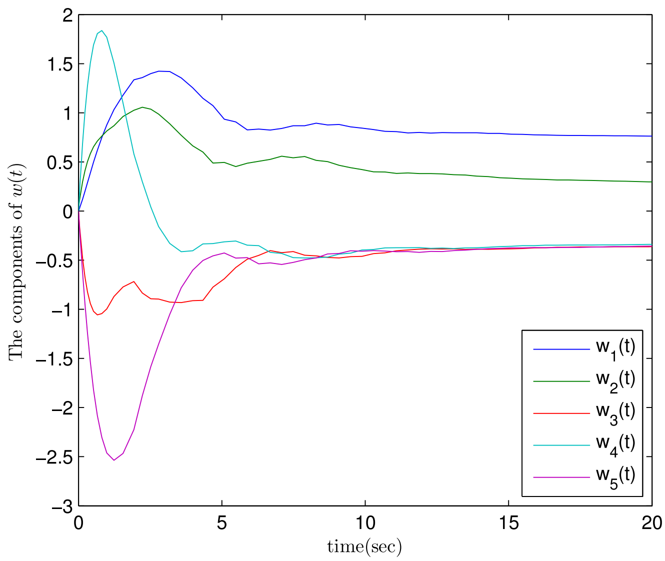

Fig. 3. The trajectories of $w_i(t)$ with $T_k = 0.01$, $\tau_k = 0.07$.

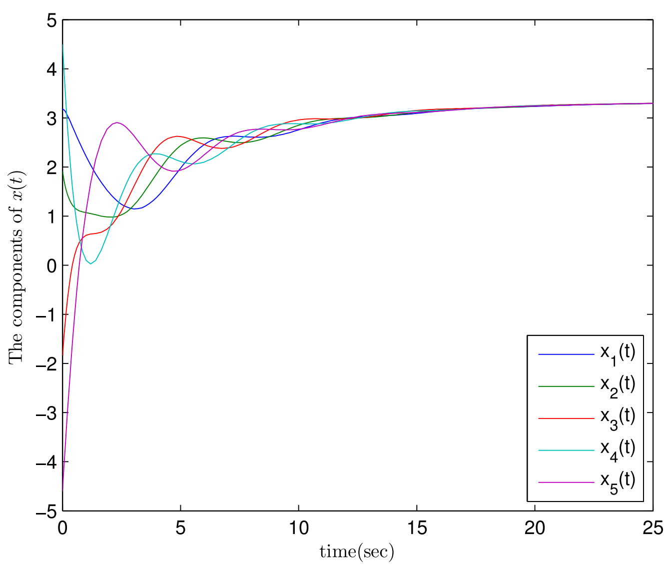

Fig. 4. The trajectories of $x_i(t)$ with $T = 0.01$, $\tau_k^i \in [0, 0.3T]$.

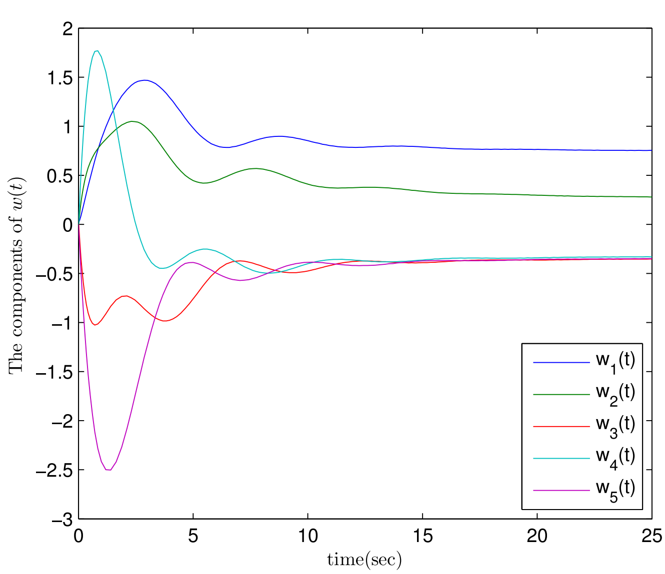

Fig. 5. The trajectories of $w_i(t)$ with $T = 0.01$, $\tau_k^i \in [0, 0.3T]$.

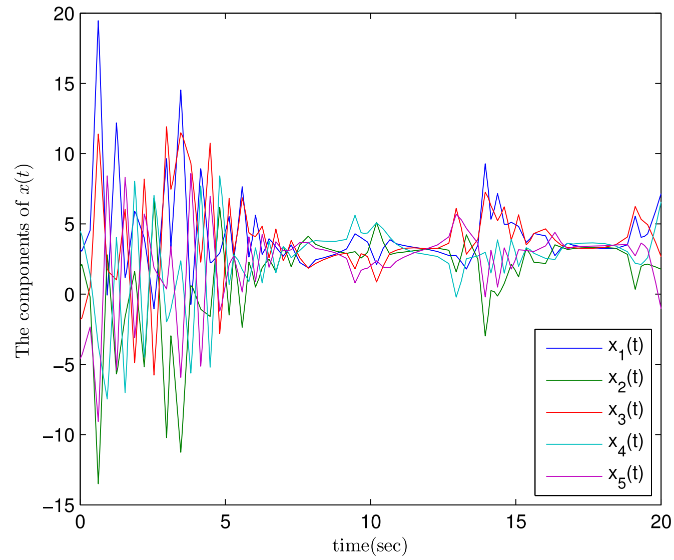

Fig. 6. The trajectories of $x_i(t)$ with $T_0 = 0.09$.

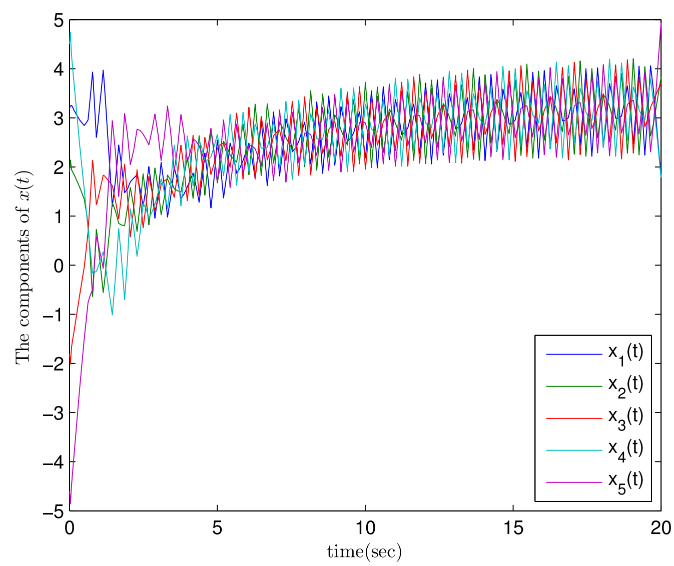

Fig. 7. The trajectories of $x_i(t)$ with $T_1 = 0.07$.

We can see that the trajectories x_i of each agent i converge to the global optimal solution $x^* = 3.1798$ of the objective function $f(x) = \sum_{i=1}^{N} f_i(x)$ and all the trajectories w_i converge to a constant, respectively, for $i = 1, 2, \ldots, 5$. The optimal value of $f(x)$ is 18.8773.

The simulation result for synchronous sampling with $T_0 = 0.09$ is depicted in Fig. 6 and asynchronous sampling with $T_1 = 0.07$ is depicted in Fig. 7, respectively, where T_0 and T_1 are larger than the upper bound $T_{\max}$ in Theorems 1 and2. Then $x_i(t)$ is not convergent.

5. Conclusion

In this paper, a distributed optimization problem of multi-agent systems with delayed sampled-data is considered. The interconnected topology is assumed to be directed, weighted-balanced and strongly connected, and the local costs are strongly convex with globally Lipschitz gradients. Two control algorithms using sampled-data with time-delay under synchronous and asynchronous sampling are presented for the multi-agent systems to reach consensus and optimal state. Based on Lyapunov theory and algebraic graph theory. Sufficient conditions are obtained to make all the agents converge to the optimal solution of the system if the design parameters are chosen properly. Finally, numerical example are given to illustrate the theoretical results.

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Junxiu Yan received her B.S. degree in Mathematics from Changzhi University, in 2011, and now she is a master degree candidate in Mathematics in China Three Gorges University. Her research interests include complex dynamical network, distributed optimization, and multi-agent systems.

Hui Yu received his bachelor's degree in mathematics from Central China Normal University, Wuhan, China in 1991, his master's degree in system analysis and integration from East China Normal University, Shanghai, China in 1999, and his Ph.D. degree in Automatic control from Huazhong University of Science and Technology, Wuhan, China in 2007. From 2011 to 2012, he was visiting the Centre of New Energy Systems, Department of Electrical, Electronics and Computer Engineering, University of Pretoria, South Africa. Since 1991, he has been with School of Science, China Three Gorges University, where he is currently a professor. His research interests include nonlinear control theory with applications to robots, multi-agent systems, energy efficiency.

Xiaohua Xia is a professor in the Electrical, Electronic and Computer Engineering at the University of Pretoria, South Africa, director of the Centre of New Energy Systems, and the director of the National Hub for the Postgraduate Programme in Energy Efficiency and Demand-side Management. He was academically affiliated with the University of Stuttgart, Germany, the Ecole Centrale de Nantes, France, and the National University of Singapore before joining the University of Pretoria in 1998. Prof. Xia is a fellow of the Institute for Electronic and Electrical Engineers (IEEE), a fellow of the South African Academy of Engineering (SAAE), a member of the Academy of Science of South Africa (ASSAF), and he has an A rating from the South African National Research Foundation (NRF). He has been an associate editor of Automatica, IEEE Transactions on Circuits and Systems II, IEEE Transactions on Automatic Control, and specialist editor (control) of the SAIEE Africa Research Journal. His research interests are modeling and control of nonlinear systems, energy systems and optimization.