Adaptive finite-time consensus in multi-agent networks

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A B S T R A C T
This paper is concerned with the finite-time consensus problem of distributed agents having non-
identical unknown nonlinear dynamics, to a leader agent that also has unknown nonlinear control input
signal. By parameterization of unknown nonlinear dynamics, a Lyapunov technique in conjunction with
homogeneity technique is presented for designing a decentralized adaptive finite-time consensus control
protocol in undirected networks. Homogeneous Lyapunov functions and homogeneous vector fields are
introduced in the stability analysis although the whole system is not homogeneous. Theoretical analysis
shows that leader-following consensus can be achieved in finite-time, meanwhile, finite-time parameter
convergence can be also guaranteed under the proposed control scheme. An example is given to validate
the theoretical results.

1. Introduction

In recent years, studies on the distributed coordination of multi-
agent systems have attracted a lot of attention in control and
robotics. Its broad applications can be found in diverse areas,
including multi-vehicle rendezvous, formation control of multi-
robots, flocking, swarming, distributed sensor fusion, attitude
alignment, and congestion control in communication networks.
One of the main challenges in cooperative control is to design
decentralized control schemes such that some group objective can
be achieved in a distributed fashion. A particularly interesting topic
in cooperative control is the consensus problem of multi-agent sys-
tems. Early well-known works on the consensus problem of multi-
agent systems can be found in [1–5], to name just a few.

An interesting topic in multi-agent systems is the finite-time
consensus problem, which is extensively studied in the litera-
tures [6–14] for multi-agent systems with single or double integra-
tor dynamics. Two finite-time consensus protocols are proposed
in [6] for continuous-time systems, under either of which, the
differential equations of the overall systems have discontinuous
right-hand sides by nonsmooth stability analysis. In [7], the results
on finite-time semistability are applied to developing finite-time
consensus protocols in nonlinear dynamical networks. The termi-
nal sliding mode technique is used in [8] to design finite-time con-

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to solve the consensus problem of multi-agents with uncertainties and external disturbances in undirected networks. In [18], the authors presented a design method for adaptive synchronization control laws for distributed systems having non-identical unknown nonlinear dynamics, and for a target dynamics to be tracked that is also nonlinear and unknown. Under some assumptions, the authors proved that the overall local cooperative error vector and the neural network weight estimation errors are both uniformly ultimately bounded. In [19], an adaptive consensus design method is presented for multi–agent systems with non-identical unknown nonlinear dynamics, and for a leader to be followed that is also nonlinear and unknown in networks with jointly connected topologies. Both consensus stability and parameter convergence are analyzed. In [20,21], the consensus problem of high order multi-agent systems with unknown nonlineardynamics; secondly, homogeneous Lyapunov techniques and homogeneity of partial terms of vector fields hold homogeneity, so the finite-time Lyapunov function is constructedinthestabilityanalysis.

However, the whole system is nonexistent and nonlinear, in which only the partial terms of the vector field hold homogeneity, so the finite-time stability theory of homogeneous systems cannot be applied directly. We use Lyapunov techniques and homogeneity, in conjunction with some inequality techniques to derive our stability results, such that finite-time consensus and finite-time parameter convergence are both achieved globally. The connectivity of multi-agent networks and the persistent excitation condition are crucial in finite-time consensus and finite-time parameter convergence, respectively.

The contributions of this paper are in three aspects. Firstly, a novel type of decentralized adaptive finite-time consensus algorithm is proposed for leader-following multi-agent systems with unknown nonlinear dynamics; secondly, homogeneous Lyapunov function and homogeneous vector fields are introduced in the finite-time stability analysis of multi-agent systems although the whole system is not homogeneous; finally, under the PE condition, finite-time parameter convergence is also guaranteed.

This paper is organized as follows. In Section 2, we establish the controller and formaly state the problem. We present our main results in Section 3, the simulation results supporting the objectives of the paper in Section 4 and the concluding remarks in Section 5.

2. Problem statement

Consider a multi-agent system consisting of N agents and a leader. The dynamics of the ith (i = 1, 2, . . ., N) agent is described by

\[ \dot{x}_i(t) = A_0 x_i(t) + b_i (f_i(x_i(t), t) + u_i(t)), \]

where

- \( A_0 \) is a matrix,
- \( b_i \) is a vector,
- \( x_i(t) \) is the state of the ith agent,
- \( f_i(x_i(t), t) \) is the nonlinear dynamics of the ith agent,
- \( u_i(t) \) is the control input of the ith agent.

More specifically, for an adaptive design by parameterizations of the unknown nonlinear dynamics, it is another problem to guarantee finite-time parameters converge in the meantime.

In this paper, we consider the finite-time consensus problem of leader-following multi-agent systems, in which the leader’s control input signal is unknown and nonlinear, the followers have unknown, non-identical, nonlinear dynamics. By parameterization of unknown nonlinear dynamics, under the assumption of connectivity of multi-agent networks and the persistent excitation assumption of the regressor matrix, a decentralized adaptive finite-time control scheme is proposed for the considered multi-agent systems to reach consensus with parameter convergence in finite-time via relative states and local consensus error feedback of neighboring agents. The stability analysis is conducted based on Lyapunov techniques and homogeneity of partial terms of vector fields. A homogeneous Lyapunov function is constructed in the stability analysis. However, the whole system is not homogeneous, in which only partial terms of the vector field hold homogeneity, so the finite-time stability theory of homogeneous systems cannot be applied directly. We use Lyapunov techniques and homogeneity, in conjunction with some inequality techniques to derive our stability results, such that finite-time consensus and finite-time parameter convergence are both achieved globally. The connectivity of multi-agent networks and the persistent excitation condition are crucial in finite-time consensus and finite-time parameter convergence, respectively.
We have
\[ \hat{u}_i(t) = \phi_i^T(t) \hat{\theta}_0, \quad i = 1, 2, \ldots, N. \]  
(4)
Similarly, the estimate of \( f(x_i(t), t) \) is expressed as
\[ \hat{f}_i(x_i(t), t) = \phi_i^T(x_i(t), t) \hat{\theta}_0, \quad i = 1, 2, \ldots, N. \]  
(5)

**Remark 1.** The unknown nonlinear dynamics of all agents are assumed to be linearly parameterized. The linearly parameterized models have been studied widely in classical adaptive control [25]. The examples of a linearly parameterized model of multi-agent systems can be found in [15, 16, 19, 21].

Let \( x(t) = \text{col}(x_1, \ldots, x_N) \) be the stack column vector of \( x_1, \ldots, x_N \). The objectives of this work are to design a decentralized adaptive finite-time consensus scheme such that leader-following consensus can be reached in finite time and finite-time parameter convergence can be guaranteed in the meantime, that is
\[
\lim_{t \to T_i} \| x_i(t) - x_0(t) \| = 0, \quad \lim_{t \to T_i} \| \hat{\theta}_0 - \theta_0 \| = 0, \\
\lim_{t \to T_i} \| \hat{\theta}_i - \theta_i \| = 0, \\
\text{for any initial condition } x_0(0), x(0), \text{where } T_i > 0 \text{ is the settling time.}
\]

**3. Main results**

Define local neighborhood consensus error [8] for agent \( i \) as
\[
\zeta_i(t) = \sum_{j \in N_i} a_j(\xi_j - \xi_i) + b_j(\xi_j - \xi_0),
\]
where \( k = 0, 1, \ldots, l-1, i = 1, 2, \ldots, N \). For agent \( i \), we propose the following \( l \)-th order control algorithm:
\[ u_i(t) = -\gamma T \hat{\theta}_i(t) - (1 - \rho) \sum_{k=0}^{l-1} c_k e^{(1-\mu)(1-\mu)} \zeta_i^{(k)} + \Phi_i^T \hat{\theta}_i, \]
and
\[ \dot{\hat{\theta}}_0 = -\gamma \phi_0(t) \Phi^T \sum_{j=0}^{N} a_j(\xi_j - e_j) + b_j \xi_j, \]
\[ \dot{\hat{\theta}}_i = -\gamma \phi_i(t) \hat{e}_i \Phi^T \sum_{j=0}^{N} a_j(\xi_j - e_j) + b_j \xi_j, \]
where \( e_i(t) = (\xi_i(0) - e_i(0), \ldots, \xi_i(l-1) - e_i(l-1)). \)

**Remark 2.** Note that controller \( u_i(t) \) defined in (8) and the adaptive laws defined in (9) are decentralized. For control purposes, we assume that the information of local consensus error vector \( e_i(t) \) of agent \( i \) is calculated in real-time and saved in its memory at each time instant by each agent and is available for its neighbors. Controller \( u_i(t) \) and the adaptive laws defined in (9) only depend on the information of relative position measurements and local consensus errors feedback from its neighboring agents. A similar method of information transmission can be found, for instance, in the literatures [8, 19, 26, 27].

Let \( [e_i]_{\mu} = \text{col}(e_i(0)^{(1-\mu)}, e_i(l)^{(1-\mu)}, e_i(1)^{(1-\mu)}, \ldots, e_i(l-1)^{(1-\mu)}) \), \( \epsilon = \text{col}(\epsilon_1, \epsilon_2, \ldots, \epsilon_N), \delta = \text{col}(\delta_1, \delta_2, \ldots, \delta_N), \Phi = \text{diag}(\Phi_1, \Phi_2, \ldots, \Phi_N), \sigma = (\alpha_0, \alpha_1, \ldots, \alpha_{t-1})^T, e(t) = \text{col}(e_1, \ldots, e_N), u = \text{col}(u_1, \ldots, u_N) \), we have
\[
u = (I_N \otimes \hat{e}_i^T) e - (1 - \rho)(I_N \otimes \hat{e}_i^T) [\epsilon]_{\mu} + \Phi^T \hat{\theta}, \]
(10)
where \( I_N \) is the \( N \times N \) identity matrix.

With (10), letting \( \Theta_i = \text{col}(\theta_0, \theta_i), \Theta = \text{col}(\Theta_1, \Theta_2, \ldots, \Theta_N), \hat{\Theta} = \text{col}(\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_N), \hat{f} = \text{col}(f_1, f_2, \ldots, f_N), \hat{x}(t) = x(t) - 1_N \otimes x_0(t) \), and noting that \( e(t) = (H \otimes I(t))^T x(t) \) and \( \hat{x}(t) = (l_N \otimes \hat{\theta}_0)x(t) + (l_N \otimes b)(f + u) \), we have
\[
\hat{e}(t) = (H \otimes I(t)) \hat{x}(t) - 1_N \otimes \hat{x}(t) = (H \otimes I(t))^T x(t) - (1 - \rho)(H \otimes \hat{\theta}_0)\epsilon + \Phi^T \hat{\theta},
\]
and
\[
\hat{e}(t) = -\gamma \Phi(t)^T e(t) - (1 - \rho)(H \otimes \hat{\theta}_0)\epsilon + \Phi^T \hat{\theta},
\]
(11)
(12)
Let \( \hat{e}(t) = e^{-1}\gamma \Phi(t)^T e(t) \), the system (11)–(12) can be rewritten as
\[
\hat{e}(t) = (l_N \otimes \hat{\theta}_0)\epsilon(t) - (H \otimes \hat{\theta}_0)\epsilon(t) - (1 - \rho)(H \otimes \hat{\theta}_0)\epsilon + \Phi^T \hat{\theta},
\]
and
\[
\hat{e}(t) = -\gamma \Phi(t)^T e(t),
\]
(13)
(14)
**Assumption 1.** Graph \( \hat{g} \) is connected.

For a symmetric matrix \( P \), by \( P > 0 \) we mean that \( P \) is positive definite. Under Assumption 1 and from Lemma 3, \( H \) is symmetric positive definite and all of eigenvalues of \( H \) are positive. Let \( \lambda_{\min}, \lambda_{\max} > 0 \) and \( \tilde{\lambda} \) be the smallest, largest eigenvalue of \( H \) and \( \tilde{P}, \) respectively. Since \( (\hat{a}_0, \hat{b}) \) is stabilizable, there exists a positive definite matrix \( P = (p_{ij})_{i,j} > 0 \) such that the following Riccati inequality
\[
A \hat{P} + P A - 2 \lambda_{\min} \hat{P} \hat{b}^T \hat{b} \hat{P} < -\lambda_{\min} \hat{P},
\]
holds with the constant vector \( \tilde{c}^T = \tilde{b}^T \hat{P} \). There also exists \( 0 < \rho_1 < 1 \) such that \( \rho_1 < \rho < 1 \), (15) and
\[
A \hat{P} + P A - 2 \lambda_{\min} \hat{P} \hat{b}^T \hat{b} \hat{P} < - \frac{1}{2} \lambda_{\min} \hat{P},
\]
hold with \( \tilde{c}^T = \tilde{b}^T \hat{P} \).

**3.1. Finite-time stability of homogeneous system**

In this subsection, we firstly consider the homogeneous part of the error system [13–14]. The following definitions are needed in the theoretical analysis.
Definition 1 ([29]). A function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is homogeneous of degree $d \in \mathbb{R}$ with respect to weights $(r_1, \ldots, r_n) \in \mathbb{R}_+^n$ if

$$V(\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n) = \lambda^d V(x_1, \ldots, x_n), \quad \forall \lambda > 0.$$  \hfill (17)

Definition 2 ([29]). A vector field $g$ is homogeneous of degree $d \in \mathbb{R}$ with respect to weights $(r_1, \ldots, r_n) \in \mathbb{R}_+^n$ if for all $1 \leq i \leq n$, the $i$th component $g_i$ of $g$ is homogeneous function of degree $n_i + d$, i.e.,

$$g_i(\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n) = \lambda^{n_i + d} g_i(x_1, \ldots, x_n), \quad \forall \lambda > 0.$$  \hfill (18)

For the homogeneous part of the error system (13)-(14), we give the following lemma.

Lemma 4. Under Assumption 1 and (15) with $\tilde{e}^T = \tilde{b}^T P$, consider system

$$\dot{e}(t) = ((b_0 \otimes A_0) e(t) - (H \otimes \tilde{b}^T) \tilde{e})^T.$$  \hfill (19)

There exists $\epsilon_0 \in (0, 1)$ such that, for every $\alpha \in (1 - \epsilon_0, 1)$, the origin is a globally finite-time stable equilibrium for system (19), \hfill (20)

with $\alpha_1 = 1$ and $\alpha_{i-1} = \alpha$.

Proof. Let $H = (h_0)_{N \times N}$, we have

$$(H \otimes \tilde{b}^T) \tilde{e} = \begin{bmatrix} H \otimes \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} & 0 \end{bmatrix} \begin{bmatrix} \epsilon_0 \\ \alpha_1 \\ \vdots \\ \alpha_{i-1} \end{bmatrix} = \begin{bmatrix} F_1(\epsilon) \\ F_2(\epsilon) \\ \vdots \\ F_N(\epsilon) \end{bmatrix},$$  \hfill (21)

where $F_i(\epsilon) = (0, \ldots, 0, \sum_{j=1}^{i-1} h_j \sum_{k=i}^{n} c_k C^{(k-1)-1}(\epsilon)^{\alpha_k})$.\hfill (22)

Let $f^\alpha$ denote the closed-loop vector field of system (19). From Proposition 8.1 of [29], it is easy to verify that, for each $\alpha > 0$, the vector field $f^\alpha$ is continuous, homogeneous of degree $\frac{d-1}{\alpha}$ with respect to the weights

$$\begin{bmatrix} 1 & 1 & \ldots & 1 \\ \frac{1}{\alpha_0} & \frac{1}{\alpha_1} & \ldots & \frac{1}{\alpha_{i-1}} \end{bmatrix},$$  \hfill (23)

where $\alpha_0, \alpha_1, \ldots, \alpha_{i-1}$ satisfy (20) with $\alpha_i = 1$ and $\alpha_{i-1} = \alpha$. Moreover, noting that $\alpha_0 = \alpha_1 = \cdots = \alpha_{i-1} = 1$ as $\alpha = 1$, the vector field $f^1$ corresponds to the system

$$(\dot{e}(t) = ((b_0 \otimes A_0) e(t) - (H \otimes \tilde{b}^T) \tilde{e}(t)).$$  \hfill (24)

Let $A = \text{diag}(\lambda_1, \ldots, \lambda_d)$ with $\lambda_1, \ldots, \lambda_d$ being eigenvalues of matrix $H$. Because $H$ is symmetric, there exists an orthogonal matrix $U$ such that $U^T H U = A$. Setting $\tilde{e}(t) = (U \otimes I) e(t)$ for the system (22), one has

$$(\dot{\tilde{e}}(t) = ((b_0 \otimes A_0) \tilde{e}(t) - (A \otimes \tilde{b}^T) \tilde{e}(t)).$$  \hfill (25)

Consider Lyapunov function candidate $W_0(\tilde{e}) = \tilde{e}^T (b_0 \otimes P) \tilde{e}$, where $P > 0$ is given in (15). Calculating the derivative along the solution of the system (23), we have

$$\frac{dW_0}{dt} = \tilde{e}^T [b_0 \otimes (A_0^T P + PA_0) - A \otimes (2P \tilde{b}^T \tilde{b})] \tilde{e}$$

$$\leq \sum_{i=1}^{N} \tilde{e}_i^T (A_0^T P + PA_0 - 2\min \hat{b}_i \tilde{b}_i^T \tilde{b}) \tilde{e}_i$$

$$\leq \sum_{i=1}^{N} \tilde{e}_i^T \tilde{e}_i = -\lambda_{\min} \sum_{i=1}^{N} \tilde{e}_i^T \tilde{e}_i < 0, \quad \forall \tilde{e} \neq 0.$$  \hfill (26)

Therefore, the origin of the system (23) is asymptotically stable.

By Theorem 6.2 of [29], there exists a positive-definite, radially unbounded, Lyapunov function $V(\epsilon(t))$ such that $\frac{d}{dt} V(\epsilon(t))$ is continuous and negative definite. Let $\mathcal{A} = \mathbb{W}^{-1} \{ (0, 1) \}$ and the boundary of set $\mathcal{A}$ be $\delta = \mathbb{W}^{-1} \{ (1) \}$. Then $\mathcal{A}$ and $\delta$ are compact since $W$ is proper and $0 \notin \delta$ since $W$ is positive definite. Define $\phi : (0, 1) \times \delta \rightarrow \mathcal{A}$ by noting that, for every $\epsilon \in (1 - \epsilon_0, 1)$, $L_\epsilon \delta$ is the $L$-derivative with respect to $\epsilon$. Then $\phi$ is continuous and satisfies $\phi(1, \epsilon) \in \{ (1 - \epsilon_0, 1) \}$ since $\phi(1, \epsilon) \in \{ (1 - \epsilon_0, 1) \}$. It follows that for $\epsilon \in (1 - \epsilon_0, 1)$, $L_\epsilon \phi \delta$ satisfies positive values on $\delta$. Thus, as is strictly positively invariant under $f^\alpha$ for every $\alpha \in (1 - \epsilon_0, 1)$. The result now follows from Theorems 7.1 and 7.3 of [29] by noting that, for every $\epsilon \in (1 - \epsilon_0, 1)$, the degree of homogeneity of $f^\alpha$ with respect to the weights (21) is negative.

Remark 3. Similar to [29], the uniqueness of solutions of system (19) is based on forward uniqueness. The vector fields considered in (19) are locally Lipschitz everywhere except on a finite collection of submanifolds. Moreover, the vector field is transversal to each such submanifold everywhere except at the origin. Hence forward uniqueness for all initial conditions except the origin follows from Remark 8.1 of [29] and references therein, while forward uniqueness at the origin follows from Lyapunov stability.

As in [30], for system (19), we construct the following homogeneous Lyapunov function

$$V_0(\epsilon) = \int_{0}^{\epsilon} \frac{1}{\lambda^{d-1} \alpha} (\alpha \circ W_0) \left( \sum_{i=1}^{d} \lambda_{\alpha}^{\frac{1}{i-1}} \epsilon_{(i-1)}^{(i-1)} \right) d\lambda_{\epsilon}, \quad \epsilon \in \mathbb{R}^N \setminus \{ 0 \}, \quad \epsilon = 0,$$  \hfill (27)

where $d > p > 2$, $p$ are positive integers, $\alpha \in (1 - \epsilon_1, 1)$ with $\epsilon_1 = \frac{p}{2p - 2}$, $W_0(\epsilon) = \epsilon^T (b_0 \otimes P) \epsilon$, in which $P$ satisfies (15), and $\alpha(s) \in C^\infty(\mathcal{R}, \mathcal{R})$ is such that $\alpha(s) = \left\{ \begin{array}{ll} 0, & s \in (1 - \epsilon_1, 1) \\ 1 - 2(s - 2)^2, & s \in (3, 2), \\ 0, & s \in (2, \infty). \end{array} \right.$  \hfill (28)

An example of function $\alpha(s)$ can be constructed as:}
Then, the derivative of function $a(s)$ is

$$a'(s) = \begin{cases} 0, & s \in (-\infty, 1), \\ 4(s - 1), & s \in \left(1, \frac{3}{2}\right], \\ -4(s - 2), & s \in \left(\frac{3}{2}, 2\right), \\ 0, & s \in (2, \infty). \end{cases}$$

(26)

**Remark 4.** Choosing different values of $p$ provides flexibility for the lower bound of $\alpha$. Large values of $p$ result in the lower bound of $\alpha$ close to 0, otherwise 1.

**Lemma 5.** If $d > p > 2$, $d, p$ are positive integers, $\alpha \in (1 - \epsilon_1, 1)$ with $\epsilon_1 = \frac{p-2}{2p+2}$, then, (i) $\min_{0 \leq k \leq l - 1} |a_{kl}| > 2$, (ii) $d > \max_{0 \leq k \leq l - 1} \left(\frac{1}{a_k}\right)$, and (iii) $\epsilon < \frac{\alpha d + 1 - \alpha d}{\alpha d - 1} < 1$ hold, where $a_{kl}$ is defined in (20).

**Proof.** From (20), it is easy to verify that

$$0 < a_{00} = \frac{\alpha}{1 - (1 - \alpha)\epsilon} < \alpha_1 < \cdots < \alpha_l = 1 < \alpha. \quad (27)$$

To prove (i), we only need to prove $p_{00} - 2 > 0$ due to (27). Since $p_{00} - 2 = (\alpha - \frac{2d}{2 + p}) - \frac{2 + 2d}{\alpha_0 - 1} \epsilon_1 > 0$, therefore, (i) holds. To prove (ii), we only need to prove $d_{00} > 1$ due to (27). It obviously holds due to $d > p > 2$ and (i). To prove (iii), we only need to prove $d > 1 - \frac{\epsilon_1}{\epsilon_2}$, which is always true. ■

**Lemma 6.** (i) $V_a(e)$ is $C^1$ on $\mathbb{R}^N$, positive definite, and homogeneous of degree $d$ with respect to the weights defined in (21); (ii) $\forall \epsilon > 0$, $V_a'(\epsilon)$ satisfies

$$\frac{dV_a(\epsilon)}{d\epsilon} \bigg|_{\epsilon = 0} = -\kappa_1V_a(0), \quad \epsilon \in \mathbb{R}^N,$$

where $\kappa_1 > 0$ is some positive constant number, $\alpha \in (1 - \epsilon_2, 1)$ with $\epsilon_2 = \min(\epsilon_0, \epsilon_1)$.

**Proof.** (i) From Lemma 5 and similar to the proof of Theorem 2 in [30], we can prove (i) and (ii) easily.

(iii) For each $\alpha > 0$, the vector field $f^\alpha$ is continuous, homogeneous of degree $\frac{d+1}{\alpha}$ with respect to the weights defined in (21).

From (i), $V_a$ is $C^1$ on $\mathbb{R}^N$ and homogeneous of degree $d$ with respect to the weights defined in (21). It follows that $\frac{dV_a}{d\epsilon}$ is continuous on $\mathbb{R}^N$ and homogeneous of degree $d + 1 - \frac{1}{\alpha}$. Lemma 4.2 of [29] and (ii) imply that there exists a constant number $\kappa_1 > 0$ such that

$$\frac{dV_a(\epsilon)}{d\epsilon} \bigg|_{\epsilon = 0} = -\kappa_1V_a(0), \quad \epsilon \in \mathbb{R}^N. \quad \blacksquare$$

### 3.2. Adaptive finite-time consensus

In this subsection, we give a theoretical analysis for adaptive finite-time consensus achievement of the system (1)–(2). To derive our main result, the following persistent excitation [25] assumption is needed to guarantee finite-parameter convergence.

**Assumption 2.** The regressor matrix $\Phi = \text{diag}[\Phi_1, \Phi_2, \ldots, \Phi_N]$ is persistently exciting (PE) [25], that is, there exist two positive real $\nu$ and $\kappa_0$ such that

$$\int_{t_1}^{t_1+\tau} \Phi^T f \tau \geq \kappa_0 \nu > 0, \quad \forall \tau \geq 0. \quad (28)$$

**Remark 5.** The PE condition is standard in classical adaptive control [31] and crucial for ensuring parameter convergence. The examples of multi-agent system can be found in [15,16,19]. The PE condition ensures the information richness of the time varying regressor matrix $\Phi$ throughout time, and guarantees parameter convergence. An intuitive interpretation of the PE condition is that when $\phi_i, i = 0, 1, \ldots, N$, rotate sufficiently in space, all parameters can be estimated with confidence of accuracy. The technical assumption may not be easily met in practice (for every $t > 0$), but it helps to indicate the most likely period for a complete estimation of parameters.

The following theorem is our main result:

**Theorem 1.** Consider the multi-agent system (1)–(2). Suppose Assumptions 1 and 2 are satisfied and $\phi_i, i = 0, 1, \ldots, N$, are continuous and uniformly bounded. Then, there exist $\epsilon \in (0, 1)$ and $0 < \rho < 1$, such that, for every $\alpha \in (1 - \epsilon, 1)$, under control law (8) and parameter adaptive law (9), (i) the system (1)–(2) reaches consensus in finite-time; (ii) finite-time parameter convergence is guaranteed in the sense of (6); (iii) the settling time $T_\epsilon < \frac{1}{\max\{\rho_1, \rho_2\}}(1 - \alpha)(1 - \alpha_0\min\{\rho_1, \rho_2\})$.

Before proving Theorem 1, we firstly prove two lemmas for the system (13)–(14).

Let $\Omega_{d, \sigma} \equiv \{e : V_{0}(e) \leq \sigma\}, \Omega_{d, \sigma}^c \equiv \{e : V_{0}(e) > \sigma\}, \Omega_{d, \sigma} = \{e : V_{0}(e) \leq \sigma\}, \Omega_{d, \sigma}^c = \{e : V_{0}(e) > \sigma\}$, $\Omega_{d, \sigma} = \{e : e^T e \leq \sigma\}$ and $\Omega_{d, \sigma}^c = \{e : e^T e > \sigma\}$.

**Lemma 7.** Under Assumption 1 and choosing $\tilde{\epsilon}(t) = \tilde{b}^TP$ satisfying (15), there exists $\epsilon \in (0, 1), \rho \in (0, 1)$, such that, for every $\alpha \in (1 - \epsilon, 1)$, (i) the system (13)–(14) is uniformly stable; (ii) the solutions $\epsilon(t)$ and $\tilde{\theta}(t)$ of the system (13)–(14) are uniformly bounded for $\forall t \geq t_0, V_0 \in \mathbb{R}^N, \tilde{\theta}_0 \in \mathbb{R}^{2\nu N}$, where $\epsilon_0 = \epsilon(t_0), \tilde{\theta}_0 = \tilde{\theta}(t_0)$; (iii) $\lim_{t \to \infty} \|\epsilon(t)\| = 0, \forall \epsilon_0 \in \mathbb{R}^N, \forall \tilde{\theta}_0 \in \mathbb{R}^{2\nu N}$; (iv) there exists $t_1^* \epsilon$ such that $\epsilon(t) \in \Omega_{d, \sigma}$, for any $t \geq t_1^*$.

**Proof.** Consider the following Lyapunov candidate function

$$W_1(\epsilon, \tilde{\theta}) = \epsilon^T (\tilde{A}_0 \otimes P) \epsilon + \frac{1}{c \gamma} \tilde{\theta}^T \tilde{\theta}, \quad (29)$$

where $P > 0$ is given by [15]. Calculating the derivative with respect to time $t$ along the solution of the system (13)–(14), we have

$$\frac{dW_1}{dt} = \epsilon^T (\tilde{A}_0 \otimes (\tilde{A}_0^T P + PA_0) - H \otimes (2P \tilde{b}^TP) \epsilon)\epsilon
- 2(1 - \rho) \epsilon^T (H \otimes \tilde{b}^TP) [\epsilon]_0^a.$$

(30)

Setting $\tilde{\epsilon}(t) = (\tilde{U} \otimes \tilde{I})\epsilon(t)$, applying (15) and noting that

$$\frac{dW_1}{dt} \leq -\lambda_{\min} \epsilon^T \epsilon + 2(1 - \rho) \epsilon^T (H \otimes \tilde{b}^TP) [\epsilon]_0^a.$$

(31)

If $\epsilon \in \mathbb{B}_1^T$, then we have

$$|\epsilon^T (H \otimes \tilde{b}^TP) [\epsilon]_0^a| \leq |\epsilon^T (H \otimes \tilde{b}^TP) [\epsilon]_0^a| \leq \lambda_{\min} \epsilon^T \epsilon,$$

and for $0 < \epsilon < \epsilon_0$,

$$\left|\epsilon^T (H \otimes \tilde{b}^TP) [\epsilon]_0^a\right| \leq \lambda_{\min} \epsilon^T \epsilon + 2(1 - \rho) \lambda_{\min} \epsilon^T \epsilon \quad (32)$$

where $0 \leq H \otimes \tilde{b}^TP < \lambda_{\min} I_N$. Then, from (31), there exists $0 < \rho_2 < 1$, such that when $\max\{\rho_1, \rho_2\} < \rho < 1$, we have

$$\frac{dW_1}{dt} \leq -\lambda_{\min} \epsilon^T \epsilon + 2(1 - \rho) \lambda_{\min} \epsilon^T \epsilon$$

$$\leq -\lambda_{\min} \epsilon^T \epsilon, \quad \epsilon \in \mathbb{B}_1^T.$$
If \( \varepsilon \in \mathcal{B}_1 \), then
\[
\frac{dW_1}{dt} = (1 - \rho)\varepsilon^T [I_N \otimes (A_0^T P + PA_0)\varepsilon - 2(H \otimes \tilde{b}\tilde{b}^T)\varepsilon] + \rho \varepsilon^T \left[ I_N \otimes (A_0^T P + PA_0) - \frac{1}{\rho} H \otimes (2P\tilde{b}\tilde{b}^T) \right] \varepsilon.
\]
(33)

Setting \( \tilde{e}(t) = (U \otimes I)\varepsilon(t) \), and applying (15), we have
\[
\lim_{a \to 1} e^T [I_N \otimes (A_0^T P + PA_0)\varepsilon - 2(H \otimes \tilde{b}\tilde{b}^T)\varepsilon] \leq -\lambda_{\min}(\varepsilon^T \varepsilon).
\]
\[
< -\lambda_{\min}(\varepsilon^T \varepsilon).
\]
(34)

Noting that the compact set \( \mathcal{B}_1 \) exists such that \( \varepsilon \in (0, 1) \) that such for \( \alpha \in (1 - \varepsilon_1, 1) \), \( e^T [I_N \otimes (A_0^T P + PA_0)\varepsilon - 2(H \otimes \tilde{b}\tilde{b}^T)\varepsilon] < \lambda_{\min}(\varepsilon^T \varepsilon) \). On the other hand, applying (16), there exists \( \rho, \max(\rho_1, \rho_2) \), \( \rho < 1 \) such that \( e^T [I_N \otimes (A_0^T P + PA_0) - \frac{1}{\rho} H \otimes (2P\tilde{b}\tilde{b}^T)] e < -\lambda_{\min}(\varepsilon^T \varepsilon) \).

Therefore,
\[
\frac{dW_1}{dt} < -\lambda_{\min}(\varepsilon^T \varepsilon), \quad \varepsilon \in \mathcal{B}_1.
\]
(35)

By Barbala's Lemma [25], we have \( \lim_{t \to \infty} \|e(t)\| = 0 \). Thus (iii) is proved.

Proof. Calculating the derivative of \( V_\varepsilon(\varepsilon) \) defined in (24) along the solution of the system (13), we have
\[
\frac{dV_\varepsilon(\varepsilon)}{dt} = (1 - \rho)\varepsilon^T \left[ (I_N \otimes A_0)\varepsilon - (H \otimes \tilde{b}\tilde{b}^T)\varepsilon \right] + \rho \varepsilon^T \left[ (I_N \otimes A_0) - \frac{1}{\rho} (H \otimes \tilde{b}\tilde{b}^T) \right] \varepsilon + \frac{1}{c} \varepsilon^T \left[ (I_N \otimes A_0) - \frac{1}{\rho} (H \otimes \tilde{b}\tilde{b}^T) \right] \varepsilon \geq 0
\]
(36)

Since \( \lim_{a \to 1} (\frac{\alpha(t)}{\alpha(t)^{-1}}) \geq 0 \), \( \varepsilon^T \left[ (I_N \otimes A_0) - \frac{1}{\rho} (H \otimes \tilde{b}\tilde{b}^T) \right] \varepsilon < 0 \), for every \( \alpha \in (1 - \min(\varepsilon_2, \varepsilon_3, \varepsilon_4), 1) \), \( \frac{\alpha(t)}{\alpha(t)^{-1}} \geq 0 \) for any \( t \in [t_1, t_2] \), and apply (16), we have
\[
\frac{dV_\varepsilon(\varepsilon)}{dt} < -\lambda_{\min}(\varepsilon^T \varepsilon), \quad \varepsilon \in \mathcal{B}_1.
\]
(37)

From Lemma 7, there exists \( t^*_\varepsilon \) such that \( \varepsilon \in \mathcal{B}_1 \), for any \( t \geq t^*_\varepsilon \). Let \( \varepsilon(t) \) be the non-trivial solution of (13). In the following, we consider three cases.

Case 1: There exists an interval \( [t_1, t_2] \subset [t^*_\varepsilon, t^*_\varepsilon + T_\varepsilon] \) such that \( \varepsilon(t) \equiv 0 \) for any \( t \in [t_1, t_2] \). It follows from (13) and (14) that \( \tilde{e}(t) \equiv 0 \) for any \( t \geq t^*_\varepsilon \). Therefore, in this case, \( \tilde{e}(t) \equiv 0 \) for any \( t \geq t^*_\varepsilon \). It follows that \( \tilde{e}(t) \) is a constant vector for any \( t \geq t^*_\varepsilon \). Now we prove \( \tilde{e}(t) \equiv 0 \) for any \( t \geq t^*_\varepsilon \). The proof is completed.

This completes the proof of (iv). □

Lemma 8. Under Assumptions 1 and 2, choose \( \varepsilon^T = \tilde{b}\tilde{b}^T P \) satisfying (15) and assume that \( \Phi \) is continuous and uniformly bounded, there exists \( \rho \in (0, 1) \) and \( \varepsilon \in (0, 1) \), for every \( \alpha \in (1 - \varepsilon, 1) \), such that the system (13)-(14) is locally finite-time stable on set \( \mathcal{B}_1 \).
Therefore, from (43)–(45), it follows that
\[
\frac{dV_{\epsilon}(\epsilon)}{dt} \bigg|_{(13)} < -\kappa_1 (1 - \rho) \nu_{\mu}^{2d - 1} + \frac{\lambda_\omega \kappa Nl}{c} \nu_{\mu} V_{\mu, \epsilon}, \quad \forall t \in [t^*_1, t^*_2 + T_3],
\]
From (43)–(45), it follows that
\[
\frac{dV_{\epsilon}(\epsilon)}{dt} < -\kappa_1 (1 - \rho) \nu_{\mu}^{2d - 1} + \frac{\lambda_\omega \kappa Nl}{c} \nu_{\mu} V_{\mu, \epsilon}, \quad \forall t \in [t^*_1, t^*_2 + T_3],
\]
Since \( \frac{dV_{\epsilon}(\epsilon)}{dt} \bigg|_{(13)} \) is continuous on \([t^*_1, t^*_2 + T_3]\), there exists \( \delta_2 > 0 \) such that, for \( 0 < \delta_0 < \delta_2 \), we have
\[
\frac{dV_{\epsilon}(\epsilon)}{dt} < -\kappa_1 (1 - \rho) \nu_{\mu}^{2d - 1} , \quad \forall t \in [t^*_1, t^*_2 + T_3].
\]
Now, we will give the lower bound of $\kappa_1$. From (26) and (49), we have

$$\kappa_1 > \frac{1}{2} \min_{\varepsilon \in \Omega_1} \left( \int_0^\infty \frac{1}{\lambda^2} \left( \bar{\alpha} \lambda^2 \varepsilon^2 \right) \lambda \min \lambda \varepsilon^2 d\lambda \right)$$

$$> \frac{1}{2} \int_0^\infty \frac{1}{\lambda^2} \left( \bar{\alpha} \lambda^2 \right) \lambda \min \lambda d\lambda.$$

$$= \frac{1}{2} \sqrt{\frac{\pi}{7}} \left( 4\varepsilon^2 - \frac{1}{\lambda^2} \right) d\lambda - \int_0^\infty \sqrt{\frac{\pi}{7}} \left( 4\varepsilon^2 - \frac{2}{\lambda^2} \right) d\lambda$$

$$= \frac{5}{2} \left( \frac{16}{3} \right)^{1/2} \left( \sqrt{6} - 1 - \sqrt{2} \right) \frac{\lambda_{\min}}{\lambda}.$$

In conclusion, we can select $\epsilon > \tilde{\epsilon}$, $\max(\rho_1, \rho_2) < \rho < 1$, $0 < \delta_0 < \min(\delta_1, \delta_2)$, $\epsilon = \min(\delta_3, \delta_4, \varepsilon_1, \varepsilon_2)$, such that, for every $\alpha \in (1 - \epsilon, 1)$, the system (13)-(14) is finite-time stable on $\Omega_{a1}$ and the settling time $T_f$ is given by

$$T_f < \frac{15 \sqrt{5a\lambda}}{\sqrt{16(\sqrt{6} - 1 - \sqrt{2})(1 - \rho)(1 - \alpha)\lambda_{\min}}}.$$ 

4. Simulations

In this section, we give an example to validate our theoretical results. Consider a multi-agent system consisting of a leader driven by control input $u_0(t) = \frac{x(t)}{\sqrt{2}} \sin t = \phi_0(t)^T \theta_0$, where $\phi_0(t) = \sin t$, $\theta_0 = \frac{\pi}{2}$, and four follower agents with nonlinear dynamics

$$f_i(x_i(t), t) = \frac{1}{3} \cos t + \exp(\xi_i^{(0)} + \hat{\xi}_i^{(1)} + \xi_i^{(2)} + \hat{\xi}_i^{(3)})$$

$$= (\cos t, \exp(\xi_i^{(0)} + \hat{\xi}_i^{(1)} + \xi_i^{(2)} + \hat{\xi}_i^{(3)})) \left( \frac{1}{3} \right)$$

$$= \phi_i(x_i(t), t)^T \theta_i, \quad i = 1, 2, 3, 4,$$

where $\phi_i = (\exp(\xi_i^{(0)} + \hat{\xi}_i^{(1)} + \xi_i^{(2)} + \hat{\xi}_i^{(3)}) \cdot \theta_i = \left( \frac{1}{3} \right), \quad i = 1, 2, 3, 4.$

Let $\lambda_{PE}(t)$ be the minimum eigenvalue of $e^{t \Phi^T} P \Phi dt$. Choosing $\nu = 5$, Fig. 2 shows that $\lambda_{PE}(t) > 0$ for all $t \geq 0$. Therefore, the PE condition (28) is satisfied with $\kappa_0 = 0.0047$. Suppose that the interconnected topology is described as in Fig. 3. We obtain that the smallest nonzero eigenvalue of $H = L + B$ is $\lambda_{\min} = 0.382$ by a straightforward calculation. Solving the Riccati inequality (15), we get a solution $P > 0$ and the largest eigenvalue of $P$ is $\lambda = 13.86$. By simple computation, we get $T_f < 75543 s$.

For the system (13)-(14), we choose the initial consensus error vector as $\bar{x}(0) = (-5.7, 7.0, 0, -0.7, -9.8, 11.7, -2.9, -0.6, -6.1, 1.2, -2.7, -3.7, -8.2, -0.6, -1.9, -7.8)^T$ and the initial parameter estimate error vector as $\hat{\theta}(0) = (0.2, -0.9, 1.4, -1.7, 0.5, -0.5, -1.7, 0.4, 1.6, -1.8, 0.4, -0.4)^T$. Under the control law (8) and the adaptive update law (9) with $\alpha = 0.85$, $\rho = 0.7$, $\mu = 0.1$, $\gamma = 0.2$, $c = 1$ and $\alpha_0 = 0.5862$, $\alpha_1 = 0.6538$, $\alpha_2 = 0.7391$, $\alpha_3 = \alpha$ according to (20), simulation is conducted in 150 s time. Fig. 4 shows sixteen components of the consensus error vector $\bar{x}(t) = \bar{x}(t) - 1_N \otimes x_0(t)$. Figs. 5 and 6 show the parameter estimate errors $\hat{\theta}_0 = \theta_0$ and $\hat{\theta} - \theta$, $i = 1, 2, 3, 4$, respectively. From the simulation, we see that all agents follow the leader with parameter convergence within simulation time duration.
5. Conclusions

In this paper, we consider the leader-following finite-time consensus problem of multi-agent systems with unknown nonlinear dynamics. A framework for designing adaptive finite-time consensus protocols, that guarantee finite-time consensus with finite-time parameter convergence, is presented. Lyapunov techniques, Riccati inequalities, homogeneous Lyapunov function, and homogeneity of vector field are applied in the control design and stability analysis. The connectivity of graph and the PE condition are the key to ensure finite-time consensus and finite-time parameter convergence, respectively.

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