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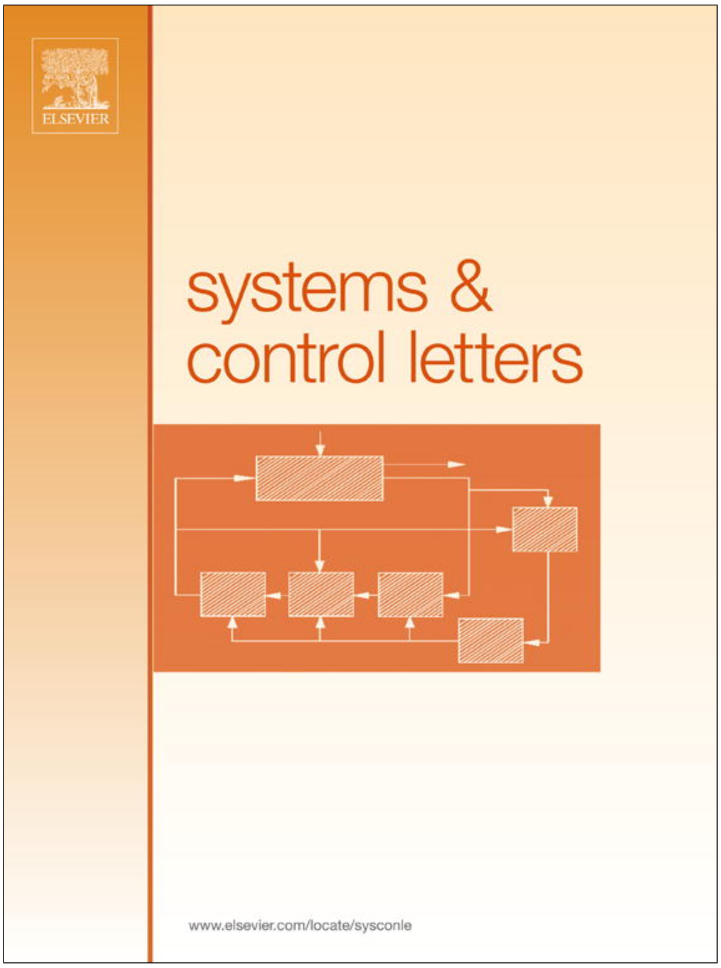

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Adaptive finite-time consensus in multi-agent networks

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ABSTRACT

This paper is concerned with the finite-time consensus problem of distributed agents having non-identical unknown nonlinear dynamics, to a leader agent that also has unknown nonlinear control input signal. By parameterization of unknown nonlinear dynamics, a Lyapunov technique in conjunction with homogeneity technique is presented for designing a decentralized adaptive finite-time consensus control protocol in undirected networks. Homogeneous Lyapunov functions and homogeneous vector fields are introduced in the stability analysis although the whole system is not homogeneous. Theoretical analysis shows that leader-following consensus can be achieved in finite-time, meanwhile, finite-time parameter convergence can be also guaranteed under the proposed control scheme. An example is given to validate the theoretical results.

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1. Introduction

In recent years, studies on the distributed coordination of multi-agent systems have attracted a lot of attention in control and robotics. Its broad applications can be found in diverse areas, including multi-vehicle rendezvous, formation control of multi-robots, flocking, swarming, distributed sensor fusion, attitude alignment, and congestion control in communication networks. One of the main challenges in cooperative control is to design decentralized control schemes such that some group objective can be achieved in a distributed fashion. A particularly interesting topic in cooperative control is the consensus problem of multi-agent systems. Early well-known works on the consensus problem of multi-agent systems can be found in [1–5], to name just a few.

An interesting topic in multi-agent systems is the finite-time consensus problem, which is extensively studied in the literatures [6–14] for multi-agent systems with single or double integrator dynamics. Two finite-time consensus protocols are proposed in [6] for continuous-time systems, under either of which, the differential equations of the overall systems have discontinuous right-hand sides by nonsmooth stability analysis. In [7], the results on finite-time semistability are applied to developing finite-time consensus protocols in nonlinear dynamical networks. The terminal sliding mode technique is used in [8] to design finite-time consensus algorithms. A finite-time formation control framework for multi-agent systems with a large population of members is developed in [9]. In [10], the finite-time consensus problem is studied in both cases of the bidirectional interaction and the unidirectional interaction, and it was proven that if the sum of time intervals, in which the interaction topology is connected, is sufficiently large, the proposed protocols will solve the finite-time consensus problems. Both first-order and second-order decentralized finite-time sliding mode estimators are proposed in [11] and employed to achieve decentralized formation tracking of multiple autonomous vehicles. The finite-time consensus algorithms for leaderless and leader–follower second-order multi-agent systems with external disturbances are addressed in [12]. A binary finite-time consensus protocol is proposed in [13], which only requires the sign information of relative measurement signals between neighboring agents' states. Finite-time weighted average consensus with respect to a monotonic function is studied in [14] for a group of kinematic agents with time-varying topology. As mentioned above, the finite-time consensus problem of multi-agent systems has been extensively studied for systems with single or double integrator dynamics using Lyapunov, homogeneity [7], nonsmooth analysis [6,13], and sliding mode techniques [8,11].

Recently, the consensus problem of multi-agent systems with unknown nonlinear dynamics [15–21] has drawn the attention of many researchers. In [15,16], the authors studied a coordination problem steering a group of agents to a formation that translates with a prescribed reference velocity. Decentralized adaptive designs are proposed for reference velocity recovery using relative position feedback in [15] and tracking of the reference velocity by incorporating relative velocity feedback in addition to relative position feedback in [16]. In [17], the authors proposed a robust decentralized adaptive control approach using a neural network

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to solve the consensus problem of multi-agents with uncertainties and external disturbances in undirected networks. In [18], the authors presented a design method for adaptive synchronization controllers for distributed systems having non-identical unknown nonlinear dynamics, and for a target dynamics to be tracked that is also nonlinear and unknown. Under some assumptions, the authors proved that the overall local cooperative error vector and the neural network weight estimation errors are both uniformly ultimately bounded. In [19], an adaptive consensus design method is presented for multi-agent systems with non-identical unknown nonlinear dynamics, and for a leader to be followed that is also nonlinear and unknown in networks with jointly connected topologies. Both consensus stability and parameter convergence are analyzed. In [20,21], the consensus problem of high order multi-agent systems with unknown nonlinear dynamics is considered using an adaptive design method. It is noted that a linear parameterization approach has been taken in [15,16,19] to deal with the unknown and complex nonlinear dynamics, and examples of applications are included in these papers.

However, when unknown nonlinear dynamics exist in the system, few works consider the finite-time consensus problem of multi-agent systems. In [15–21], only asymptotical stability is considered using the adaptive design method. To the best knowledge of the authors, for networks of multiple agents with unknown nonlinear dynamics, it is still an open problem to design decentralized control laws such that the whole system reaches consensus in finite time. Moreover, for an adaptive design by parameterizations of the unknown nonlinear dynamics, it is another problem to guarantee finite-time parameters converge in the meantime.

In this paper, we consider the finite-time consensus problem of leader-following multi-agent systems, in which the leader's control input signal is unknown and nonlinear, the followers have unknown, non-identical, nonlinear dynamics. By parameterization of unknown nonlinear dynamics, under the assumption of connectivity of multi-agent networks and the persistent excitation assumption of the regressor matrix, a decentralized adaptive finite-time control scheme is proposed for the considered multi-agent systems to reach consensus with parameter convergence in finite-time via relative states and local consensus error feedback of neighboring agents. The stability analysis is conducted based on Lyapunov techniques and homogeneity of partial terms of vector fields. A homogeneous Lyapunov function is constructed in the stability analysis. However, the whole system is not homogeneous, in which only partial terms of the vector field hold homogeneity, so the finite-time stability theory of homogeneous systems cannot be applied directly. We use Lyapunov techniques and homogeneity, in conjunction with some inequality techniques to derive our stability results, such that finite-time consensus and finite-time parameter convergence are both achieved globally. The connectivity of multi-agent networks and the persistent excitation condition are crucial in finite-time consensus and finite-time parameter convergence, respectively.

The contributions of this paper are in three aspects. Firstly, a novel type of decentralized adaptive finite-time consensus algorithm is proposed for leader-following multi-agent systems with unknown nonlinear dynamics; secondly, homogeneous Lyapunov function and homogeneous vector fields are introduced in the finite-time stability analysis of multi-agent systems although the whole system is not homogeneous; finally, under the PE condition, finite-time parameter convergence is also guaranteed.

This paper is organized as follows. In Section 2, we establish the notation and formally state the problem. We present our main results in Section 3, the simulation results supporting the objectives of the paper in Section 4 and the concluding remarks in Section 5.

2. Problem statement

Consider a multi-agent system consisting of N agents and a leader. The dynamics of the ith ($i = 1, 2, \dots, N$) agent is described by

$$\dot{x}_i(t) = A_0 x_i(t) + \bar{b}[f_i(x_i(t), t) + u_i(t)], \tag{1}$$

where $A_0 = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$, $\bar{b} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$, $x_i(t) = (\xi_i^{(0)}, \xi_i^{(1)}, \dots, \xi_i^{(l-1)})^T$, $\xi_i^{(k)}(t) \in \mathcal{R}$, $k = 0, 1, \dots, l-1$, denoting the kth derivative of ξ_i with $\xi_i^{(0)} = \xi_i$, are the states of the ith agent, $u_i(t) \in \mathcal{R}$ is the control input of the ith agent, and smooth function $f_i(x_i(t), t)$ is the nonlinear dynamics of agent i, which is assumed to be unknown. Standard assumptions for the existence of unique solutions are made, i.e., $f_i(x_i(t), t)$, $i = 1, 2, \dots, N$, is continuous in t and Lipschitz in $x_i(t)$. We assume that the leader's dynamics of the considered multi-agent system is as follows:

$$\dot{x}_0(t) = A_0 x_0(t) + \bar{b} u_0(t), \tag{2}$$

where $x_0(t) = (\xi_0^{(0)}, \xi_0^{(1)}, \dots, \xi_0^{(l-1)})^T$, $\xi_0^{(k)}(t) \in \mathcal{R}$, $k = 0, 1, \dots, l-1$ are the states of the leader, the control input $u_0(t)$ of the leader agent is also assumed to be unknown.

With regarding the N agents as the nodes in $\mathcal{V} = \{1, 2, \dots, N\}$, the relationships between N agents can be described by a simple and undirected graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ [22] consisting of a node set $\mathcal{V}$ and an edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ of unordered pair $(i, j) \in \mathcal{E}$. The set of neighbors of node i is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} | (i, j) \in \mathcal{E}, j \neq i\}$. A path is a sequence of connected edges in a graph. If there is a path between any two nodes of a graph $\mathcal{G}$, then $\mathcal{G}$ is said to be connected, otherwise disconnected. To describe the information transmission between N agents and the leader, we need to define another graph $\bar{\mathcal{G}}$ on nodes $0, 1, 2, \dots, N$, which consists of graph $\mathcal{G}$, node 0 representing the leader agent and edges between the leader and its neighbors [23]. The adjacency matrix of graph $\mathcal{G}$ is denoted by $A = [a_{ij}] \in R^{N\times N}$, whose (ij)th entry is 1 if (i, j) is an edge of graph $\mathcal{G}$ and 0 if it is not. The index number between agent i, $i = 1, 2, \dots, N$, and the leader is denoted by b_i, which is defined to be 1 whenever the leader agent is agent i's neighbor and 0 otherwise. The degree matrix $D \in R^{N\times N}$ of graph $\mathcal{G}$ is a diagonal matrix with the ith diagonal element being $|\mathcal{N}_i|$. The Laplacian of graph $\mathcal{G}$ is defined as $L = D - A$, which is symmetric and has the following well-known result in algebraic graph theory [22].

Lemma 1. *The Laplacian L of graph $\mathcal{G}$ has at least one zero eigenvalue with $\mathbf{1}_N = (1, 1, \dots, 1)^T \in \mathcal{R}^N$ as its eigenvector, and all the nonzero eigenvalues of L are positive. Laplacian L has a simple zero eigenvalue if and only if graph $\mathcal{G}$ is connected.*

The following lemma is also needed in deriving our main results.

Lemma 2 ([24]). *Consider the system $\dot{x}(t) = f(x(t))$, $f(0) = 0$, $x \in \mathcal{R}^n$, $x(0) = x_0$. Suppose that there exist a positive definite continuous function $V(x) : \mathcal{D} \to \mathcal{R}$, real numbers $c_\alpha > 0$ and $\alpha \in (0, 1)$, and an open neighborhood $\mathcal{H} \subset \mathcal{D}$ of the origin such that $\dot{V}(x) \leq -c_\alpha V(x)^\alpha$, $x \in \mathcal{H} \setminus \{0\}$. Then the origin is a finite-time stable equilibrium of the system. In addition, the settling time T satisfies that $T \leq \frac{1}{c_\alpha(1-\alpha)} V(x)^{1-\alpha}$.*

Suppose that the unknown nonlinear dynamics $f_i(x_i(t), t)$, are parameterized as

$$f_i(x_i(t), t) = \phi_i^T(x_i(t), t)\theta_i, \quad i = 1, 2, \dots, N, \tag{3}$$

and the leader's unknown control input signal is parameterized as $u_0(t) = \phi_0^T(t)\theta_0$, where $\phi_0(t), \phi_i(x_i(t), t) \in \mathcal{R}^m$ are basis function column vectors and $\theta_0, \theta_i \in \mathcal{R}^m$ are constant true parameter column vectors to be estimated.

Because θ_0 is unavailable to each agent, for the purpose of designing a decentralized controller, the ith agent estimates the unknown parameter vector θ_0 by $\hat{\theta}_{0i}$ and $u_0(t)$ by $\hat{u}_i(t)$ respectively.

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We have

$$\hat{u}_i(t) = \phi_0^T(t)\hat{\theta}_{0i}, \quad i = 1, 2, \ldots, N. \tag{4}$$

Similarly, the estimate of $f_i(x_i(t), t)$ is expressed as

$$\hat{f}_i(x_i(t), t) = \phi_i^T(x_i(t), t)\hat{\theta}_i, \quad i = 1, 2, \ldots, N. \tag{5}$$

Remark 1. The unknown nonlinear dynamics of all agents are assumed to be linearly parameterized. The linearly parameterized models have been studied widely in classical adaptive control [25]. The examples of a linearly parameterized model of multi-agent systems can be found in [15,16,19,21].

Let $x(t) = \text{col}(x_1, \ldots, x_N)$ be the stack column vector of $x_1, \ldots, x_N$, the objectives of this work are to design a decentralized adaptive finite-time consensus scheme such that leader-following consensus can be reached in finite time and finite-time parameter convergence can be guaranteed in the meantime, that is

$$\lim_{t\to T_s} \|x_i(t) - x_0(t)\| = 0, \qquad \lim_{t\to T_s} \|\hat{\theta}_{0i} - \theta_0\| = 0, \qquad \lim_{t\to T_s} \|\hat{\theta}_i - \theta_i\| = 0, \tag{6}$$

for any initial condition $x_0(0), x(0)$, where $T_s > 0$ is the settling time.

3. Main results

Define local neighborhood consensus error [8] for agent i as

$$\zeta_i^{(k)}(t) = \sum_{j\in\mathcal{N}_i} a_{ij}(\xi_i^{(k)} - \xi_j^{(k)}) + b_i(\xi_i^{(k)} - \xi_0^{(k)}), \tag{7}$$

where $k = 0, 1, \ldots, l-1, i = 1, 2, \ldots, N$. For agent i, we propose the following lth-order consensus control algorithm:

$$u_i(t) = -\bar{c}^T e_i(t) - (1-\rho)\sum_{k=0}^{l-1} c_k c^{(1-\alpha_k)(1-\mu)}\lceil\zeta_i^{(k)}\rfloor^{\alpha_k} + \Phi_i^T\hat{\Theta}_i, \tag{8}$$

and

$$\dot{\hat{\theta}}_{0i} = -c^{-1}\gamma\phi_0(t)\bar{c}^T\left[\sum_{j\in\mathcal{N}_i} a_{ij}(e_i - e_j) + b_i e_i\right], \qquad \dot{\hat{\theta}}_i = -c^{-1}\gamma\phi_i(x_i, t)\bar{c}^T\left[\sum_{j\in\mathcal{N}_i} a_{ij}(e_i - e_j) + b_i e_i\right], \tag{9}$$

where $e_i(t) = (\zeta_i^{(0)}, \zeta_i^{(1)}, \ldots, \zeta_i^{(l-1)})^T$, $\lceil\zeta_i^{(k)}\rfloor^{\alpha_k} = |\zeta_i^{(k)}|^{\alpha_k}\text{sgn}(\zeta_i^{(k)})$, $\text{sgn}(\cdot)$ is the sign function, $\Phi_i = \text{col}(\phi_0, -\phi_i)$, $\hat{\Theta}_i = \text{col}(\hat{\theta}_{0i}, \hat{\theta}_i)$, $0 < \rho < 1, 0 < \mu < 1, c \ge 1, \alpha_k > 0$ are positive constant numbers, $\bar{c} = (c_0, \ldots, c_{l-1})^T$ is a constant vector to be designed.

Remark 2. Note that controller $u_i(t)$ defined in (8) and the adaptive laws defined in (9) are decentralized. For control purposes, we assume that the information of local consensus error vector $e_i(t)$ of agent i is calculated in real-time and saved in its memory at each time instant by each agent and is available for its neighbors. Controller $u_i(t)$ and the adaptive laws defined in (9) only depend on the information of relative position measurements and local consensus errors feedback from its neighboring agents. A similar method of information transmission can be found, for instance, in the literatures [8,19,26,27].

Let $\lceil e_i\rfloor^{\bar{\alpha}} = \text{col}(c^{(1-\alpha_0)(1-\mu)}\lceil\zeta_i^{(0)}\rfloor^{\alpha_0}, \ldots, c^{(1-\alpha_{l-1})(1-\mu)}\lceil\zeta_i^{(l-1)}\rfloor^{\alpha_{l-1}})$, $\lceil e\rfloor^{\bar{\alpha}} = \text{col}(\lceil e_1\rfloor^{\bar{\alpha}}, \ldots, \lceil e_N\rfloor^{\bar{\alpha}})$, $\hat{\Theta} = \text{col}(\hat{\Theta}_1, \hat{\Theta}_2, \ldots, \hat{\Theta}_N)$, $\Phi = \text{diag}\{\Phi_1, \Phi_2, \ldots, \Phi_N\}$, $\bar{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_{l-1})^T$, $e(t) = \text{col}(e_1, \ldots, e_N)$, $u = \text{col}(u_1, \ldots, u_N)$, we have

$$u = -(I_N \otimes \bar{c}^T)e - (1-\rho)(I_N \otimes \bar{c}^T)\lceil e\rfloor^{\bar{\alpha}} + \Phi^T\hat{\Theta}, \tag{10}$$

where I_N is the $N \times N$ identity matrix.

With (10), letting $\Theta_i = \text{col}(\theta_0, \theta_i)$, $\Theta = \text{col}(\Theta_1, \Theta_2, \ldots, \Theta_N)$, $\bar{\Theta} = \hat{\Theta} - \Theta$, $f = \text{col}(f_1, f_2, \ldots, f_N)$, $\bar{x}(t) = x(t) - \mathbf{1}_N \otimes x_0(t)$, and noting that $e(t) = (H \otimes I_l)\bar{x}(t)$ and $\dot{x}(t) = (I_N \otimes A_0)x(t) + (I_N \otimes \bar{b})(f + u)$, we have

$$\begin{aligned}
\dot{e}(t) &= (H \otimes I_l)(\dot{x}(t) - \mathbf{1}_N \otimes \dot{x}_0(t)) \\
&= (H \otimes I_l)[(I_N \otimes A_0)x(t) + (I_N \otimes \bar{b})(f + u) \\
&\quad - \mathbf{1}_N \otimes (A_0 x_0(t) + \bar{b}u_0(t))] \\
&= (H \otimes I_l)[(I_N \otimes A_0)\bar{x}(t) + (I_N \otimes \bar{b})(f + u) - \mathbf{1}_N \otimes \bar{b}u_0(t)] \\
&= (H \otimes I_l)\{(I_N \otimes A_0)\bar{x}(t) + (I_N \otimes \bar{b})[-(I_N \otimes \bar{c}^T)e \\
&\quad - (1-\rho)(I_N \otimes \bar{c}^T)\lceil e\rfloor^{\bar{\alpha}} + \Phi^T\bar{\Theta}]\} \\
&= (I_N \otimes A_0)e(t) - (H \otimes \bar{b}\bar{c}^T)e(t) - (1-\rho)(H \otimes \bar{b}\bar{c}^T)\lceil e\rfloor^{\bar{\alpha}} \\
&\quad + (H \otimes \bar{b})\Phi^T\bar{\Theta}
\end{aligned}$$

where $H = L + B, B = \text{diag}(b_1, b_2, \ldots, b_N)$, matrix H satisfied the following well-known lemma [28]:

Lemma 3. (i) *Matrix H has nonnegative eigenvalues;* (ii) *Matrix H is positive definite if and only if graph $\bar{\mathcal{G}}$ is connected.*

We obtain the following error system:

$$\begin{aligned}
\dot{e}(t) &= (I_N \otimes A_0)e(t) - (H \otimes \bar{b}\bar{c}^T)e(t) \\
&\quad - (1-\rho)(H \otimes \bar{b}\bar{c}^T)\lceil e\rfloor^{\bar{\alpha}} + (H \otimes \bar{b})\Phi^T\bar{\Theta}
\end{aligned} \tag{11}$$

and

$$\dot{\bar{\Theta}} = -c^{-1}\gamma\Phi(H \otimes \bar{c}^T)e. \tag{12}$$

Let $\varepsilon(t) = c^{-1}e(t)$, the system (11)–(12) can be rewritten as

$$\begin{aligned}
\dot{\varepsilon}(t) &= (I_N \otimes A_0)\varepsilon(t) - (H \otimes \bar{b}\bar{c}^T)\varepsilon(t) \\
&\quad - (1-\rho)(H \otimes \bar{b}\bar{c}^T)\lceil\varepsilon\rfloor^{\bar{\alpha}} + c^{-1}(H \otimes \bar{b})\Phi^T\bar{\Theta},
\end{aligned} \tag{13}$$

and

$$\dot{\bar{\Theta}} = -\gamma\Phi(H \otimes \bar{c}^T)\varepsilon, \tag{14}$$

where $\lceil\varepsilon_i\rfloor^{\bar{\alpha}} = \text{col}(c^{(\alpha_0-1)\mu}\lceil\varepsilon_i^{(0)}\rfloor^{\alpha_0}, c^{(\alpha_1-1)\mu}\lceil\varepsilon_i^{(1)}\rfloor^{\alpha_1}, \ldots, c^{(\alpha_{l-1}-1)\mu}\lceil\varepsilon_i^{(l-1)}\rfloor^{\alpha_{l-1}})$, $i = 1, 2, \ldots, N$, $\lceil\varepsilon\rfloor^{\bar{\alpha}} = \text{col}(\lceil\varepsilon_1\rfloor^{\bar{\alpha}}, \lceil\varepsilon_2\rfloor^{\bar{\alpha}}, \ldots, \lceil\varepsilon_N\rfloor^{\bar{\alpha}})$.

Assumption 1. Graph $\bar{\mathcal{G}}$ is connected.

For a symmetric matrix P, by $P > 0$ we mean that P is positive definite. Under Assumption 1 and from Lemma 3, H is symmetric positive definite and all of eigenvalues of H are positive. Let $\lambda_{\min}, \lambda_{\max} > 0$ and $\underline{\lambda}, \bar{\lambda} >$ be the smallest, largest eigenvalue of H and P, respectively. Since $(A_0, \bar{b})$ is stabilizable, there exists a positive definite matrix $P = (p_{ij})_{l\times l} > 0$ such that the following Riccati inequality

$$A_0^T P + PA_0 - 2\lambda_{\min}P\bar{b}\bar{b}^T P < -\lambda_{\min}I_l \tag{15}$$

holds with the constant vector $\bar{c}$ in (10) being designed as $\bar{c}^T = \bar{b}^T P$. There also exists $0 < \rho_1 < 1$ such that $\rho_1 < \rho < 1$, (15) and

$$A_0^T P + PA_0 - \frac{2}{\rho}\lambda_{\min}P\bar{b}\bar{b}^T P < -\frac{1}{2}\lambda_{\min}I_l \tag{16}$$

hold with $\bar{c}^T = \bar{b}^T P$.

3.1. Finite-time stability of homogeneous system

In this subsection, we firstly consider the homogeneous part of the error system (13)–(14). The following definitions are needed in the theoretical analysis.

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Definition 1 ([29]). A function $V : \mathcal{R}^n \to \mathcal{R}$ is homogeneous of degree $d \in \mathcal{R}$ with respect to weights $(r_1, \dots, r_n) \in \mathcal{R}^n_+$ if

$$V(\lambda^{r_1}x_1, \dots, \lambda^{r_n}x_n) = \lambda^d V(x_1, \dots, x_n), \quad \forall \lambda > 0. \tag{17}$$

Definition 2 ([29]). A vector field g is homogeneous of degree $d \in \mathcal{R}$ with respect to the weights $(r_1, \dots, r_n) \in \mathcal{R}^n_+$ if for all $1 \le i \le n$, the ith component g_i of g is homogeneous function of degree $r_i + d$, i.e.,

$$g_i(\lambda^{r_1}x_1, \dots, \lambda^{r_n}x_n) = \lambda^{r_i+d} g_i(x_1, \dots, x_n), \quad \forall \lambda > 0. \tag{18}$$

For the homogeneous part of the error system (13)–(14), we give the following lemma.

Lemma 4. *Under Assumption 1 and (15) with $\bar{c}^T = \bar{b}^T P$, consider system*

$$\dot{\varepsilon}(t) = (I_N \otimes A_0)\varepsilon(t) - (H \otimes \bar{b}\bar{c}^T)\lceil \varepsilon \rfloor^{\bar{\alpha}}. \tag{19}$$

There exists $\epsilon_0 \in (0, 1)$ such that, for every $\alpha \in (1-\epsilon_0, 1)$, the origin is a globally finite-time stable equilibrium for system (19), where $\alpha_0, \alpha_1, \dots, \alpha_{l-1}$ satisfy

$$\alpha_{k-1} = \frac{\alpha_k \alpha_{k+1}}{2\alpha_{k+1} - \alpha_k}, \quad k = 1, \dots, l-1, \tag{20}$$

with $\alpha_l = 1$ and $\alpha_{l-1} = \alpha$.

Proof. Let $H = (h_{ij})_{N\times N}$, we have

$$(H \otimes \bar{b}\bar{c}^T)\lceil \varepsilon \rfloor^{\bar{\alpha}} = \left[H \otimes \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ c_0 & c_1 & \cdots & c_{l-1} \end{pmatrix} \right] \lceil \varepsilon \rfloor^{\bar{\alpha}} = \begin{pmatrix} F_1(\varepsilon) \\ F_2(\varepsilon) \\ \vdots \\ F_N(\varepsilon) \end{pmatrix},$$

where $F_i(\varepsilon) = (0, \dots, 0, \sum_{j=1}^N h_{ij} \sum_{k=0}^{l-1} c_k c^{(\alpha_k - 1)\mu} \lceil \varepsilon_j^{(k)} \rfloor^{\alpha_k})^T$, $i = 1, 2, \dots, N$.

Let f^α denote the closed-loop vector field of system (19). From Proposition 8.1 of [29], it is easy to verify that, for each $\alpha > 0$, the vector field f^α is continuous, homogeneous of degree $\frac{\alpha-1}{\alpha}$ with respect to the weights

$$\Bigg\{ \underbrace{\underbrace{\frac{1}{\alpha_0}, \frac{1}{\alpha_1}, \dots, \frac{1}{\alpha_{l-1}}}, \dots, \underbrace{\frac{1}{\alpha_0}, \frac{1}{\alpha_1}, \dots, \frac{1}{\alpha_{l-1}}}}_{N} \Bigg\}, \tag{21}$$

where $\alpha_0, \alpha_1, \dots, \alpha_{l-1}$ satisfy (20) with $\alpha_l = 1$ and $\alpha_{l-1} = \alpha$. Moreover, noting that $\alpha_0 = \alpha_1 = \cdots = \alpha_{l-1} = 1$ as $\alpha = 1$, the vector field f^1 corresponds to the system

$$\dot{\varepsilon}(t) = (I_N \otimes A_0)\varepsilon(t) - (H \otimes \bar{b}\bar{c}^T)\varepsilon(t). \tag{22}$$

Let $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_N)$ with $\lambda_1, \lambda_2, \dots, \lambda_N$ being eigenvalues of matrix H. Because H is symmetric, there exists an orthogonal matrix U such that $UHU^T = \Lambda$. Setting $\tilde{\varepsilon}(t) = (U \otimes I_l)\varepsilon(t)$ for the system (22), one has

$$\dot{\tilde{\varepsilon}}(t) = (I_N \otimes A_0)\tilde{\varepsilon}(t) - (\Lambda \otimes \bar{b}\bar{c}^T)\tilde{\varepsilon}(t). \tag{23}$$

Consider Lyapunov function candidate $W_0(\tilde{\varepsilon}) = \tilde{\varepsilon}^T(I_N \otimes P)\tilde{\varepsilon}$, where $P > 0$ is given in (15). Calculating the derivative along the solution of the system (23), we have

$$\begin{aligned} \left.\frac{dW_0}{dt}\right|_{(23)} &= \tilde{\varepsilon}^T[I_N \otimes (A_0^T P + PA_0) - \Lambda \otimes (2P\bar{b}\bar{b}^T P)]\tilde{\varepsilon} \\ &= \sum_{i=1}^N \tilde{\varepsilon}_i^T (A_0^T P + PA_0 - 2\lambda_i P\bar{b}\bar{b}^T P)\tilde{\varepsilon}_i \\ &\le \sum_{i=1}^N \tilde{\varepsilon}_i^T (A_0^T P + PA_0 - 2\lambda_{\min} P\bar{b}\bar{b}^T P)\tilde{\varepsilon}_i \\ &< -\lambda_{\min} \sum_{i=1}^N \tilde{\varepsilon}_i^T \tilde{\varepsilon}_i = -\lambda_{\min}\tilde{\varepsilon}^T\tilde{\varepsilon} < 0, \quad \forall \tilde{\varepsilon} \ne 0. \end{aligned}$$

Therefore, the origin of the system (23) is an asymptotically stable equilibrium. By Theorem 6.2 of [29], there exists a positive-definite, radially unbounded, Lyapunov function $W(\varepsilon(t))$ such that $\frac{dW}{dt}$ is continuous and negative definite. Let $\mathcal{A} = W^{-1}([0, 1])$ and the boundary of set $\mathcal{A}$ be $\mathcal{S} = W^{-1}(\{1\})$. Then $\mathcal{A}$ and $\mathcal{S}$ are compact since W is proper and $0 \notin \mathcal{S}$ since W is positive definite. Define $\varphi : (0, 1] \times \mathcal{S} \to \mathcal{R}$ by $\varphi(\alpha, z) = L_{f^\alpha} W(z)$, where $L_{f^\alpha} W(z)$ is the *Lie-derivative* with respect to f^α. Then φ is continuous and satisfies $\varphi(1, z) < 0$ for all $z \in \mathcal{S}$, that is, $\varphi(\{1\} \times \mathcal{S}) \subset (-\infty, 0)$. Since $\mathcal{S}$ is compact, there exists $\epsilon_0 > 0$ such that $\varphi((1-\epsilon_0, 1] \times \mathcal{S}) \subset (-\infty, 0)$. It follows that for $\alpha \in (1-\epsilon_0, 1]$, $L_{f^\alpha} W$ takes negative values on $\mathcal{S}$. Thus, $\mathcal{A}$ is strictly positively invariant under f^α for every $\alpha \in (1-\epsilon_0, 1]$. By Theorem 6.1 of [29] the origin is globally asymptotically stable equilibrium under f^α for every $\alpha \in (1-\epsilon_0, 1]$. The result now follows from Theorems 7.1 and 7.3 of [29] by noting that, for every $\alpha \in (1-\epsilon_0, 1)$, the degree of homogeneity of f^α with respect to the weights (21) is negative. ■

Remark 3. Similar to [29], the uniqueness of solutions of system (19) is based on forward uniqueness. The vector fields considered in (19) are locally Lipschitz everywhere except on a finite collection of submanifolds. Moreover, the vector field is transverse to each such submanifold everywhere except at the origin. Hence forward uniqueness for all initial conditions except the origin follows from Remark 8.1 of [29] and references therein, while forward uniqueness at the origin follows from Lyapunov stability.

As in [30], for system (19), we construct the following homogeneous Lyapunov function

$$V_\alpha(\varepsilon) = \begin{cases} \displaystyle\int_0^\infty \frac{1}{\lambda^{d+1}}(a \circ W_0) \\ \quad \times (\lambda^{\frac{1}{\alpha_0}}\varepsilon_1^{(0)}, \dots, \lambda^{\frac{1}{\alpha_{l-1}}}\varepsilon_1^{(l-1)}, \dots, \\ \quad \lambda^{\frac{1}{\alpha_0}}\varepsilon_N^{(0)}, \dots, \lambda^{\frac{1}{\alpha_{l-1}}}\varepsilon_N^{(l-1)})d\lambda, & \varepsilon \in \mathcal{R}^{Nl} \setminus \{0\} \\ 0, & \varepsilon = 0, \end{cases} \tag{24}$$

where $d > p > 2$, d, p are positive integers, $\alpha \in (1-\epsilon_1, 1)$ with $\epsilon_1 = \frac{p-2}{2l+p-2}$, $W_0(\varepsilon) = \varepsilon^T(I_N \otimes P)\varepsilon$, in which P satisfies (15), and $a(s) \in C^\infty(\mathcal{R}, \mathcal{R})$ is such that $a(s) = \begin{cases} 0, & s \in (-\infty, 1], \\ 1, & s \in [2, +\infty), \end{cases}$ and its-derivative satisfies $a'(s) \ge 0$ on $\mathcal{R}$.

An example of function $a(s)$ can be constructed as:

$$a(s) = \begin{cases} 0, & s \in (-\infty, 1], \\ 2(s-1)^2, & s \in \left(1, \frac{3}{2}\right], \\ 1 - 2(s-2)^2, & s \in \left(\frac{3}{2}, 2\right], \\ 1, & s \in (2, \infty). \end{cases} \tag{25}$$

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Then, the derivative of function $a(s)$ is

$$a'(s) = \begin{cases} 0, & s \in (-\infty, 1], \\ 4(s-1), & s \in \left(1, \frac{3}{2}\right], \\ -4(s-2), & s \in \left(\frac{3}{2}, 2\right] \\ 0, & s \in (2, \infty). \end{cases} \tag{26}$$

Remark 4. Choosing different values of p provides flexibility for the lower bound of α. Large values of p result in the lower bound of α close to 0, otherwise 1.

Lemma 5. *If $d > p > 2$, d, p are positive integers, $\alpha \in (1-\epsilon_1, 1)$ with $\epsilon_1 = \frac{p-2}{2l+p-2}$, then,* (i) $\min_{0\le k\le l-1}\{p\alpha_k\} > 2$, (ii) $d > \max_{0\le k\le l-1}\{\frac{1}{\alpha_k}\}$, *and* (iii) $0 < \frac{\alpha d+\alpha-1}{\alpha d} < 1$ *hold, where α_k is defined in* (20).

Proof. From (20), it is easy to verify that

$$0 < \alpha_0 = \frac{\alpha}{l-(l-1)\alpha} < \alpha_1 < \cdots < \alpha_{l-1} = \alpha < 1. \tag{27}$$

To prove (i), we only need to prove $p\alpha_0 - 2 > 0$ due to (27). Since $p\alpha_0 - 2 = (\alpha - \frac{2l}{p-2+2l})\frac{p-2+2l}{l-(l-1)\alpha} = (\alpha - 1 + \epsilon_0)\frac{p-2+2l}{l-(l-1)\alpha} > 0$, therefore, (i) holds. To prove (ii), we only need to prove $d\alpha_0 > 1$ due to (27). It obviously holds due to $d > p > 2$ and (i). To prove (iii), we only need to prove $d > \frac{1}{\alpha} - 1 > \frac{p-2}{2l}$, which is always true. ■

Lemma 6. (i) *$V_\alpha(\varepsilon)$ is C^1 on $\mathcal{R}^{Nl}$, positive definite, and homogeneous of degree d with respect to the weights defined in* (21); (ii) $\forall \varepsilon \neq 0$, $\frac{dV_\alpha(\varepsilon)}{dt}|_{(19)} < 0$; (iii) *Homogeneous Lyapunov function $V_\alpha(\varepsilon)$ satisfies* $\left.\frac{dV_\alpha(\varepsilon)}{dt}\right|_{(19)} \le -\kappa_1 V_\alpha^{\frac{\alpha d+\alpha-1}{\alpha d}}$, $\varepsilon \in \mathcal{R}^{Nl}$, *where $\kappa_1 > 0$ is some positive constant number, $\alpha \in (1-\epsilon_2, 1)$ with $\epsilon_2 = \min\{\epsilon_0, \epsilon_1\}$.*

Proof. (i) From Lemma 5 and similar to the proof of Theorem 2 in [30], we can prove (i) and (ii) easily.

(iii) For each $\alpha > 0$, the vector field f^α is continuous, homogeneous of degree $\frac{\alpha-1}{\alpha}$ with respect to the weights defined in (21). From (i), V_α is C^1 on $\mathcal{R}^{Nl}$ and homogeneous of degree d with respect to the weights defined in (21). It follows that $\frac{dV_\alpha}{dt}$ is continuous on $\mathcal{R}^{Nl}$ and homogeneous of degree $d+1-\frac{1}{\alpha}$. Lemma 4.2 of [29] and (ii) imply that there exists a constant number $\kappa_1 > 0$ such that $\left.\frac{dV_\alpha(\varepsilon)}{dt}\right|_{(19)} \le -\kappa_1 V_\alpha^{\frac{\alpha d+\alpha-1}{\alpha d}}$, $\varepsilon \in \mathcal{R}^{Nl}$. ■

3.2. Adaptive finite-time consensus

In this subsection, we give a theoretical analysis for adaptive finite-time consensus achievement of the system (1)–(2). To derive our main result, the following persistent excitation [25] assumption is needed to guarantee finite-time parameter convergence.

Assumption 2. The regressor matrix $\Phi = \mathrm{diag}\{\Phi_1, \Phi_2, \ldots, \Phi_N\}$ is persistently exciting (PE) [25], that is there exist two positive real ν and κ_0 such that

$$\int_t^{t+\nu} \Phi\Phi^T d\tau \ge \kappa_0 I > 0, \quad \forall t \ge 0. \tag{28}$$

Remark 5. The PE condition is standard in classical adaptive control [31] and crucial for ensuring parameter convergence. The examples of multi-agent system can be found in [15,16,19]. The PE condition ensures the information richness of the time varying regressor matrix Φ throughout time, and guarantees parameter convergence. An intuitive interpretation of the PE condition is that when ϕ_i, $i = 0, 1, \ldots, N$, rotate sufficiently in space, all parameters can be estimated with confidence of accuracy. The technical assumption may not be easily met in practice (for every $t > 0$), but it helps to indicate the most likely period for a complete estimation of parameters.

The following theorem is our main result:

Theorem 1. *Consider the multi-agent system* (1)–(2). *Suppose Assumptions 1 and 2 are satisfied and ϕ_i, $i = 0, 1, \ldots, N$, are continuous and uniformly bounded. Then, there exist $\epsilon \in (0, 1)$ and $0 < \rho < 1$, such that, for every $\alpha \in (1-\epsilon, 1)$, under control law* (8) *and parameter adaptive law* (9), (i) *the system* (1)–(2) *reaches consensus in finite-time;* (ii) *finite-time parameter convergence is guaranteed in the sense of* (6); (iii) *the settling time* $T_s < \frac{15\sqrt[3]{5}\alpha\bar{\lambda}}{\sqrt[3]{16}(\sqrt{6}-1-\sqrt{2})(1-\rho)(1-\alpha)\lambda_{\min}}$.

Before proving Theorem 1, we firstly prove two lemmas for the system (13)–(14).

Let $\Omega_\sigma \triangleq \{\varepsilon : W_0(\varepsilon) \le \sigma\}$, $\Omega_\sigma^c \triangleq \{\varepsilon : W_0(\varepsilon) > \sigma\}$, $\Omega_{\alpha,\sigma} \triangleq \{\varepsilon : V_\alpha(\varepsilon) \le \sigma\}$, $\Omega_{\alpha,\sigma}^c \triangleq \{\varepsilon : V_\alpha(\varepsilon) > \sigma\}$, $\mathcal{B}_\sigma \triangleq \{\varepsilon : \varepsilon^T\varepsilon \le \sigma\}$ and $\mathcal{B}_\sigma^c \triangleq \{\varepsilon : \varepsilon^T\varepsilon > \sigma\}$.

Lemma 7. *Under Assumption 1 and choosing $\bar{c}^T = \bar{b}^T P$ satisfying* (15), *there exists $\epsilon \in (0, 1)$, $\rho \in (0, 1)$, such that, for every $\alpha \in (1-\epsilon, 1)$,* (i) *the system* (13)–(14) *is uniformly stable;* (ii) *the solutions $\varepsilon(t)$ and $\bar{\Theta}(t)$ of the system* (13)–(14) *are uniformly bounded for $\forall t \ge t_0$, $\forall \varepsilon_0 \in \mathcal{R}^{Nl}$, $\bar{\Theta}_0 \in \mathcal{R}^{2Nm}$, where $\varepsilon_0 = \varepsilon(t_0)$, $\bar{\Theta}_0 = \bar{\Theta}(t_0)$;* (iii) $\lim_{t\to\infty}\|\varepsilon(t)\| = 0$, $\forall \varepsilon_0 \in \mathcal{R}^{Nl}$, $\forall \bar{\Theta}_0 \in \mathcal{R}^{2Nm}$; (iv) *there exists t_1^* such that $\varepsilon(t) \in \Omega_{\alpha,1}$, for any $t \ge t_1^*$.*

Proof. Consider the following Lyapunov candidate function

$$W_1(\varepsilon, \bar{\Theta}) = \varepsilon^T(I_N \otimes P)\varepsilon + \frac{1}{c\gamma}\bar{\Theta}^T\bar{\Theta}, \tag{29}$$

where $P > 0$ is given by (15). Calculating the derivative with respect to time t along the solution of the system (13)–(14), we have

$$\frac{dW_1}{dt} = \varepsilon^T[I_N \otimes (A_0^T P + PA_0) - H \otimes (2P\bar{b}\bar{b}^T P)]\varepsilon - 2(1-\rho)\varepsilon^T(H \otimes P\bar{b}\bar{b}^T P)\lceil\varepsilon\rfloor^{\bar{\alpha}}. \tag{30}$$

Setting $\tilde{\varepsilon}(t) = (U \otimes I_l)\varepsilon(t)$, applying (15) and noting that $\tilde{\varepsilon}^T\tilde{\varepsilon} = \varepsilon^T\varepsilon$, one has

$$\frac{dW_1}{dt} \le -\lambda_{\min}\varepsilon^T\varepsilon + 2(1-\rho)|\varepsilon^T(H \otimes P\bar{b}\bar{b}^T P)\lceil\varepsilon\rfloor^{\bar{\alpha}}|. \tag{31}$$

If $\varepsilon \in \mathcal{B}_1^c$, then we have

$$\begin{aligned} &|\varepsilon^T(H \otimes P\bar{b}\bar{b}^T P)\lceil\varepsilon\rfloor^{\bar{\alpha}}| \\ &< [\varepsilon^T(H \otimes P\bar{b}\bar{b}^T P)\varepsilon]^{\frac{1}{2}}[(\lceil\varepsilon\rfloor^{\bar{\alpha}})^T(H \otimes P\bar{b}\bar{b}^T P)\lceil\varepsilon\rfloor^{\bar{\alpha}}]^{\frac{1}{2}} \\ &< \lambda_F(\varepsilon^T\varepsilon)^{\frac{1}{2}}[(\lceil\varepsilon\rfloor^{\bar{\alpha}})^T\lceil\varepsilon\rfloor^{\bar{\alpha}}]^{\frac{1}{2}} \\ &\le \lambda_F(\varepsilon^T\varepsilon)^{\frac{1}{2}}\left(\sum_{i=1}^N\sum_{k=0}^{l-1}|\varepsilon_i^{(k)}|^{2\alpha_k}\right)^{\frac{1}{2}} \\ &\le \lambda_F(\varepsilon^T\varepsilon)^{\frac{1}{2}}\left(\sum_{i=1}^N\sum_{k=0}^{l-1}|\varepsilon_i^{(k)}|^2\right)^{\frac{1}{2}} \le \lambda_F\varepsilon^T\varepsilon, \end{aligned}$$

where $0 \le H \otimes P\bar{b}\bar{b}^T P < \lambda_F I_{Nl}$. Then, from (31), there exists $0 < \rho_2 < 1$, such that when $\max\{\rho_1, \rho_2\} < \rho < 1$, we have

$$\frac{dW_1}{dt} \le -\lambda_{\min}\varepsilon^T\varepsilon + 2(1-\rho)\lambda_F\varepsilon^T\varepsilon < -\frac{1}{2}\lambda_{\min}\varepsilon^T\varepsilon, \quad \varepsilon \in \mathcal{B}_1^c. \tag{32}$$

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If $\varepsilon \in \mathcal{B}_1$, then

$$\frac{dW_1}{dt} = (1-\rho)\varepsilon^T[I_N \otimes (A_0^T P + PA_0)\varepsilon - 2(H \otimes P\bar{b}\bar{b}^T P)\lceil\varepsilon\rfloor^{\tilde{\alpha}}] + \rho\varepsilon^T\left[I_N \otimes (A_0^T P + PA_0) - \frac{1}{\rho}H \otimes (2P\bar{b}\bar{b}^T P)\right]\varepsilon. \tag{33}$$

Setting $\tilde{\varepsilon}(t) = (U \otimes I_l)\varepsilon(t)$, and applying (15), we have

$$\lim_{\alpha\to 1}\varepsilon^T[I_N \otimes (A_0^T P + PA_0)\varepsilon - 2(H \otimes P\bar{b}\bar{b}^T P)\lceil\varepsilon\rfloor^{\tilde{\alpha}}] < -\lambda_{\min}\varepsilon^T\varepsilon.$$

Noting that $\mathcal{B}_1$ is a compact set, there exists $\epsilon_3 \in (0,1)$ such that for $\alpha \in (1-\epsilon_3, 1)$, $\varepsilon^T[I_N \otimes (A_0^T P + PA_0)\varepsilon - 2(H \otimes P\bar{b}\bar{b}^T P)\lceil\varepsilon\rfloor^{\tilde{\alpha}}] < -\lambda_{\min}\varepsilon^T\varepsilon$.

On the other hand, applying (16), there exists ρ, $\max\{\rho_1,\rho_2\} < \rho < 1$ such that $\varepsilon^T[I_N \otimes (A_0^T P + PA_0) - \frac{1}{\rho}H \otimes (2P\bar{b}\bar{b}^T P)]\varepsilon < -\frac{1}{2}\lambda_{\min}\varepsilon^T\varepsilon$.

Therefore,

$$\frac{dW_1}{dt} < -\frac{1}{2}\lambda_{\min}\varepsilon^T\varepsilon, \quad \varepsilon \in \mathcal{B}_1. \tag{34}$$

By the same arguments as in [25], and choosing $\epsilon = \min\{\epsilon_2, \epsilon_3\}$, $\max\{\rho_1,\rho_2\} < \rho < 1$, we obtain (i) and (ii).

From (32) and (34), we have $\lim_{t\to\infty} W_1(\varepsilon(t), \bar{\Theta}(t)) = W_1|_{t=\infty}$ and

$$\frac{1}{2}\lambda_{\min}\lim_{t\to\infty}\int_{t_0}^{t}\varepsilon(\tau)^T\varepsilon(\tau)d\tau < W_1|_{t=t_0} - W_1|_{t=\infty}.$$

By Barbalat's Lemma [25], we have $\lim_{t\to\infty}\|\varepsilon(t)\| = 0$. Thus (iii) is proved.

Construct a homogeneous function $U_\alpha(\varepsilon) = \sum_{i=1}^{N}\sum_{k=0}^{l-1}(\varepsilon_i^{(k)})^{p\alpha_k}$. Since $U_\alpha(\varepsilon)$ and $V_\alpha(\varepsilon)$ are homogeneous of degree p and d with respect to the weights defined in (21), respectively. Then, Lemma 4.2 in [29] ensures that there exists a constant $\kappa_2 > 0$ such that $V_\alpha(\varepsilon) < \kappa_2(U_\alpha(\varepsilon))^{\frac{d}{p}}$.

Note that $U_\alpha(\varepsilon) < \varepsilon^T\varepsilon < \frac{1}{\underline{\lambda}}\varepsilon^T(I_N \otimes P)\varepsilon$ due to $p\alpha_k > 2$, $k = 0, 1, \ldots, l-1$, as $\varepsilon \in \mathcal{B}_1$. It follows that $V_\alpha(\varepsilon) < \frac{1}{2}$ as $\varepsilon \in \Omega_{\sigma_1}$ with $\sigma_1 = \min\{\underline{\lambda}(\frac{\min\{\kappa_2,\frac{1}{2}\}}{\kappa_2})^{p/d}, 1\}$, due to $d \geq p$.

On the other hand, from (32) and (34), and $c \geq 1$, it follows that

$$\begin{aligned}\underline{\lambda}\varepsilon^T\varepsilon &< \varepsilon^T(I_N \otimes P)\varepsilon < W_1(\varepsilon, \bar{\Theta}) \\ &< \varepsilon_0^T(I_N \otimes P)\varepsilon_0 + \gamma^{-1}\bar{\Theta}_0^T\bar{\Theta}_0 \\ &\quad - \frac{\lambda_{\min}\sigma_1}{2\bar{\lambda}}(t-t_0), \quad \varepsilon \in \mathcal{R}^{Nl}\setminus\Omega_{\sigma_1}.\end{aligned} \tag{35}$$

Then, from (35), we can obtain that there exists $t_1^* > 0$,

$$t_1^* = \begin{cases} t_0 + \dfrac{2\bar{\lambda}\Delta}{\lambda_{\min}\sigma_1}, & \text{if } \Delta > 0, \\ t_0, & \text{otherwise,}\end{cases} \tag{36}$$

which is independent of c such that $\varepsilon \in \Omega_{\alpha,1}$ when $t \geq t_1^*$, where $\Delta = \varepsilon_0^T(I_N \otimes P)\varepsilon_0 + \gamma^{-1}\bar{\Theta}_0^T\bar{\Theta}_0 - \sigma_2$, $\sigma_2 = \min\{\underline{\lambda}(\frac{\min\{\kappa_2,1\}}{\kappa_2})^{p/d}, 1\}$. We also have

$$\bar{\Theta}^T(t)\bar{\Theta}(t) < c\gamma\sigma_2, \quad t \geq t_1^*. \tag{37}$$

This completes the proof of (iv). ■

Lemma 8. *Under Assumptions 1 and 2, choose $\bar{c}^T = \bar{b}^T P$ satisfying (15) and assume that Φ is continuous and uniformly bounded, there exists $\rho \in (0,1)$ and $\epsilon \in (0,1)$, for every $\alpha \in (1-\epsilon, 1)$, such that the system (13)–(14) is locally finite-time stable on set $\Omega_{\alpha,1}$.*

Proof. Calculating the derivative of $V_\alpha(\varepsilon)$ defined in (24) along the solution of the system (13), we have

$$\begin{aligned}\left.\frac{dV_\alpha(\varepsilon)}{dt}\right|_{(13)} &= (1-\rho)\left(\frac{\partial V_\alpha(\varepsilon)}{\partial\varepsilon}\right)^T[(I_N \otimes A_0)\varepsilon - (H \otimes \bar{b}\bar{c}^T)\lceil\varepsilon\rfloor^{\tilde{\alpha}}] \\ &\quad + \rho\left(\frac{\partial V_\alpha(\varepsilon)}{\partial\varepsilon}\right)^T\left[(I_N \otimes A_0) - \frac{1}{\rho}(H \otimes \bar{b}\bar{c}^T)\right]\varepsilon \\ &\quad + \frac{1}{c}\left(\frac{\partial V_\alpha(\varepsilon)}{\partial\varepsilon}\right)^T(H \otimes \bar{b})\Phi^T\bar{\Theta}, \quad \varepsilon \in \Omega_{\alpha,1} \\ &< -\kappa_1(1-\rho)V_\alpha^{\frac{\alpha d+\alpha-1}{\alpha d}} + \frac{1}{c}\left(\frac{\partial V_\alpha(\varepsilon)}{\partial\varepsilon}\right)^T(H \otimes \bar{b})\Phi^T\bar{\Theta} \\ &\quad + \rho\left(\frac{\partial V_\alpha(\varepsilon)}{\partial\varepsilon}\right)^T\left[(I_N \otimes A_0) - \frac{1}{\rho}(H \otimes \bar{b}\bar{c}^T)\right]\varepsilon, \quad \varepsilon \in \Omega_{\alpha,1}.\end{aligned} \tag{38}$$

Since $\lim_{\alpha\to 1}(\frac{\partial V_\alpha(\varepsilon)}{\partial\varepsilon})^T[(I_N \otimes A_0) - \frac{1}{\rho}(H \otimes \bar{b}\bar{c}^T)]\varepsilon < 0$, there exists $\epsilon_4 \in (0,1)$ such that, for every $\alpha \in (1-\min\{\epsilon_2, \epsilon_3, \epsilon_4\}, 1)$, $(\frac{\partial V_\alpha(\varepsilon)}{\partial\varepsilon})^T[(I_N \otimes A_0) - \frac{1}{\rho}(H \otimes \bar{b}\bar{c}^T)]\varepsilon < 0$, and then, for $\varepsilon \in \Omega_{\alpha,1}$,

$$\left.\frac{dV_\alpha(\varepsilon)}{dt}\right|_{(13)} < -\kappa_1(1-\rho)V_\alpha^{\frac{\alpha d+\alpha-1}{\alpha d}} + \frac{1}{c}\left(\frac{\partial V_\alpha(\varepsilon)}{\partial\varepsilon}\right)^T \times (H \otimes \bar{b})\Phi^T\bar{\Theta}. \tag{39}$$

From Lemma 7, there exists t_1^* such that $\varepsilon \in \Omega_{\alpha,1}$, for any $t \geq t_1^*$. Let $\varepsilon(t, t_0, \varepsilon_0)$ be the non-trivial solution of (13). In the following, we consider three cases.

Case 1: There exists an interval $[t_1, t_2] \subset [t_1^*, t_1^*+T_2]$ $(t_2 > t_1, T_2 = \frac{2\alpha d}{\kappa_1(1-\rho)(1-\alpha)})$ such that $\varepsilon(t, t_0, \varepsilon_0) = 0$ for any $t \in [t_1, t_2]$.

It follows from (13) and (14) that $\bar{\Theta}(t)$ is a constant vector for $t \in [t_1, t_2]$. Therefore, in this case, $\varepsilon(t) \equiv 0$ and then $\dot{\bar{\Theta}} \equiv 0$ for any $t > t_1$. It follows that $\bar{\Theta}$ is a constant vector for any $t > t_1$.

Now we prove $\bar{\Theta}(t) \equiv 0$ for any $t > t_1$ by contradiction. Construct function $\Psi(\bar{\Theta}(t), t) = \frac{1}{2}[\bar{\Theta}^T(t+T_1)\bar{\Theta}(t+T_1) - \bar{\Theta}^T(t)\bar{\Theta}(t)]$ with $T_1 > 0$. It is obvious that $\Psi(\bar{\Theta}(t), t) \equiv 0, \forall t > t_1$, due to the fact that $\bar{\Theta}(t)$ is constant, then $\frac{d\Psi(\bar{\Theta}(t),t)}{dt} \equiv 0, \forall t > t_1$. Supposing $\bar{\Theta}(t) \neq 0$ and calculating the derivative of function $\Psi(\bar{\Theta}(t), t)$ with respect to time, we have

$$\begin{aligned}\frac{d\Psi(\bar{\Theta}(t), t)}{dt} &= \bar{\Theta}^T(t+T_1)\dot{\bar{\Theta}}(t+T_1) - \bar{\Theta}^T(t)\dot{\bar{\Theta}}(t) \\ &= \int_t^{t+T_1}\frac{d}{d\tau}(\bar{\Theta}^T(\tau)\dot{\bar{\Theta}}(\tau))d\tau \\ &= -\gamma\int_t^{t+T_1}\frac{d}{d\tau}[\bar{\Theta}^T(\tau)\Phi(H \otimes \bar{c}^T)\varepsilon]d\tau \\ &= -\gamma\int_t^{t+T_1}\bar{\Theta}^T\Phi(H^2 \otimes \bar{c}^T\bar{b})\Phi^T\bar{\Theta}d\tau \\ &= -\gamma\int_t^{t+T_1}\bar{\Theta}^T\Phi(H^2 \otimes p_{ll})\Phi^T\bar{\Theta}d\tau \\ &\leq -\gamma p_{ll}\lambda_{\max}^2\int_t^{t+T_1}\bar{\Theta}^T\Phi\Phi^T\bar{\Theta}d\tau \\ &= -\gamma p_{ll}\lambda_{\max}^2\bar{\Theta}^T\left(\int_t^{t+T_1}\Phi\Phi^T d\tau\right)\bar{\Theta},\end{aligned} \tag{40}$$

where $p_{ll} > 0$ is the ll-th entry of positive definite matrix $P = (p_{ij})_{l\times l}$. Applying the PE condition defined in (28) to the last

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equation in (40), we have

$$\frac{d\Psi(\bar{\Theta}(t),t)}{dt} < -\gamma p_{ll}\kappa_0\lambda_{\max}^2\|\bar{\Theta}\|^2 < 0, \quad \forall t > t_1, \tag{41}$$

which contradicts that $\frac{d\Psi(\bar{\Theta}(t),t)}{dt} \equiv 0, \forall t > t_1$. Therefore, $\bar{\Theta} \equiv 0$, $\forall t > t_1$. In this case, the system (13) and (14) is finite-time stable.

Case 2: $\varepsilon(t,t_0,\varepsilon_0)$ only passes through 0 and do not stay on 0 as $t \in [t_1^*, t_1^*+T_2]$.

Let $\{t_i : i \in \mathcal{I}\} \subset [t_1^*, t_1^*+T_2]$ be a time sequence that $\varepsilon(t,t_0,\varepsilon_0)$ passes through 0 at each t_i, where $\mathcal{I}$ is an index set. Without loss of generality, we suppose that $t_i < t_{i+1}, i, i+1 \in \mathcal{I}$. We firstly illustrate that $\{t_i : i \in \mathcal{I}\}$ must be a finite set. Otherwise, there exists a infinite subsequence $\{t_{i_k} : i_k \in \mathcal{I}\}$ of $\{t_i : i \in \mathcal{I}\}$ such that $\lim_{k\to\infty} t_{i_k} = t^*, t^* \in [t_1^*, t_1^*+T_2]$. Then, for any $\delta > 0$, there exists $k^* > 0$ such that as $k > k^*$, $|t_{i_k} - t^*| < \delta$. Since $\varepsilon(t,t_0,\varepsilon_0)$ and $\dot{\varepsilon}(t,t_0,\varepsilon_0)$ are continuous, then for any $t \in (t^*-\delta, t^*)$, we have $\varepsilon(t,t_0,\varepsilon_0) \equiv 0$. This contradicts our assumption. Let $\{t_i\}_{i=1}^n$ be a finite set that $\varepsilon(t,t_0,\varepsilon_0)$ passes through 0 at each t_i. There exists $\delta_1 > 0$ such that set $\mathcal{T} = \cup_{i=1}^n (t_i-\delta_0, t_i+\delta_0)$ is nonempty open set for any $\delta_0 > 0$, $0 < \delta_0 < \delta_1$. Then, $\varepsilon(t,t_0,\varepsilon_0) \neq 0$ on the compact set $[t_1^*, t_1^*+T_2] \setminus \mathcal{T}$. Note that $\varepsilon(t,t_0,\varepsilon_0)$ is continuous on $[t_1^*, t_1^*+T_2] \setminus \mathcal{T}$, therefore, there exists a constant $\kappa_3 > 0$ such that, $U_\alpha(\varepsilon) = \sum_{i=1}^N \sum_{k=0}^{l-1} (\varepsilon_i^{(k)})^{p\alpha_k} \geq \kappa_3, \forall t \in [t_1^*, t_1^*+T_2] \setminus \mathcal{T}$.

It follows from (37) that there exists $\tilde{c} > 1$, $0 < \mu < 1$ such that for $c > \tilde{c}$,

$$\frac{1}{c^{2-2\mu}}\bar{\Theta}^T\bar{\Theta} < \kappa_3 \leq U_\alpha(\varepsilon), \quad \forall t \in [t_1^*, t_1^*+T_2] \setminus \mathcal{T}, \tag{42}$$

and

$$\frac{c^\mu \kappa_1(1-\rho)}{\lambda_\theta \kappa_4 N l} > 1, \tag{43}$$

where κ_4 will be given later, $\lambda_\theta > 0$ is given such that $\Phi(H^2 \otimes \bar{b}^T\bar{b})\Phi^T < \lambda_\theta^2 I$ due to Φ being uniformly bounded. It follows from (39) and (42) that

$$\begin{aligned} \left.\frac{dV_\alpha(\varepsilon)}{dt}\right|_{(13)} &< -\kappa_1(1-\rho)V_\alpha^{\frac{\alpha d+\alpha-1}{\alpha d}} + \frac{\lambda_\theta}{c}\left(\frac{\partial V_\alpha^T}{\partial \varepsilon}\frac{\partial V_\alpha}{\partial \varepsilon}\right)^{\frac{1}{2}} (\bar{\Theta}^T\bar{\Theta})^{\frac{1}{2}} \\ &< -\kappa_1(1-\rho)V_\alpha^{\frac{\alpha d+\alpha-1}{\alpha d}} + \frac{\lambda_\theta}{c^\mu}\left(\frac{\partial V_\alpha^T}{\partial \varepsilon}\frac{\partial V_\alpha}{\partial \varepsilon}\right)^{\frac{1}{2}} (U_\alpha)^{\frac{1}{2}}, \\ &\forall t \in [t_1^*, t_1^*+T_2] \setminus \mathcal{T}. \end{aligned} \tag{44}$$

Note that $(\frac{\partial V_\alpha}{\partial \varepsilon_i^{(k)}})^2 U_\alpha(\varepsilon)$ is homogeneous of degree $2(d-\frac{1}{\alpha_k})+p$ and $\frac{d-\frac{1}{\alpha_k}+\frac{p}{2}}{d} > 1$, then, there exists $\kappa_4 > 0$ [29] such that

$$\begin{aligned} \left|\sum_{i=1}^N \sum_{k=0}^{l-1} \left(\frac{\partial V_\alpha}{\partial \varepsilon_i^{(k)}}\right)^2 \cdot U_\alpha(\varepsilon)\right| &< \sum_{i=1}^N \sum_{k=0}^{l-1} \kappa_4^2 V_\alpha^{\frac{2(d-\frac{1}{\alpha_k}+\frac{p}{2})}{d}} \\ &< \kappa_4^2 N^2 l^2 V_\alpha^2. \end{aligned} \tag{45}$$

Therefore, from (43)–(45), it follows that

$$\begin{aligned} \left.\frac{dV_\alpha(\varepsilon)}{dt}\right|_{(13)} &< -\kappa_1(1-\rho)V_\alpha^{\frac{\alpha d+\alpha-1}{\alpha d}} + \frac{\lambda_\theta \kappa_4 N l}{c^\mu}V_\alpha \\ &< -\frac{1}{2}\kappa_1(1-\rho)V_\alpha^{\frac{\alpha d+\alpha-1}{\alpha d}}, \\ &\forall t \in [t_1^*, t_1^*+T_2] \setminus \mathcal{T}. \end{aligned} \tag{46}$$

Since $\left.\frac{dV_\alpha(\varepsilon)}{dt}\right|_{(13)}$ is continuous on $[t_1^*, t_1^*+T_2]$, there exists $\delta_2 > 0$ such that, for $0 < \delta_0 < \delta_2$, we have

$$\left.\frac{dV_\alpha(\varepsilon)}{dt}\right|_{(13)} < -\frac{1}{2}\kappa_1(1-\rho)V_\alpha^{\frac{\alpha d+\alpha-1}{\alpha d}}, \quad \forall t \in [t_1^*, t_1^*+T_2]. \tag{47}$$

By Lemma 2, the system (13)–(14) is finite-time convergent with settling time

$$T_s < \frac{2\alpha d V_\alpha(t_1^*)^{\frac{1-\alpha}{\alpha d}}}{\kappa_1(1-\rho)(1-\alpha)} < \frac{2\alpha d}{\kappa_1(1-\rho)(1-\alpha)}. \tag{48}$$

This implies $\varepsilon(t,t_0,\varepsilon_0) = 0$ for any $t \in [t_1^*+T_0, t_1^*+T_2]$, which contradicts our assumption. Therefore, the second case does not happen.

Case 3: $\varepsilon(t,t_0,\varepsilon_0) \neq 0$ for any $t \in [t_1^*, t_1^*+T_2]$.

Using the same method, we can derive that the third case does not happen either.

In the following, we give one of explicit estimation of the bound of T_s. From (48), the bound of the setting time T_s is dependent on κ_1. Once the lower bound of κ_1 is given, then, the bound of T_s can be obtained. By Lemma 4.2 of [29], we have

$$\left.\frac{dV_\alpha(\varepsilon)}{d\varepsilon}\right|_{(19)} < \left[\max_{\{\varepsilon:V_\alpha(\varepsilon)=1\}} \left.\frac{dV_\alpha(\varepsilon)}{d\varepsilon}\right|_{(19)}\right] V_\alpha(\varepsilon)^{\frac{\alpha d+\alpha-1}{\alpha d}}.$$

Using the same method as in [32], we have

$$\begin{aligned} \lim_{\alpha\to 1}\left[\max_{\{\varepsilon:V_\alpha(\varepsilon)=1\}} \left.\frac{dV_\alpha(\varepsilon)}{d\varepsilon}\right|_{(19)}\right] &= \max_{\{\varepsilon:V_1(\varepsilon)=1\}} \left.\frac{dV_1(\varepsilon)}{d\varepsilon}\right|_{(22)} \\ &\leq \max_{\{\varepsilon:V_1(\varepsilon)=1\}} \int_0^\infty \frac{1}{\lambda^{d-1}} a'(\lambda^2\varepsilon^T(I_N\otimes P)\varepsilon)(-\lambda_{\min}\varepsilon^T\varepsilon)d\lambda \\ &\leq -\min_{\{\varepsilon:V_1(\varepsilon)=1\}} \int_0^\infty \frac{1}{\lambda^{d-1}} a'(\lambda^2\varepsilon^T(I_N\otimes P)\varepsilon)\lambda_{\min}\varepsilon^T\varepsilon d\lambda, \end{aligned}$$

where $V_1(\varepsilon) = V_\alpha(\varepsilon)|_{\alpha=1} = \int_0^\infty \frac{1}{\lambda^4} a(\lambda^2\varepsilon^T(I_N\otimes P)\varepsilon)d\lambda$.

Then, there exists $0 < \epsilon_5 < 1$ such that when $\alpha \in (1-\epsilon_5, 1)$, we have

$$\kappa_1 > \frac{1}{2}\min_{\{\varepsilon:V_1(\varepsilon)=1\}} \int_0^\infty \frac{1}{\lambda^{d-1}} a'(\lambda^2\varepsilon^T(I_N\otimes P)\varepsilon)\lambda_{\min}\varepsilon^T\varepsilon d\lambda. \tag{49}$$

Moreover, we can select that $d = 3$ and $a(s)$ as constructed in (25) and consider the set $\{\varepsilon : \varepsilon^T(I_N\otimes P)\varepsilon = r^2, r > 0\}$. We will determine the value of r such that $V_1(\varepsilon) = 1$. Then,

$$\begin{aligned} 1 &= \int_0^\infty \frac{1}{\lambda^4} a(\lambda^2 r^2)d\lambda \\ &= \int_{\frac{1}{r}}^{\sqrt{\frac{3}{2}}\frac{1}{r}} \frac{2}{\lambda^4}(\lambda^2r^2-1)^2 d\lambda \\ &\quad + \int_{\sqrt{\frac{3}{2}}\frac{1}{r}}^{\frac{\sqrt{2}}{r}} \frac{1}{\lambda^4}(1-2(\lambda^2r^2-2)^2)d\lambda + \int_{\frac{\sqrt{2}}{r}}^\infty \frac{1}{\lambda^4}d\lambda. \end{aligned}$$

By simple computation, we have

$$\frac{16}{3}(\sqrt{6}-1-\sqrt{2})r^3 = 1.$$

Then,

$$r = \frac{16^{1/3}}{3^{1/3}(\sqrt{6}-1-\sqrt{2})^{1/3}}.$$

Note that $\{\varepsilon : \varepsilon^T(I\otimes P)\varepsilon = r_0^2\} \subset \{\varepsilon : V_1(\varepsilon) = 1\}$. On the other hand, for any $0 \neq \varepsilon \in \{\varepsilon : V_1(\varepsilon) = 1\}$, there exists $r' > 0$ such that $\varepsilon^T(I_N\otimes P)\varepsilon = (r')^2$. Then, we have $r = r'$. Therefore,

$$\{\varepsilon : V_1(\varepsilon) = 1\} = \{\varepsilon : \varepsilon^T(I_N\otimes P)\varepsilon = r^2\}.$$

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Now, we will give the lower bound of κ_1. From (26) and (49), we have

$$\begin{aligned}\kappa_1 &> \frac{1}{2}\min_{\{\varepsilon:V_1(\varepsilon)=1\}}\int_0^\infty \frac{1}{\lambda^2}a'(\lambda^2\varepsilon^T(I_N\otimes P)\varepsilon)\lambda_{\min}\varepsilon^T\varepsilon d\lambda\\ &> \frac{1}{2}\int_0^\infty \frac{1}{\lambda^2}a'(\lambda^2 r^2)\frac{\lambda_{\min}}{\bar\lambda}r d\lambda\\ &= \frac{1}{2}\int_{\frac{1}{r}}^{\sqrt{\frac{3}{2}}\frac{1}{r}}\left(4r^2-\frac{1}{\lambda^2}\right)d\lambda-\int_{\sqrt{\frac{3}{2}}\frac{1}{r}}^{\frac{\sqrt{2}}{r}}\left(4r^2-\frac{2}{\lambda^2}\right)d\lambda\\ &= \frac{5}{2}\left(\frac{16}{5}\right)^{1/3}(\sqrt{6}-1-\sqrt{2})\frac{\lambda_{\min}}{\bar\lambda}.\end{aligned}$$

In conclusion, we can select $c > \tilde c$, $\max\{\rho_1,\rho_2\} < \rho < 1$, $0 < \delta_0 < \min\{\delta_1,\delta_2\}$, $\epsilon = \min\{\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5\}$, such that, for every $\alpha \in (1-\epsilon, 1)$, the system (13)–(14) is finite-time stable on $\Omega_{\alpha,1}$ and the settling time T_s is given by

$$T_s < \frac{15\sqrt[3]{5}\alpha\bar\lambda}{\sqrt[3]{16}(\sqrt{6}-1-\sqrt{2})(1-\rho)(1-\alpha)\lambda_{\min}}. \quad \blacksquare \tag{50}$$

In summary, we give a flow chart in Fig. 1 to show how to decompose the original problem. In Section 3.1, we obtain the finite-time stability of the homogeneous system (19). Then, by constructing a homogeneous Lyapunov function (24), we obtain an important inequality in Lemma 6. In Section 3.2, based on the homogeneous Lyapunov function (24) and the important inequality obtained in Lemma 6, we prove that the state of the nonhomogeneous system (13)–(14) can enter into the set $\Omega_{\alpha,1}$ in finite-time for any initial state $\varepsilon(t_0)$ in Lemma 7 and the nonhomogeneous system (13)–(14) is locally finite-time stable on set $\Omega_{\alpha,1}$ in Lemma 8. Therefore, the result of Theorem 1 follows from Lemmas 7 and 8 directly.

4. Simulations

In this section, we give an example to validate our theoretical results. Consider a multi-agent system consisting of a leader driven by control input $u_0(t) = \frac{\sqrt{2}}{2}\sin t = \phi_0(t)^T\theta_0$, where $\phi_0(t) = \sin t$, $\theta_0 = \frac{\sqrt{2}}{2}$, and four follower agents with nonlinear dynamics

$$\begin{aligned}f_i(x_i(t),t) &= \frac{i}{3}\cos t + \exp(\xi_i^{(0)}+\xi_i^{(1)}+\xi_i^{(2)}+\xi_i^{(3)})\\ &= (\cos t, \exp(\xi_i^{(0)}+\xi_i^{(1)}+\xi_i^{(2)}+\xi_i^{(3)}))\begin{pmatrix}\frac{i}{3}\\ 1\end{pmatrix}\\ &= \phi_i(x_i(t),t)^T\theta_i, \quad i = 1,2,3,4,\end{aligned}$$

where $\phi_i = \begin{pmatrix}\cos t\\ \exp(\xi_i^{(0)}+\xi_i^{(1)}+\xi_i^{(2)}+\xi_i^{(3)})\end{pmatrix}$, $\theta_i = \begin{pmatrix}\frac{i}{3}\\ 1\end{pmatrix}$, $i = 1,2,3,4$.

Let $\lambda_{PE}(t)$ be the minimum eigenvalue of $\int_t^{t+\nu}\Phi\Phi^T d\tau$. Choosing $\nu = 5$, Fig. 2 shows that $\lambda_{PE}(t) > 0$ for all $t \geq 0$. Therefore, the PE condition (28) is satisfied with $\kappa_0 = 0.0047$. Suppose that the interconnected topology is described as in Fig. 3. We obtain that the smallest nonzero eigenvalue of $H = L+B$ is $\lambda_{\min} = 0.382$ by a straightforward calculation. Solving the Riccati inequality (15), we get a solution $P > 0$ and the largest eigenvalue of P is $\bar\lambda = 13.86$. By simple computation, we get $T_s < 75543$ s.

For the system (13)–(14), we choose the initial consensus error vector as $\bar x(0) = (-5.7, 7.0, 0, -0.7, -9.8, 11.7, -2.9, -0.6, -6.1, 1.2, -2.7, -3.7, -8.2, -0.6, -1.9, -7.8)^T$ and the initial parameter estimate error vector as $\tilde\Theta(0) = (0.2, -0.9, 1.4, -1.7, 0.5, -0.5, -1.7, 0.4, 1.6, -1.8, 0.4, -0.4)^T$. Under the control law (8) and the adaptive update law (9) with $\alpha = 0.85$, $\rho = 0.7$, $\mu = 0.1$, $\gamma = 0.2$, $c = 1$ and $\alpha_0 = 0.5862$, $\alpha_1 = 0.6538$, $\alpha_2 = 0.7391$, $\alpha_3 = \alpha$ according to (20), simulation is conducted in 150 s time. Fig. 4 shows sixteen components of the consensus error vector $\bar x(t) = x(t) - \mathbf{1}_N \otimes x_0(t)$. Figs. 5 and 6 show the parameter estimate errors $\hat\theta_{0i} - \theta_0$ and $\hat\theta_i - \theta_i$, $i = 1,2,3,4$, respectively. From the simulation, we see that all agents follow the leader with parameter convergence within simulation time duration.

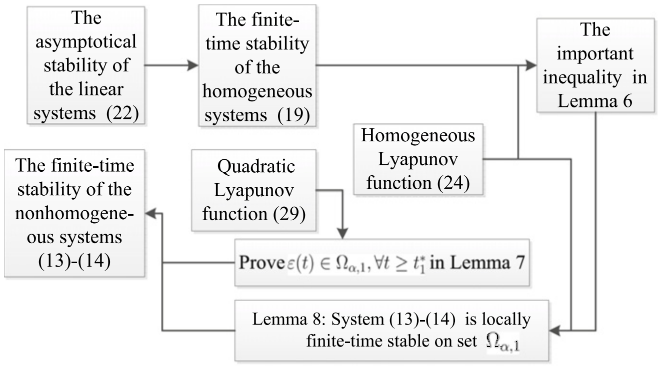

Fig. 1. Problem decomposition.

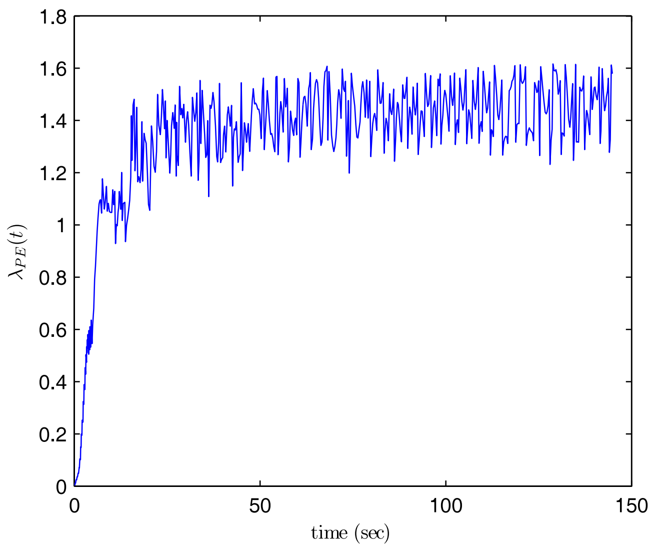

Fig. 2. The PE condition is satisfied.

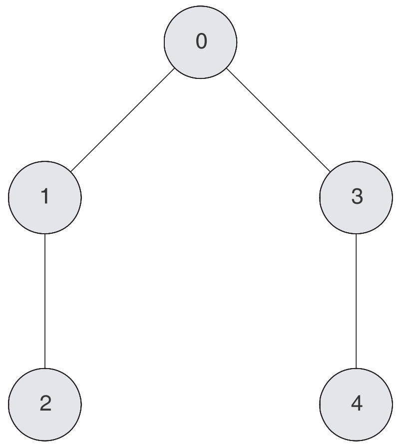

Fig. 3. Connected graph.

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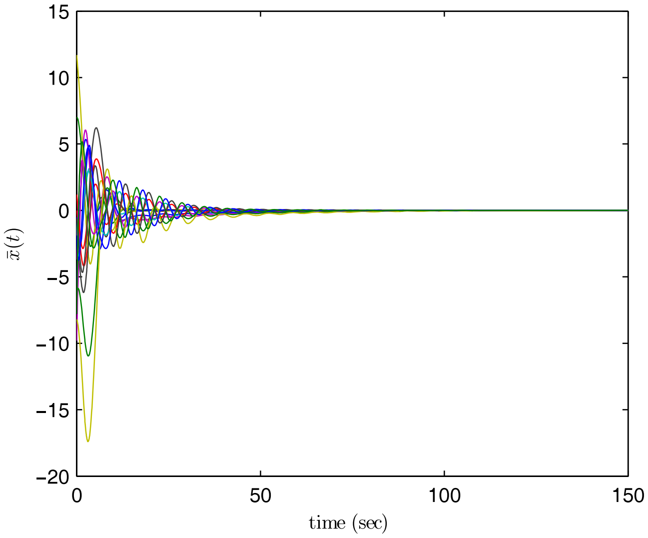

Fig. 4. Sixteen components of consensus error vector $\bar{x}(t)$.

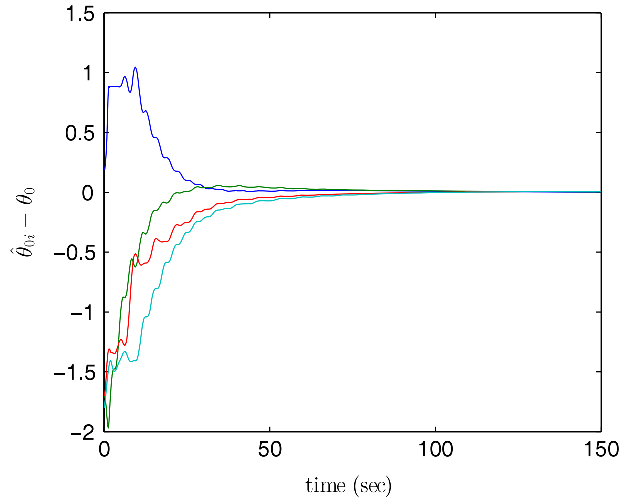

Fig. 5. Parameter estimate errors $\hat{\theta}_{0i} - \theta_0$, for $i = 1, 2, 3, 4$.

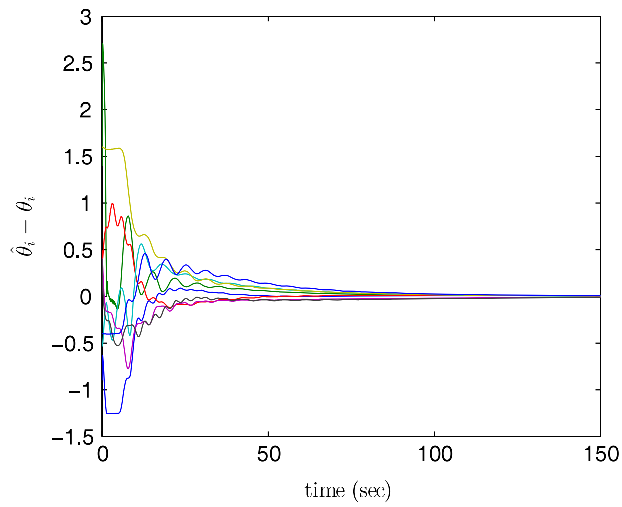

Fig. 6. Parameter estimate errors $\hat{\theta}_i - \theta_i$, for $i = 1, 2, 3, 4$.

5. Conclusions

In this paper, we consider the leader-following finite-time consensus problem of multi-agent systems with unknown nonlinear dynamics. A framework for designing adaptive finite-time consensus protocols, that guarantee finite-time consensus with finite-time parameter convergence, is presented. Lyapunov techniques, Riccati inequalities, homogeneous Lyapunov function, and homogeneity of vector field are applied in the control design and stability analysis. The connectivity of graph and the PE condition are the key to ensure finite-time consensus and finite-time parameter convergence, respectively.

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