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Brief Paper

Analysis of nonlinear time-delay systems using modules over non-commutative rings ☆

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Abstract

The theory of non-commutative rings is introduced to provide a basis for the study of nonlinear control systems with time delays. The left Ore ring of non-commutative polynomials defined over the field of meromorphic function is suggested as the framework for such a study. This approach is then generalized to a broader class of nonlinear systems with delays that are called generalized Roesser systems. Finally, the theory is applied to analyze nonlinear time-delay systems. A weak observability is defined and characterized, generalizing the well-known linear result. Properties of closed submodules are then developed to obtain a result on the accessibility of such systems.

Keywords: Accessibility; Non-commutative rings; Nonlinear systems; Observability; Ore rings; Roesser model; Time delay

1. Introduction

The module-theoretical approach has turned out to be a very powerful tool in the study of control systems; see for instance Conte and Perdon (1984), Conte and Perdon (1998), Fuhrmann (1976), Kalman, Falb, and Arbib (1969), Morse (1976), Conte, Moog and Perdon (1999), Grizzle (1993), and Kotta (1995). In the above-mentioned literature, either a field or a commutative ring is used. This is because only time-invariant linear systems (with or without delays) and time-invariant nonlinear systems without delays are considered. In the case of time-varying linear systems, or nonlinear systems with delays, it has been shown in Ježek (1996a), Moog, Castro-Linares, Velasco-Villa and Márquez-Martínez (2000) that the non-commutative operators have to be applied.

Some applications of the theory of non-commutative rings to control theory are due to Ježek (Ježek, 1996a, b). However, Ježek's works are rather focused on the background mathematics of non-commutative rings than control theory itself. An explicit application to control systems is done in Moog et al. (2000) where a class of nonlinear time-delay systems is studied. The disturbance decoupling problem for nonlinear time-delay systems is tackled in Moog et al. (2000) and the system inversion of nonlinear time-delay systems is discussed in Márquez-Martínez, Moog and Velasco-Villa (2000b).

In this paper, the theory of non-commutative rings is introduced to provide a basis for the study of nonlinear control systems with time delays. The left Ore ring of non-commutative polynomials defined over the field of meromorphic functions is suggested as the framework for such a study. This approach is then generalized to a broader class of nonlinear systems with delays that are called generalized Roesser systems. This is done with the help of rings of fractions constructed over the left Ore ring. Finally, the theory is applied to analyze nonlinear time-delay systems. A weak observability is defined and characterized, generalizing the well-known linear result. Properties of closed submodules are then developed to obtain a result on the accessibility of such systems.

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The layout of the paper is as follows: the framework is presented in Section 2. The necessary background on non-commutative algebra is collected in Appendix A. Section 3 is devoted to application of the theory developed to nonlinear time-delay systems. Conclusions are drawn in Section 4.

2. The algebraic framework

An algebraic approach was taken in Conte et al. (1999), Márquez-Martínez et al. (2000b), Moog et al. (2000) to deal with nonlinear systems with time delays:

$$\begin{aligned} \dot{x}(t) &= F(t) := f(x(t-i), i \in S\_) \\ &\quad + \sum_{j=0}^{s} g_j(x(t-i), i \in S\_) u(t-j), \\ y(t) &= h(x(t-i), i \in S\_), \\ x(t) &= \varphi(t); u(t) = u_0, \quad \forall t \in [t_0 - s, t_0] \end{aligned} \tag{1}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, output $y \in \mathbb{R}^p$, and $\mathbb{R}$ standing for the field of real numbers. f, g_j and h are meromorphic functions, $S\_ := \{0, 1, \ldots, s\}$ is a finite set of constant time delays, $f(x(t-i), i \in S\_) := f(x(t), x(t-1), \ldots, x(t-s))$, and φ denotes a continuous function of initial conditions. Without loss of generality, assume that the time delays are integers, because any commensurate time delays can be transformed into integer ones. To exclude singularities, assume that

$$\phi(x(t-i), u(t-i), \ldots, u^{(k)}(t-i)) \equiv 0 \tag{2}$$

holds for no non-trivial meromorphic function ϕ.

One of the objectives of this paper is to generalize the approach to a broader class of systems

$$\begin{aligned} \dot{x}(t) &= \mathscr{F}(t) := f(x(t), z(t), u(t)), \\ z(t+1) &= g(x(t), z(t), u(t)), \\ y(t) &= h(x(t), z(t), u(t)) \end{aligned} \tag{3}$$

in which the discrete state $z \in \mathbb{R}^q$.

This class of systems can be seen as a nonlinear extension of the Roesser model (Roesser, 1975) studied in the theory of linear 2D systems. Systems (1) can also be written in the above format. Actually, if $q \in N$ is the maximal delay occurring in the equations of (1), defining, for $i = 1, \ldots, q$, $z_{1i}(t) = x(t-i)$, $z_{2i}(t) = u(t-i)$, then one has the following discrete-time dynamics for the delay system:

$$z_1(t+1) = A_1 z_1(t) + B_1 x(t), \tag{4}$$

$$z_2(t+1) = A_2 z_2(t) + B_2 u(t) \tag{5}$$

in which (A_i, B_i) are (block) controllability pairs.

To generalize the algebraic approach described in Conte et al. (1999), Márquez-Martínez et al. (2000b), Moog et al. (2000) to system (3), let $\mathscr{C} := \{x_j(t-i), u_l^{(k)}(t-i), j = 1, 2, \ldots, n, \ l = 1, 2, \ldots, m, \ k, i \in Z^+\}$, where Z^+ denotes the set of nonnegative integers, and let $\mathscr{K}$ be the field of meromorphic functions of a finite number of the variables listed in $\mathscr{C}$.

Let $\mathscr{K}[s]$ denote the commutative ring of polynomials in the indeterminate s over $\mathscr{K}$ and let $\mathscr{K}(\delta]$ denote the set of polynomials of the form

$$a(\delta] = a_0(t) + a_1(t)\delta + \cdots + a_{r_a}(t)\delta^{r_a} \tag{6}$$

in which $a_i(t) \in \mathscr{K}$. If addition in $\mathscr{K}(\delta]$ is defined as usually, while multiplication is given by

$$a(\delta] \cdot b(\delta] = \sum_{k=0}^{r_a+r_b} \sum_{i+j=k}^{i \leqslant r_a, j \leqslant r_b} a_i(t) b_j(t-i) \delta^k \tag{7}$$

then $\mathscr{K}(\delta]$ is a *non-commutative* ring. In addition, this ring is not a skew polynomial ring, due to the definition of $\mathscr{K}$ (Ježek, 1996b). Despite these particularities, the ring $\mathscr{K}(\delta]$ permits to establish a module theory over it, due to the next lemma which shows that the ring $\mathscr{K}(\delta]$ is a left Ore ring; see Ore (1931, 1933), Cohn (1971), Ježek (1996b) and the references therein.

Lemma 1. *For any $a(\delta], b(\delta] \in \mathscr{K}(\delta]$, there exist non-zero polynomials $c(\delta], d(\delta] \in \mathscr{K}(\delta]$ such that $c(\delta] \cdot a(\delta] = d(\delta] \cdot b(\delta]$.*

Proof. Since

$$c(\delta] \cdot a(\delta] = \sum_{k=0}^{r_c+r_a} \sum_{i+j=k}^{i \leqslant r_c, j \leqslant r_a} c_i(t) a_j(t-i) \delta^k,$$

$$d(\delta] \cdot b(\delta] = \sum_{k=0}^{r_d+r_b} \sum_{i+j=k}^{i \leqslant r_d, j \leqslant r_b} d_i(t) b_j(t-i) \delta^k,$$

where r_a is the *degree* of $a(\delta]$. To have $c(\delta] \cdot a(\delta] = d(\delta] \cdot b(\delta]$, one only needs to choose r_c and r_d such that

$$r_c + r_a = r_d + r_b, \tag{8}$$

and for $k = 0, \ldots, r_c + r_a$

$$\sum_{i+j=k}^{i \leqslant r_c, j \leqslant r_a} c_i(t) a_j(t-i) \delta^k = \sum_{i+j=k}^{i \leqslant r_d, j \leqslant r_b} d_i(t) b_j(t-i) \delta^k. \tag{9}$$

For these $r_c + r_a + 1$ independent equations and $r_c + r_d + 2$ variables consisting in the coefficients of $c(\delta]$ and $d(\delta]$, there exist $(r_c + r_d + 2) - (r_c + r_a + 1) = r_c - r_b + 1$ independent solutions. Thus, choosing $r_c > r_b$ and r_d satisfying (8), ensures that (9) has a non-zero solution.

Further, the ring $\mathscr{K}(\delta]$ has many important features as the ring $\mathscr{K}[s]$. Some of them are described below and some of them are summarized in Appendix A.

Theorem 1. $\mathscr{K}(\delta]$ *is a Noetherian ring.*

The proof follows basically the same line as the proof of the Hilbert Base Theorem (see for instance Farb and Dennis (1993)) and therefore is omitted.

Next, as $\mathscr{K}(\delta]$ is a left Ore ring and also a (left) domain of integrity, it is possible to construct a (left) ring of fractions, hereafter denoted as $\mathscr{K}\langle\delta)$. An element of $\mathscr{K}\langle\delta)$, say $w\langle\delta)$, is called a rational function and denoted as $b(\delta]\backslash a(\delta]\in\mathscr{K}\langle\delta)$, or by $b^{-1}(\delta]a(\delta]$. For a left Ore ring $\mathscr{R}$ and given set of symbols $\{\mathrm{d}x, \mathrm{d}u, \mathrm{d}\dot{u}, \dots, \mathrm{d}u^{(k)}, \dots,\}$, k is finite, we define a left $\mathscr{R}$-module $\mathscr{M}_k := \mathrm{span}_{\mathscr{R}}\{\mathrm{d}x, \mathrm{d}u, \mathrm{d}\dot{u}, \dots, \mathrm{d}u^{(k)}\}$. Let $\mathscr{M}$ denote a union of all such finitely generated modules $\mathscr{M}_k$. In this paper, $\mathscr{R}$ can be either $\mathscr{K}(\delta]$ or $\mathscr{K}\langle\delta)$. Following the algebraic tradition, those modules over the division ring $\mathscr{K}\langle\delta)$ are called vector spaces over $\mathscr{K}\langle\delta)$. As indicated by the name, these vector spaces share many properties of the vector spaces defined over commutative fields.

A special kind of submodules over $\mathscr{K}(\delta]$ is very important for this paper. First, the concept of closure of a submodule introduced in Conte and Perdon (1984) can be generalized to submodules over $\mathscr{K}(\delta]$.

Let $\mathscr{N}$ be a submodule of $\mathscr{M}$. The closure of $\mathscr{N}$ in $\mathscr{M}$ over $\mathscr{K}(\delta]$ is the submodule

$$\bar{\mathscr{N}} = \{\omega\in\mathscr{M} \mid \exists 0\neq a(\delta]\in\mathscr{K}(\delta],\ \text{such that } a(\delta]\omega\in\mathscr{N}\}.$$

If $\mathscr{N}$ coincides with $\bar{\mathscr{N}}$, $\mathscr{N}$ is called closed in $\mathscr{M}$.

The following is a list of results about closure and closeness of submodules. The proof of these results is either straightforward or follows the same lines as in Conte and Perdon (1984), thus omitted.

Lemma 2. (1) *The closure $\bar{\mathscr{N}}$ of $\mathscr{N}$ in $\mathscr{M}$ is the smallest closed submodule of $\mathscr{M}$ containing $\mathscr{N}$*;

(2) *For any finitely generated submodule $\mathscr{N}$ of $\mathscr{M}$, one has* $\mathrm{rank}_{\mathscr{K}(\delta]}\mathscr{N} = \mathrm{rank}_{\mathscr{K}(\delta]}\bar{\mathscr{N}}$.

(3) *$\bar{\mathscr{N}}$ is the largest submodule of $\mathscr{M}$ containing $\mathscr{N}$ and having a rank equaling to* $\mathrm{rank}_{\mathscr{K}(\delta]}\mathscr{N}$.

Lemma 3. *Let $\mathscr{N}$ submodule of $\mathscr{M}$ over $\mathscr{K}(\delta]$, and $\mathscr{S}$ its base. If $\mathrm{span}_{\mathscr{K}(\delta]}\{\mathscr{S}\}$ is closed, then $\mathscr{N} = \mathrm{span}_{\mathscr{K}(\delta]}\{\mathscr{S}\}$.*

Lemma 4. *For any two submodules $\mathscr{N}_1$ and $\mathscr{N}_2$ of $\mathscr{M}$*

$$\overline{\mathscr{N}_1+\mathscr{N}_2} \supset \overline{\mathscr{N}_1}+\overline{\mathscr{N}_2}, \tag{10}$$

$$\overline{\mathscr{N}_1\cap\mathscr{N}_2} = \overline{\mathscr{N}_1}\cap\overline{\mathscr{N}_2}. \tag{11}$$

The modules over $\mathscr{K}(\delta]$ do not have the so-called Artinian property: a counterexample is $\mathscr{G}_i = \mathrm{span}_{\mathscr{K}(\delta]}\{\delta^i\,\mathrm{d}x\}$.

However, there is a positive result for closed submodules.

Theorem 2. *Any decreasing sequence of closed submodules of a finitely generated module $\mathscr{M}$ over $\mathscr{K}(\delta]$ stabilizes.*

Proof. Suppose $\mathscr{M}_1 \supset \mathscr{M}_2 \supset \cdots \supset \mathscr{M}_k \supset \cdots$ to be a chain of closed submodule over $\mathscr{K}(\delta]$. Since $\mathrm{rank}\,\mathscr{M} < \infty$,

$$\cdots \mathrm{rank}\,\mathscr{M}_k \leqslant \cdots \leqslant \mathrm{rank}\,\mathscr{M}_2 \leqslant \mathrm{rank}\,\mathscr{M}_1 < \infty.$$

So there is a $k\in Z^+$ such that $\mathrm{rank}\,\mathscr{M}_k = \mathrm{rank}\,\mathscr{M}_i$, for $i\geqslant k$. By Lemma 2(3), $\mathscr{M}_i = \mathscr{M}_k$ for $i\geqslant k$. □

The standard differentials of the functions in $\mathscr{K}$ span over the field $\mathscr{K}$ a vector space $\mathscr{E}$, that is, $\mathscr{E} = \mathrm{span}_{\mathscr{K}}\,\mathrm{d}\mathscr{K}$, and any differential one form $\omega\in\mathscr{E}$ can be associated with an element in $\mathscr{M}$. For simplicity, the notation $\mathrm{d}\alpha$ will be used to denote also its association in $\mathscr{M}$.

If there is an element $\Phi\in\mathscr{K}$ such that when $z(t) = \Phi$ satisfies the second equation of system (3), then system (3) is said to be well-posed.

It can be easily seen that the nonlinear time-delay system (1) is well-posed, because the solutions to the corresponding Eqs. (4) and (5) are $z_{1i}(t) = x(t-i-1)$, $z_{2i}(t) = u(t-i-1)$.

Next the operation of differentiation is extended to elements in $\mathscr{K}\langle\delta)$ and vectors in $\mathscr{M}$:

- For any element $a(x(t-i), u^{(k)}(t-i), i\in N, k\geqslant 0)\in\mathscr{K}$, the derivative along the dynamics of system (3) is defined as usual, in which $z(t)$ is replaced by Φ in the expression of $\mathscr{F}(t)$.
- For any polynomial $a(\delta] = \sum_{i=0}^{r_a} a_i\delta^i\in\mathscr{K}(\delta]$, the derivative $\dot{a}$ along the dynamics of system (3) is a fraction in $\mathscr{K}\langle\delta]$ defined by $\dot{a}\langle\delta] = \sum_{i=0}^{r_a}\dot{a}_i\delta^i$, in which $\dot{a}_i$ is the derivative of $a_i\in\mathscr{K}$ along the dynamics of (3).
- For any element $w\langle\delta)=b^{-1}(\delta]a(\delta]\in\mathscr{K}\langle\delta)$, the derivative along the dynamics of (3) is defined by

$$\dot{w}\langle\delta) = (db)^{-1}(d\dot{a} - ca), \tag{12}$$

  where c, d satisfy $cb = d\dot{b}$.
- For any vector $\omega\in\mathscr{M}_k$ (or $\mathscr{M}$), the derivative $\dot{\omega}$ of $\omega = \kappa_{-1}\,\mathrm{d}x + \sum_{i=0}^{k}\kappa_i\,\mathrm{d}u^{(i)}$ along the dynamics of (3) is a vector of some $\mathscr{M}_s$, $s > k$ (or $\mathscr{M}$) defined by $\dot{\omega} = \dot{\kappa}_{-1}\,\mathrm{d}x + \sum_{i=0}^{k}\dot{\kappa}_i\,\mathrm{d}u^{(i)} + \kappa_{-1}\,\mathrm{d}f + \sum_{i=0}^{k}\kappa_i\,\mathrm{d}u^{(i+1)}$, in which $\dot{\kappa}_i$, for $i = -1, 0, 1, \dots, k$, is the derivative of κ_i along the dynamics of (3), $\mathrm{d}f\in\mathscr{M}_s$ is the association of differential of f.

The validity of the definition of (12) is proved in Theorem 13 of Ježek (1996a).

From the second equation of system (3), one obtains: $\mathrm{d}z = (I-\delta a)^{-1}(\delta b\,\mathrm{d}x + \delta c\,\mathrm{d}u)$, where I is the identity matrix, and $a = \partial g/\partial z(t)$, $b = \partial g/\partial x(t)$, $c = \partial g/\partial u(t)$. So in many practical cases, the $\mathrm{d}q$ for a function (e.g., some output) of the form $q = q(x(t), z(t), u(t))$ can be found as, without solving for $z(t) = \Phi$,

$$\mathrm{d}q = \frac{\partial q}{\partial x(t)}\,\mathrm{d}x + \frac{\partial q}{\partial z(t)}\,\mathrm{d}z + \frac{\partial q}{\partial u(t)}\,\mathrm{d}u.$$

Applying the above to the discrete dynamics (4) and (5), one has that $\mathrm{d}z_{1i} = \delta^i\mathrm{d}x$, $\mathrm{d}z_{2i} = \delta^i\mathrm{d}u$. In this case $\mathrm{d}z_{ij}$ are linear combinations of $\mathrm{d}x$ and $\mathrm{d}u$ with coefficients in the ring of polynomials $\mathscr{K}(\delta]$. This explains why modules over $\mathscr{K}(\delta]$ were used to study nonlinear time-delay systems (Moog et al. 2000).

3. Analysis of nonlinear time-delay systems

Observability is a basic system property of time-delay systems. It is also naturally associated with the observer design problem as indicated in Watanabe and Oguchi (1985) for linear time-delay systems and in Márquez-Martínez, Moog, and Velasco-Villa (2000a) for nonlinear time-delay systems.

There are now two approaches to study the observability of the nonlinear time-delay system (1). The first one sees the system as one over $\mathscr{K}(\delta]$.

Define

$$\mathscr{Y}_k = \mathrm{span}_{\mathscr{K}(\delta]}\{\mathrm{d}y, \mathrm{d}\dot{y}, \ldots, \mathrm{d}y^{(k)}\},$$

$$\mathscr{U} = \mathrm{span}_{\mathscr{K}(\delta]}\{\mathrm{d}u, \mathrm{d}\dot{u}, \ldots\},$$

and $\mathscr{X} = \mathrm{span}_{\mathscr{K}(\delta]}\{\mathrm{d}x\}$. Then

$$(\mathscr{Y}_0 + \mathscr{U}) \cap \mathscr{X} \subset (\mathscr{Y}_1 + \mathscr{U}) \cap \mathscr{X}$$
$$\subset \cdots \subset (\mathscr{Y}_k + \mathscr{U}) \cap \mathscr{X} \subset \cdots$$

is an increasing sequence of submodules of $\mathscr{X}$. By Theorems 1 and A.2, for $k \geqslant n$

$$(\mathscr{Y}_k + \mathscr{U}) \cap \mathscr{X} = (\mathscr{Y}_n + \mathscr{U}) \cap \mathscr{X}.$$

Denote $\mathscr{O} = \overline{(\mathscr{Y}_n + \mathscr{U})} \cap \mathscr{X}$, where $\overline{(\mathscr{Y}_n + \mathscr{U})}$ is the closure of $(\mathscr{Y}_n + \mathscr{U})$, and $\mathscr{O}$ is called the polynomial observation submodule of system (1).

The second approach sees the system as one over $\mathscr{K}\langle\delta)$, and similarly define $\bar{\mathscr{Y}}_k = \mathrm{span}_{\mathscr{K}\langle\delta)}\{\mathrm{d}y, \mathrm{d}\dot{y}, \ldots, \mathrm{d}y^{(k)}\}$, $\bar{\mathscr{U}} = \mathrm{span}_{\mathscr{K}\langle\delta)}\{\mathrm{d}u, \mathrm{d}\dot{u}, \ldots\}$, and $\bar{\mathscr{X}} = \mathrm{span}_{\mathscr{K}\langle\delta)}\{\mathrm{d}x\}$. Then, the corresponding increasing sequence of submodules of $\bar{\mathscr{X}}$

$$(\bar{\mathscr{Y}}_0 + \bar{\mathscr{U}}) \cap \bar{\mathscr{X}} \subset (\bar{\mathscr{Y}}_1 + \bar{\mathscr{U}}) \cap \bar{\mathscr{X}}$$
$$\subset \cdots \subset (\bar{\mathscr{Y}}_k + \bar{\mathscr{U}}) \cap \bar{\mathscr{X}} \subset \cdots$$

will stabilize in a finite number of steps. Thus,

$$(\bar{\mathscr{Y}}_k + \bar{\mathscr{U}}) \cap \bar{\mathscr{X}} = (\bar{\mathscr{Y}}_n + \bar{\mathscr{U}}) \cap \bar{\mathscr{X}}.$$

Denote $\bar{\mathscr{O}} = (\bar{\mathscr{Y}}_n + \bar{\mathscr{U}}) \cap \bar{\mathscr{X}}$, and define $\bar{\mathscr{O}}$ as the rational observation submodule of system (1).

System (1) is said to be weakly observable if $\mathrm{rank}_{\mathscr{K}\langle\delta)} \bar{\mathscr{O}} = n$. This definition reduces to the one given in Lee and Olbrot (1981) for linear time-delay systems. The concept is defined over $\mathscr{K}\langle\delta)$ rather than over $\mathscr{K}(\delta]$, because "inverse" exists on $\mathscr{K}\langle\delta)$ while it may not on $\mathscr{K}(\delta]$.

The main result in this section is the following theorem.

Theorem 3. *System* (1) *is weakly observable if and only if* $\mathrm{rank}_{\mathscr{K}(\delta]} \mathscr{O} = n$.

Proof. Actually one can prove that

$$\mathrm{rank}_{\mathscr{K}(\delta]} \mathscr{O} = \mathrm{rank}_{\mathscr{K}\langle\delta)} \bar{\mathscr{O}}. \tag{13}$$

One has

$$\bar{\mathscr{O}} = \mathscr{K}\langle\delta) \otimes \mathscr{O}$$

since first of all $\mathscr{K}\langle\delta) \otimes \mathscr{O}$ is a submodule over $\mathscr{K}\langle\delta)$ and is contained in $\bar{\mathscr{O}}$. And since any element $\omega \in \bar{\mathscr{O}}$ may be written as $\omega = b^{-1}(\delta]\sum a_i(\delta]\omega_i$, that is, the fraction $b(\delta]\backslash 1$ times an element of $\mathscr{O}$, where $b(\delta]$ and $a_i(\delta]$ are polynomials, and ω_i are the basis vectors of $\mathscr{O}$. By Proposition 9.1 (Cohn, 1971, Chapter 0), $\bar{\mathscr{O}} \subset \mathscr{K}\langle\delta) \otimes \mathscr{O}$, thus the equation. And the natural mapping

$$\mathscr{O} \to \mathscr{K}\langle\delta) \otimes \mathscr{O} = \bar{\mathscr{O}}$$

is an imbedding, since $\mathscr{O}$ is obviously torsion free.

Having this, any set of independent vectors (over $\mathscr{K}(\delta]$) in $\mathscr{O}$ can be regarded as a set of vectors of $\bar{\mathscr{O}}$. It is easily seen that they must be independent over $\mathscr{K}\langle\delta)$. Thus, the inequality

$$\mathrm{rank}_{\mathscr{K}(\delta]} \mathscr{O} \leqslant \mathrm{rank}_{\mathscr{K}\langle\delta)} \bar{\mathscr{O}}.$$

On the other hand, denote $r = \mathrm{rank}_{\mathscr{K}(\delta]} \mathscr{O}$ and $\{\omega_1, \ldots, \omega_r\}$ is a base for $\mathscr{O}$. For any $\bar{\omega} \in \bar{\mathscr{O}}$, $\bar{\omega} = b^{-1}(\delta]\omega$ for some $b(\delta] \in \mathscr{K}(\delta]$ and $\omega \in \mathscr{O}$. Since $\{\omega_1, \ldots, \omega_r\}$ is a base for $\mathscr{O}$, there are polynomials $p(\delta], a_1(\delta], \ldots, a_r(\delta] \in \mathscr{K}(\delta]$, $p(\delta] \neq 0$ such that

$$p(\delta]\omega = a_1(\delta]\omega_1 + \cdots + a_r(\delta]\omega_r.$$

Hence, $\bar{\omega}$ can be expressed over $\mathscr{K}\langle\delta)$ as

$$\bar{\omega} = b^{-1}(\delta]p^{-1}(\delta](a_1(\delta]\omega_1 + \cdots + a_r(\delta]\omega_r).$$

From this, $r = \mathrm{rank}_{\mathscr{K}(\delta]} \mathscr{O} \geqslant \mathrm{rank}_{\mathscr{K}\langle\delta)} \bar{\mathscr{O}}$. That ends the proof. □

This result says that, to characterize the rank of the submodule $\bar{\mathscr{O}}$ over the ring of fractions $\mathscr{K}\langle\delta)$, it is enough to check the rank of $\mathscr{O}$ over the ring of polynomials $\mathscr{K}(\delta]$. This is an extension of the well-known result for linear time-delay systems (Theorem 9 of Lee and Olbrot (1981)). This result is even more important in the nonlinear case: it would be much more complex to check $\mathrm{rank}_{\mathscr{K}\langle\delta)}\bar{\mathscr{O}} = n$ because it involves calculation of derivatives of "fractions" as defined in (12).

The submodules over $\mathscr{K}\langle\delta)$ are purposely denoted as $\bar{\mathscr{Y}}_k$, $\bar{\mathscr{U}}$, $\bar{\mathscr{X}}$ and $\bar{\mathscr{O}}$. They are actually very closely related to the concept of closure. To show this, remember that $\mathscr{O}$ is imbedded in $\bar{\mathscr{O}}$, if the closure of the submodule spanned by $\mathscr{O}$ over $\mathscr{K}\langle\delta)$ is denoted as $cls\,\mathscr{O}$, then it is easy to prove that

$$cls\,\mathscr{O} = \mathscr{K}\langle\delta) \otimes \mathscr{O} = \bar{\mathscr{O}}.$$

Similar arguments apply to $\bar{\mathscr{Y}}_k$, $\bar{\mathscr{U}}$, $\bar{\mathscr{X}}$. Having this association, it is easy to show that (comparing Lemma 4)

$$\overline{\mathscr{Y}_k + \mathscr{U}} = \bar{\mathscr{Y}}_k + \bar{\mathscr{U}}.$$

Accessibility of nonlinear time-delay system can be dealt with by applying Theorem 2. Recall that the filtration of a decreasing sequence of submodules $\mathscr{H}_0 \supset \mathscr{H}_1 \supset \cdots \supset \mathscr{H}_k \supset \cdots$ defined by

$$\mathscr{H}_0 = \mathrm{span}_{\mathscr{K}(\delta]}\{\mathrm{d}x, \mathrm{d}u\},$$

$$\mathscr{H}_{j+1} = \mathrm{span}_{\mathscr{K}(\delta]}\{\omega \in \mathscr{H}_j \mid \dot{\omega} \in \mathscr{H}_j\}$$

was extended to the case of time-delay systems in Márquez-Martínez (1999) to study the accessibility of the system. It was shown that the condition $\mathscr{H}_\infty = 0$ guarantees the accessibility of system (1) (or non-existence of uncontrollable dynamics). The sequence $\mathscr{H}_i$ was shown to stabilize to $\mathscr{H}_\infty$ by an argument of relative degrees of one forms. Having Theorem 2, the same can be guaranteed by the closeness of the submodules $\mathscr{H}_i$.

Proposition 1. *The submodules $\mathscr{H}_i$ are closed.*

Proof. Since $\mathscr{H}_0$ and $\mathscr{H}_1 = \text{span}_{\mathscr{K}(\delta]}\{\mathrm{d}x\}$ are obviously closed. Now suppose $\mathscr{H}_j$ is closed, it is shown that $\mathscr{H}_{j+1}$ is closed.

For any $\alpha \in \mathscr{K}(\delta]$ and one form ω, if $\alpha\omega \in \mathscr{H}_{j+1}$, then by definition, $\alpha\omega \in \mathscr{H}_j$ and $\dot{\widehat{\alpha\omega}} \in \mathscr{H}_j$. By the closeness of $\mathscr{H}_j$, $\alpha\omega \in \mathscr{H}_j$ implies that $\omega \in \mathscr{H}_j$. Since $\dot{\widehat{\alpha\omega}} = \dot{\alpha}\omega + \alpha\dot{\omega}$, and $\omega \in \mathscr{H}_j$, one has $\dot{\alpha}\omega \in \mathscr{H}_j$, thus $\alpha\dot{\omega} \in \mathscr{H}_j$. Also by the closeness of $\mathscr{H}_j$, $\alpha\dot{\omega} \in \mathscr{H}_j$ implies that $\dot{\omega} \in \mathscr{H}_j$. That is, $\omega \in \mathscr{H}_{j+1}$.

From the previous proposition, and Lemma 2, it is easy to prove that this sequence converges in at most n steps. Thus $\mathscr{H}_\infty$ may be characterized in a finite number of steps. □

Theorem 4 (Márquez-Martínez, 1999). *System* (1) *is accessible if $\mathscr{H}_\infty = 0$.*

To illustrate the calculations, an example is taken from Márquez-Martínez et al. (2000b) where "canonical form observer" was designed.

$$\dot{x}_1(t) = 0.2x_1(t-1) + 0.1x_2(t-1) + 0.5x_2(t-1)x_2(t-2) + 0.2x_2^2(t-2) + u(t-1),$$

$$\dot{x}_2(t) = -0.25x_2(t-1),$$

$$y(t) = x_1(t-1) - x_2^2(t-2).$$

For an observability analysis, one computes

$$\mathrm{d}y = \delta\,\mathrm{d}x_1 - 2x_2(t-2)\delta^2\,\mathrm{d}x_2, \tag{14}$$

$$\mathrm{d}\dot{y} = 0.2\delta^2\,\mathrm{d}x_1 + a(\delta]\,\mathrm{d}x_2 + \delta^2\,\mathrm{d}u, \tag{15}$$

$$b(\delta]\,\mathrm{d}x_2 = \mathrm{d}\dot{y} - 0.2\delta\,\mathrm{d}y - \delta^2\,\mathrm{d}u, \tag{16}$$

$$c(\delta]\,\mathrm{d}x_1 = 2x_2(t-2)x_2(t-3)(\mathrm{d}\dot{y} - \delta^2\mathrm{d}u) + d(\delta]\mathrm{d}y, \tag{17}$$

where

$$a(\delta] = (x_2(t-2) + 0.4x_2(t-3))\delta^3 + (0.1 + x_2(t-3))\delta^2,$$

$$b(\delta] = (x_2(t-2) + 0.8x_2(t-3))\delta^3 + (0.1 + x_2(t-3))\delta^2,$$

$$c(\delta] = (x_2^2(t-2) + 0.8x_2(t-2)x_2(t-3))\delta^2 + x_2^2(t-3)\delta + x_2(t-3),$$

$$d(\delta] = (x_2(t-2)(x_2(t-2) + 0.4x_2(t-3))\delta + x_2(t-3)(0.1 + x_2(t-3)).$$

Relations (16) and (17) are obtained from (14) and (15) by using the property of left Ore rings. Direct inverting (14) and (15) is impossible in $\mathscr{K}(\delta]$. From (16) and (17), one sees that $\mathscr{X} \subset \overline{(\mathscr{Y} + \mathscr{U})}$, implying that $\text{rank}_{\mathscr{K}(\delta]}\,\mathscr{O} = \mathscr{X} \cap \overline{(\mathscr{Y} + \mathscr{U})} = 2$. The system is weakly observable.

4. Conclusion

In this paper, a precise mathematical treatment was given for the algebraic approach to nonlinear time-delay systems. This is accomplished by introducing the theory of non-commutative rings as a basis for the study of nonlinear time-delay systems. The approach was also extended to a class of generalized Roesser systems by making use of the classical division ring of fractions of the left Ore ring. This wider class of systems is a more suitable framework to deal with feedback design problems which are topics of current investigation. Applications of the theories to the study of accessibility and observability of nonlinear time-delay systems have been carried out. A well-known result on weak observability has been generalized to the nonlinear case.

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Appendix A. non-commutative algebra

General references on non-commutative algebra are Cohn (1971), Farb and Dennis (1993). References for Ore rings are Ore (1931), Ore (1933), Ježek (1996a), Ježek (1996b).

A ring R is said to satisfy the left Ore condition, and is called a left Ore ring, if for all $a, b \in R$, both non-zero, there exist $a', b' \in R$ (both non-zero) so that $a'a = b'b$; that is, a and b have a common left multiple.

Let $S = \{(a,b)\colon a, b \in R,\ b \neq 0\}$, and define an equivalence relation $\sim$ on S by setting $(a,b) \sim (c,d)$ if $b'a = d'c$, where $b'b = d'd$ via the left Ore condition. Let $b\backslash a$ denote the equivalent class associated with (a,b). Let D be the set of equivalence classes in S. Define an addition by

$$b\backslash a + d\backslash c = b'b\backslash(b'a + d'c),$$

where $b'b = d'd$ via the left Ore condition, and define a multiplication by

$$b\backslash a \cdot d\backslash c = a'b\backslash d'c,$$

where $a'a = d'd$ via the left Ore condition. Then D is a division ring, and R is embedded in D, that is, there is an injective homomorphism from R to D.

For any submodule $\mathcal{N} \subset \mathcal{M}$, a subset $\mathcal{S} \subset \mathcal{N}$ is called linearly independent if $a_1\omega_1 + \cdots + a_n\omega_n = 0$, for any $\omega_1, \ldots, \omega_n \in \mathcal{N}$, $a_1, \ldots, a_n \in R$, $n \geqslant 1$, implies $a_1 = \cdots = a_n = 0$. Otherwise the subset $\mathcal{S}$ is called linearly dependent. A base of $\mathcal{N}$ is a subset $\mathcal{S}$ of $\mathcal{N}$ which is linearly independent and for every element $\omega \in \mathcal{N}$ the subset $\{\omega\} \cup \mathcal{S}$ is linearly dependent.

The cardinality of $\mathcal{S}$ is denoted by $|\mathcal{S}|$.

The rank of a submodule of $\mathcal{M}$ over left Ore ring is well defined.

Theorem A.1. *Suppose $\mathcal{M}$ and $\mathcal{N}$ are two modules over $\mathcal{K}(\delta]$, and $\mathcal{N}$ is a submodule of $\mathcal{M}$. $\mathcal{S}$ and $\mathcal{T}$ are two bases of $\mathcal{N}$. Then $|\mathcal{S}| = |\mathcal{T}|$.*

A module is called Noetherian if any increasing sequence of submodules stabilizes in a finite number of steps. A module is called Artinian if any decreasing sequence of submodules stabilizes in a finite number of steps.

Theorem A.2. *Any finitely generated submodule $\mathcal{N}$ of $\mathcal{M}$ over a Noetherian left Ore ring is Noetherian.*

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