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Brief paper

Δ-Modulated feedback in discretization of sliding mode control☆

Xiaohua Xia[a,*], Alan S.I. Zinober[b]

[a]Department of Electrical, Electronic and Computer Engineering, University of Pretoria, South Africa
[b]Department of Applied Mathematics, The University of Sheffield, Sheffield S10 2TN, UK

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Abstract

Modulated feedback control introduces periodicity. The global attracting property of the periodic points is established for a simple scalar discrete-time system under Δ-modulated feedback. Attracting regions of the periodic points are also characterized. When the discretization effects of the equivalent control-based sliding mode control systems are studied, we show that the zero-order-hold discretization gives rise to Δ-modulation in the sliding mode direction. The global attracting property of Δ-modulated feedback offers a vivid illustration of the way sliding is achieved. Interestingly, we find that a ZOH discretization scheme of the equivalent control-based sliding mode control system with relative degree one results in only 2-periodic orbits.

Keywords: Delta-modulation; Periodic points; Sliding mode control

1. Introduction

Discrete sliding mode control (SMC) arises in two different situations: one associated with SMC of discrete-time systems and the other resulting from discretization of the SMC of continuous-time systems. Studies of both cases have been reported in the literature (see Corradini & Orlando, 1998; Koshkouei & Zinober, 2000; Wu, Drakunov, & Ozguner, 2000; Yu, 1998; Yu & Chen, 2003, and references therein). A different line of research is sigma–delta (or Δ-) modulation and Δ-modulated feedback of discrete signals and/or systems. Sigma–delta modulation first appeared in electronic circuits as a method of analog-to-digital conversion (Inose & Yasuda, 1963). More recent studies, motivated by the renewed interest in hybrid systems with hard nonlinearities, include Δ-modulated feedback control systems and the associated complexities (Gai, Xia, & Chen, 2003; Xia, Chen, Gai, & Zinober, 2004; Xia, Gai, & Chen, 2004; Xia & Zinober, 2004).

The link between SMC and Δ-modulated control was first noted in Zinober and Xia (2004). This paper explores the connection further. We show that the zero-order-hold (ZOH) discretization of the equivalent control-based SMC system gives rise to Δ-modulation in the sliding mode direction. To illustrate vividly how sliding is achieved, we first present a detailed investigation of the global attracting properties of a scalar discrete-time system under Δ-modulated feedback. Global attractiveness of equivalent control-based SMC is then realized by the modulation in the sliding direction and followed by the absorption of the stable zero dynamics of the system. Another interesting result is that a ZOH discretization scheme of the equivalent control-based SMC system with relative degree one produces only 2-periodic orbits, understandably due to sampling.

Note that we are simply using the analogy of SMC, and we assume that the system has a relative degree one with respect to the sliding surface. ZOH discretization of classical SMC is usually carefully avoided for its drawbacks, and higher-order and dynamic SMC are used for design (Zinober, 1994). Our method does not intend to add to SMC design theory, instead, our interest here is in the discretization effect on a continuous-time SMC system, as in Corradini and Orlando (1998) and Yu and Chen (2003).

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* Corresponding author. Tel.: +27 12 420 2165; fax: +27 12 362 5000.
E-mail address: xxia@postino.up.ac.za (X. Xia).

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The layout of the paper is as follows. In Section 2, we investigate the global attracting properties of a scalar discrete-time system under Δ-modulated feedback. The study of the discretization of the equivalent control-based SMC system is in Section 3. The last section contains some concluding remarks.

2. Delta-modulated control

In this section, we study the periodic orbits of the following scalar, discrete-time linear system:

$$x^+ = ax - \Delta\,\mathrm{sgn}(ax), \tag{1}$$

where $x \in R$ is the state variable, x^+ denotes the system state at the next discrete time, and a is a real number. Δ is a positive real number and $\mathrm{sgn}(x)$ is the function defined by

$$\mathrm{sgn}(x) = \begin{cases} 1 & \text{when } x \geqslant 0, \\ -1 & \text{when } x < 0. \end{cases}$$

Here, we will be concerned only with the case $|a| \leqslant 1$. The existence of periodic points for the case $|a| > 1$ has been discussed in Gai et al. (2003), Xia, Chen et al. (2004), and Xia, Gai et al. (2004). It was found that any periods can happen when $|a| \geqslant 2$, and when $1 < |a| < 2$, a period of a certain order can happen if and only if $|a|$ is greater than a certain value in the interval $(1, 2)$.

The case $|a| \leqslant 1$ provides a relatively thorough investigation of the periodicity and its attractiveness. A detailed analysis in the next section gives a vivid illustration of SMC.

Theorem 1. (1) *When* $|a|=1$, $\Omega=[-\Delta, \Delta]$ *is a global attractor on* $(-\infty, \infty)$;

(2) *when* $|a| < 1$, *the global attractor is the following set of two points*:

$$\{-\Delta/(1+|a|), \Delta/(1+|a|)\}; \tag{2}$$

(3) *when* $0 \leqslant a < 1$, *the two points in* (2) *are 2-periodic points*; *when* $-1 < a < 0$, *the two points in* (2) *are* (1-*periodic*) *fixed points*.

Proof. The proof of the cases $|a| = 1$ and $a = 0$ are straightforward and omitted. Denote the closed-loop system as

$$x^+ = ax - \Delta\,\mathrm{sgn}(ax) \stackrel{\text{def}}{=} f_c(x). \tag{3}$$

Denote $\mathscr{D} = (-\Delta/|a|, \Delta/|a|)$. Let $x(0) = x_0$ be the initial condition. It can be easily checked that when $x_0 \in \mathscr{D}$, $x(k)$ satisfies the relation: $\mathrm{sgn}(ax(k)) = \mathrm{sgn}(ax_0)(-\mathrm{sgn}(a))^k$, for all $k \geqslant 1$. In this case, Eq. (3) transforms as follows:

$$x^+ = ax - \Delta\,\mathrm{sgn}(ax_0)(-\mathrm{sgn}(a))^k. \tag{4}$$

Applying the $\mathscr{Z}$-transformation to Eq. (4) one obtains

$$zX(z) - zx_0 = aX(z) - \Delta\,\mathrm{sgn}(ax_0)\frac{z}{z+\mathrm{sgn}(a)}.$$

Solving with respect to $X(z)$ one obtains:

$$\begin{aligned} X(z) &= \frac{z}{(z-a)}x_0 - \Delta\,\mathrm{sgn}(ax_0)\frac{z}{(z-1)(z+\mathrm{sgn}(a))} \\ &= \frac{z}{(z-a)}x_0 - \frac{\Delta\,\mathrm{sgn}(x_0)}{(1+|a|)}\left[\frac{z}{(z-a)} - \frac{z}{(z+\mathrm{sgn}(a))}\right], \end{aligned}$$

from which

$$x(k) = x_0 a^k - \Delta_a\,\mathrm{sgn}(x_0)(a^k - (-\mathrm{sgn}(a))^k), \tag{5}$$

where $\Delta_a = \Delta/(1+|a|)$. From this, it follows that when $x_0 = \pm\Delta_a (\in \mathscr{D})$:

$$\begin{aligned} x(k) &= \pm\,\Delta_a a^k \mp \Delta_a(a^k - (-\mathrm{sgn}(a))^k) \\ &= \pm\,\Delta_a(-\mathrm{sgn}(a))^k. \end{aligned}$$

This proves that the two points $-\Delta_a$ and Δ_a are 2-periodic points when $0 \leqslant a < 1$, and 1-periodic points when $-1 < a < 0$.

Since $|a| < 1$, $a^k \to 0$, and from (5), when $k \to \infty$, the trajectory $x(k)$ tends to

$$x(k) \simeq \Delta_a\,\mathrm{sgn}(x_0)(-\mathrm{sgn}(a))^k,$$

i.e., the set $\{-\Delta_a, \Delta_a\}$ is a global attractor for the domain $\mathscr{D}$.

Let us now consider the case $x_0 \notin \mathscr{D}$, and let us suppose that $x(k) \notin \mathscr{D}$. Then it can easily be verified that

$$|x(k+1)| = |a||x(k)| - \Delta,$$

and therefore

$$|x(k+1)| < |x(k)| - \Delta.$$

This proves that $|x(k)|$ is strictly decreasing (with a step size greater than Δ) outside $\mathscr{D}$. Hence, for any $x_0 \notin \mathscr{D}$, there exists a finite instant k such that $x(k) \in \mathscr{D}$. □

Since the periodic points are globally attractive, it is interesting to find out the attracting region for each of the periodic points.

First, we introduce a new concept. For any real number x and $a \neq 0$ (the case $a = 0$ is trivial), the characteristic index κ is defined as the following non-negative integer:

$$\kappa = \left\lfloor \log_{|a|}\left(\frac{\Delta}{\Delta + (1-|a|)|x|}\right)\right\rfloor,$$

where $\lfloor * \rfloor$ denotes the floor, i.e., the maximal integer bounded above by the real number $*$.

Lemma 1. (i) *For any* x, *the characteristic index* κ *is the smallest non-negative integer* m *such that*

$$|f_c^{(m)}| < \frac{\Delta}{|a|}.$$

(iia) *For* $-1 < a < 0$, κ *is the smallest non-negative integer* m *such that* $f_c^{(m)}$ *and* $f_c^{(m+1)}$ *have the same sign.*

(iib) *For* $0 < a < 1$, κ *is the smallest non-negative integer* m *such that* $f_c^{(m)}$ *and* $f_c^{(m+1)}$ *have opposite signs.*

Proof. We prove the result only for the case $0 < a < 1$. Proofs for other cases can be worked out along similar lines, and are therefore omitted.

If $0 < a < 1$, it follows that

$$f_c^{(m+1)} = af_c^{(m)} - \operatorname{sgn}(f_c^{(m)})\Delta = \begin{cases} af_c^{(m)} - \Delta, & f_c^{(m)} \geqslant 0, \\ af_c^{(m)} + \Delta, & f_c^{(m)} < 0. \end{cases} \tag{6}$$

It is easy to see that $|f_c^{(m)}| < \Delta/a$ if and only if $f_c^{(m)}$ and $f_c^{(m+1)}$ have different signs.

Note that for $m \leqslant \kappa$, it is easy to obtain that for $x > 0$, $f_c^{(m)}(x) = a^m|x| - ((1-a^m)/(1-a))\Delta$, and for $x \leqslant 0$, $f_c^{(m)}(x) = -a^m|x| + ((1-a^m)/(1-a))\Delta$.

It is then straightforward to verify that the real number $s = \log_a \Delta/(\Delta + (1-a)|x|)$ satisfies

$$a^s|x| - \frac{(1-a^s)}{(1-a)}\Delta = 0.$$

Therefore, $\kappa = \lfloor s \rfloor$ is the smallest integer such that $f_c^{(m)}$ changes sign.

This completes the proof of the lemma. □

The analysis given in the proof can be useful in finding the limiting periodic points. We will carry this out separately for the two types of systems with $-1 < a < 0$ and $0 < a < 1$, respectively.

If $-1 < a < 0$, then we have

$$f_c^{(m+1)}(x) = f_c(f_c^{(m)})(x) = af_c^{(m)}(x) + \operatorname{sgn}(f_c^{(m)}(x))\Delta.$$

By (iia) of Lemma 1, $f_c^{(m)}$ has the same sign as $f_c^{(\kappa)}$, for $m \geqslant \kappa$. Therefore, we have, for $m \geqslant \kappa$,

$$f_c^{(m+1)}(x) = af_c^{(m)}(x) + \operatorname{sgn}(f_c^{(\kappa)}(x))\Delta.$$

Hence, by denoting the limit of $f_c^{(m)}$ by x^*, we can solve for x^* from $x^* = ax^* + \operatorname{sgn}(f_c^{(\kappa)})\Delta$, to obtain

$$x^* = \frac{\operatorname{sgn}(f_c^{(\kappa)})\Delta}{1-a}.$$

If $0 < a < 1$, then first let κ_e be the next even integer (or zero) greater than κ (i.e., $\kappa_e = \kappa$ if κ is even or zero, and $\kappa_e = \kappa + 1$ if κ is odd). Then, from (iib) of Lemma 1, $f_c^{(2m)}$ has the same sign as $f_c^{(\kappa_e)}$, for $m \geqslant \kappa_e/2$. Therefore, we have, for $2m \geqslant \kappa_e$,

$$f_c^{(2(m+1))} = a^2 f_c^{(m)} - a\operatorname{sgn}(f_c^{(\kappa_e)})\Delta + \operatorname{sgn}(f_c^{(\kappa_e)})\Delta.$$

Hence, if we denote the limit of $f_c^{(2m)}$ by x^*, then

$$x^* = a^2x^* - a\operatorname{sgn}(f_c^{(\kappa_e)})\Delta + \operatorname{sgn}(f_c^{(\kappa_e)})\Delta,$$

and

$$x^* = \frac{\operatorname{sgn}(f_c^{(\kappa_e)})\Delta}{1+a}.$$

Summarizing the above, we have the following characterization of the attracting region of a periodic point.

Theorem 2. *For any x, denote its characteristic index as κ.* (i) *For $-1 < a < 0$, x belongs to the attracting region of $\Delta/(1-a)$ $(-\Delta/(1-a))$ if and only if $\operatorname{sgn}(x^{(\kappa)}) = 1$ $(\operatorname{sgn}(x^{(\kappa)}) = -1)$.* (ii) *For $0 \leqslant a < 1$, x belongs to the attracting region of $\Delta/(1+a)$ $(-\Delta/(1+a))$ if and only if $\operatorname{sgn}(x^{(\kappa)}) = (-1)^\kappa$ $(\operatorname{sgn}(x^{(\kappa)}) = (-1)^{\kappa+1})$.*

3. Application to discretized SMC systems

Consider a continuous time system

$$\dot{x} = Ax + bu, \tag{7}$$

where $x \in R^n$, A is an $n \times n$ matrix, and b is an n-dimensional vector. A basic SMC design (Zinober, 1994) is to seek a sliding mode defined by $s = c^{\mathrm{T}}x$, where c is an n-dimensional vector, such that $c^{\mathrm{T}}x$ has relative degree 1 with respect to system (7), i.e., $c^{\mathrm{T}}b \neq 0$.

In this case, a sliding mode controller is obtained as

$$u = -\alpha c^{\mathrm{T}}x - \frac{1}{c^{\mathrm{T}}b}c^{\mathrm{T}}Ax - \frac{\beta}{c^{\mathrm{T}}b}\operatorname{sgn}(c^{\mathrm{T}}x), \tag{8}$$

in which $\alpha \geqslant 0$ and $\beta > 0$ are tuning parameters. There are three parts:

$$u_r = -\alpha c^{\mathrm{T}}x,$$
$$u_{eq} = -\frac{1}{c^{\mathrm{T}}b}c^{\mathrm{T}}Ax,$$
$$u_s = -\frac{\beta}{c^{\mathrm{T}}b}\operatorname{sgn}(c^{\mathrm{T}}x).$$

The *equivalent control* u_{eq} (Zinober, 1994) is derived by solving $\dot{s} = 0$, where $\dot{s} = c^{\mathrm{T}}(Ax + bu)$ is the derivative of s along the dynamics of (7). The *switching control* u_s is designed to satisfy the sliding condition $s\dot{s} \leqslant 0$.

The *reaching control* u_r adds some reaching manipulability to avoid the chattering problem (Gao & Hung, 1993).

The SMC design is applicable to system (7) when it is *minimal phase*, with $c^{\mathrm{T}}x$ as an output (Byrnes & Isidori, 1988).

To study the discretization effects on the SMC, we assume that the controller u is *digitized* through a ZOH at the sampling moments:

$$\begin{aligned} u(t) = u_k &\overset{\text{def}}{=} u(kh) \\ &= -\alpha c^{\mathrm{T}}x(kh) - \frac{1}{c^{\mathrm{T}}b}c^{\mathrm{T}}Ax(kh) - \frac{\beta}{c^{\mathrm{T}}b}\operatorname{sgn}(c^{\mathrm{T}}x(kh)) \\ &\triangleq -\alpha c^{\mathrm{T}}x(k) - \frac{1}{c^{\mathrm{T}}b}c^{\mathrm{T}}Ax(k) - \frac{\beta}{c^{\mathrm{T}}b}\operatorname{sgn}(c^{\mathrm{T}}x(k)), \end{aligned} \tag{9}$$

for all $t \in [kh, (k+1)h)$, where $h > 0$ is the sampling period. A discrete-time conversion of the system (7) under ZOH is obtained as

$$x(k+1) = \mathrm{e}^{Ah}x(k) + \int_0^h \mathrm{e}^{A\tau}b\,\mathrm{d}\tau\, u_k, \tag{10}$$

where u_k is given in (9).

To reveal the special structure of the discretization of the system, let us first make the coordinate transformation on the original (closed-loop) system (7) under feedback (8):

$$z_1 = c^{\mathrm{T}} x,$$

and choose $c_2, c_3, \ldots, c_n \in R^n$ satisfying $c_i^{\mathrm{T}} b = 0$, for $i = 2, 3, \ldots, n$, and $\{c, c_2, c_3, \ldots, c_n\}$ is a linearly independent set. This is always possible due to $c^{\mathrm{T}} b \neq 0$.

Hence, let $z_i = c_i^{\mathrm{T}} x$, for $i = 2, 3, \ldots, n$. It is easily seen that system (7) under SMC is written in the new coordinates as

$$\dot{z}_1 = -\alpha z_1 - \beta \,\mathrm{sgn}(z_1),$$
$$\dot{\tilde{z}} = \Psi \tilde{z} + p z_1,$$

in which we denote $\tilde{z} = (z_2, z_3, \ldots, z_n)^{\mathrm{T}}$, $\Psi \in R^{(n-1)\times(n-1)}$ is a stable matrix, due to the assumption that the system is minimal phase, and $p \in R^{n-1}$.

Applying a zero-order hold discretization to the system in coordinates z, we obtain

$$z_1^+ = \kappa z_1 - \Delta \,\mathrm{sgn}(z_1), \tag{11}$$
$$\tilde{z}^+ = \Phi \tilde{z} + \gamma z_1, \tag{12}$$

in which

$$\kappa = \mathrm{e}^{-\alpha h}, \quad \Phi = \mathrm{e}^{\Psi h},$$
$$\Delta = \begin{cases} \beta(1 - \mathrm{e}^{-\alpha h})/\alpha, & \text{when } \alpha \neq 0, \\ \beta h & \text{when } \alpha = 0, \end{cases}$$
$$\gamma = (-\alpha h I_{n-1} - \Psi h)^{-1}(\mathrm{e}^{-\alpha h} I_{n-1} - \mathrm{e}^{\Psi h}) p,$$

and $h > 0$ is the sampling period. These equations are readily derived by applying (10). As a matter of fact, the system (11)–(12) is the transformed version of (10) under the same coordinate transformation.

We note that the dynamics of z_1 is decoupled from that of $\tilde{z}$. It is exactly in a form that has been considered in the previous section. Since $0 < \kappa \leqslant 1$, we know from Theorem 1 that when $0 < \kappa < 1$, $\{\pm \Delta/(1+\kappa)\}$ is the only (2-) periodic orbit, and it is globally attracting; when $\kappa = 1$, every point in $(-\Delta, \Delta]$ is 2-periodic, any point is attracted to one pair of these 2-periodic points.

The following result concerning periodic orbits from externally asymptotically periodic excitation is an easy generalization of a well-known result (see also Xia & Zinober, 2004) and the proof is omitted.

Theorem 3. *Consider a discrete-time system of order n,*

$$x^+ = Ax + bu, \tag{13}$$

where $x \in R^n$ is the state, x^+ denotes the system state at the next discrete-time step, $u \in R$ is the scalar input, A is an $n \times n$ matrix of real numbers, and b is a column vector of n real numbers. A is a stable matrix, i.e., the eigenvalues of A lie within the unit circle. (i) *For an asymptotically L-periodic input sequence, there is a periodic orbit of period L for system* (13). (ii) *This periodic orbit is globally attracting.*

To find the periodic points of (11) and (12), first of all we note that z_1 can only be 2-periodic. Therefore, z_1 in (12) can be regarded as a 2-periodic (modulated) orbit, in order to find the periodic orbit for the overall system. Since z_1 is asymptotically 2-periodic and Φ is stable, we can apply Theorem 3 to conclude that there is/are only 2-periodic orbit(s) arising from discretization of SMC. These are summarized in the following theorem.

Theorem 4. *Discretization of SMC results in only 2-periodic orbits. When $\alpha > 0$, there is a unique 2-periodic orbit determined by (in z coordinates): $\{P, -P\}$, in which*

$$P = \Delta/(1+\kappa) \begin{bmatrix} -1 \\ (I_{n-1} + \Phi)^{-1}\gamma \end{bmatrix}. \tag{14}$$

This pair is globally attracting.

When $\alpha = 0$, each pair of points of the following form (in z coordinates) is a 2-periodic orbit, for $\varphi \in [-\Delta, \Delta)$:

$$\left\{ \varphi \begin{bmatrix} -1 \\ (I_{n-1} + \Phi)^{-1}\gamma \end{bmatrix}, -\varphi \begin{bmatrix} -1 \\ (I_{n-1} + \Phi)^{-1}\gamma \end{bmatrix} \right\}.$$

Proof. When $\alpha \neq 0$, by Theorem 1, z_1 is globally attracted to two periodic points $\pm\Delta/(1+\kappa)$. It is easily verified that the two points in (14) are the only 2-periodic points for the system. The $\tilde{z}$ part in (12) has exactly the structure discussed in Theorem 3, therefore, these two points are globally attracting. Similar arguments apply to the case when $\alpha = 0$. □

This result clearly indicates how sliding mode is achieved: first of all the system is dragged towards the sliding mode (z_1) by a Δ-modulation mechanism, then it is *absorbed* by its stable zero dynamics (the matrix Φ is stable). From the expressions given in the theorem, the two periodic points are on two different sides of the sliding mode hyperplane (defined by $z_1 = 0$). It is noticed that the component-wise distance of the two periodic points is ordered at $O(\Delta)$. And from the expression of Δ following (12), in both situations $\alpha = 0$ and $\alpha \neq 0$, $\Delta \sim 2h$. So eventually, we conclude that the distance of any two corresponding components of the two periodic points is ordered at $O(h)$. Hence, the chattering of SMC still exists in its ZOH sampling implementation, but it is regulated by the sampling period h. When h is very small, chattering becomes "invisible".

Note that a crucial assumption in arriving at Theorem 4 is that system (1) has relative degree one taking the sliding surface as an output. If this is not the case, much more complicated situations can arise (Yu, 1998).

Suppose there are matched uncertainties in system (7). Using well-known SMC invariance results (see, e.g., Zinober, 1994), we can establish the attractiveness of trajectories to the neighborhood of the 2-periodic points, for systems with matched perturbations and uncertainties because of the global attractiveness of these 2-periodic points. The robustness is demonstrated in the following example illustrating our result in Theorem 4.

Consider the system (7), $A = A_0 + \Delta A$, where A_0 is the nominal known system matrix and ΔA is an unknown structural disturbance

$$A_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 20 & -14 & 4 \end{bmatrix}, \quad \Delta A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -0.12 & -0.3 & 0.22 \end{bmatrix},$$

and $b = (0\ \ 0\ \ 1)^{\mathrm{T}}$, $c = (1\ \ 1\ \ 1)^{\mathrm{T}}$.

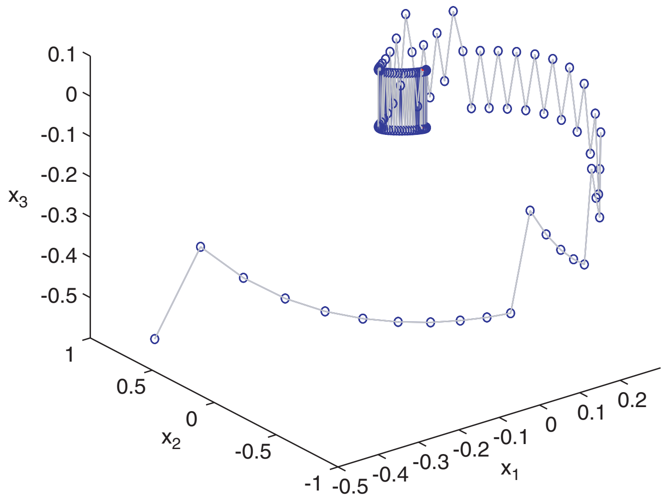

Fig. 1. Discretized SMC, with $h = 0.1$.

The SMC is based upon (8) and is designed for the nominal system A_0, with $\alpha = 1$ and $\beta = 1$. The sampling period is chosen as $h = 0.1$, and the initial condition is $x(0) = (-2, 1, 1)$. The simulation is done for $N = 150$ steps, and the graph in Fig. 1 shows the iterations after step 15. It can be seen from the graph that the orbit first approaches the sliding surface and then slides within a "band" of the sliding surface and towards the two periodic points. The width of the band is about 0.2, two times the sampling period. □

4. Concluding remarks

It has been shown that global attractiveness of equivalent control-based SMC is realized by the modulation in the sliding direction and followed by the absorption of the stable zero dynamics of the system. Another interesting result we have found in the paper is that a ZOH discretization scheme of the equivalent control-based SMC system with relative degree one results in only 2-periodic orbits, due to sampling. The periodic orbits are regulated by the sampling period to move in the close vicinity of the sliding mode hyperplane.

Preliminary research indicates that the approach is extendable to a class of multi-input multi-output systems with a uniform relative degree one for each output and with a nonsingular Falb–Wolovich matrix. After further research we intend publishing these results.

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Xiaohua Xia obtained his Ph.D. degree at Beijing University of Aeronautics and Astronautics, Beijing, China, in 1989. He stayed at the University of Stuttgart, Germany, as an Alexander von Humboldt fellow in May 1994 and for two years, followed by two short visits to Ecole Centrale de Nantes, France and National University of Singapore during 1996 and 1997, both as a post-doctoral fellow. He joined the University of Pretoria, South Africa, since 1998, and became a full professor in 2000. He also held visiting positions as an invited professor at IRCCYN, Nantes, France, in 2001, 2004 and 2005, respectively. He is a Senior IEEE member, served as the South African IEEE Section/Control Chapter Chair. He also serves for IFAC as the vice-chair of the Technical Committee of Nonlinear Systems. He is an Associate Editor of Automática, IEEE Transactions on Circuits and Systems II, and the Specialist Editor (Control) of the Transactions of the SAIEE. He has been a registered Professional Engineer with the Engineering Council of South Africa, since 1998, and elected a fellow of the South African Academy of Engineering, since 2005. His research interests include: non-linear feedback control, observer design, time-delay systems, hybrid systems, modeling and control of HIV/AIDS, and control and handling of heavy-haul trains.

Alan S. I. Zinober after obtaining his Ph.D. degree from Cambridge University in 1974, was appointed Lecturer in the Department of Applied and Computational Mathematics at the University of Sheffield in 1974, and promoted to Senior Lecturer in 1990, Reader in Applied Mathematics in 1993 and Professor in 1995. He has been the recipient of a number of EPSRC research grants and has published over 180 journal and conference publications as well as editing three research monographs. He has undertaken research in many areas of sliding mode control as well as other areas of nonlinear control theory. He has been Chairman of the IEEE (UK and RI) Control Systems Chapter.