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On the bound of the Lyapunov exponents for continuous systems

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In this paper, both upper bounds and lower bounds for all the Lyapunov exponents of continuous differential systems are determined. Several examples are given to show the application of the estimates derived here. © *2004 American Institute of Physics.* [DOI: 10.1063/1.1768911]

The Lyapunov exponents, first introduced by Oseledec, play an important role in nonlinear systems, especially in chaotic systems, mainly due to the fact that chaotic systems can be characterized with the positivity of the leading (or the largest) Lyapunov exponent. For parameterized dynamical systems, the Lyapunov exponents are of benefit to identification of some kinds of bifurcations. For instance, if the leading Lyapunov exponent is zero but the rest are negative for some bifurcation parameter value, then the Hopf bifurcation occurs, if the first two (i.e., the largest and the second largest) Lyapunov exponents are equal to zero but the rest are negative for some parameter value, then two-torus bifurcation occurs, and so on. In this paper, the prior estimates for all the Lyapunov exponents of a given continuous differential system are derived. Several examples show that these estimates are valid.

I. INTRODUCTION

Lyapunov exponents, introduced by Oseledec,[1] play a crucial role in analyzing dynamics of evolutionary systems, especially in chaotic and/or bifurcative systems. Until now, many analyses and algorithms exist for the Lyapunov exponents of a given system.[2–11] Recently, Li and Chen derived a bound for the Lyapunov exponents of discrete-time systems, with numerical examples showing the validity of the derived result.[12] These estimates are easily computable, and are different from the existing ones.[11] In the present paper, we further consider continuous systems. In similar spirits, we obtain effective estimates of both upper bounds and lower bounds of their Lyapunov exponents. Several numerical examples are shown to indicate the application of the prior estimates.

Consider the following nonautonomous system:

$$\frac{\mathrm{d}x}{\mathrm{d}t}=f(x,t),\quad (x,t)\in\Omega\times(t_0,+\infty)\subset R^n\times(t_0,+\infty),$$
$$x(t_0)=x_0, \tag{1}$$

where $f(x,t)=(f_1(x,t),\dots,f_n(x,t))'\in R^n$, $f_x(x,t)$ is assumed to be piece-wise continuous with respect to x in Ω for any t, and the Jacobian $f_x(x,t)$ is bounded, i.e., $\|f_x(x,t)\|\leqslant M$, in which $\|\cdot\|$ is an arbitrary norm, $(x,t)\in\Omega\times(t_0,+\infty)$, and $x_0\in\Omega$.

The fundamental solution matrix Φ solves the following initial-value problem:

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t}=f_x(x,t)\Phi,$$
$$\Phi(t_0)=I. \tag{2}$$

Here Φ also depends upon x, but we still use $\Phi(t)$ for notational simplicity. We will see this will not influence the discussion later on, I is the unit matrix.

Recall the definition of the Lyapunov exponents below.

Definition:[1,4,8] Let $\mu_k(t)$, $k=1,2,\dots,n$, be the eigenvalues of Φ from Eq. (2), which satisfy $|\mu_1(t)|\leqslant|\mu_2(t)|\leqslant\cdots\leqslant|\mu_n(t)|$. The Lyapunov exponents ℓ_k of the trajectory $x(t)$ solving (1) are defined by

$$\ell_k=\limsup_{t\to+\infty}\frac{1}{t}\ln|\mu_k(t)|,\quad k=1,2,\dots,n.$$

These exponents ℓ_k, $k=1,2,\dots,n$, are real numbers. The existence of this limit was established.[1] If $\mu_1(t)=0$ for arbitrary t, then Φ is not invertible, and $\ell_1=-\infty$ which does not happen in general. Hence, here and hereafter, we assume that $\mu_1(t)$ is not (identically) equal to zero. Therefore, Φ is always supposed to be invertible.

II. ESTIMATE OF THE BOUND

At first, several lemmas are introduced below.

Lemma 1 (Gronwall's Inequality):[13] Let $u(t):[0,\alpha]\to R$ be continuous and nonnegative. Suppose $C\geqslant 0$, $K\geqslant 0$ are such that

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$$u(t)\leqslant C+\int_0^t Ku(s)\mathrm{d}s$$

for all $t\in[0,\alpha]$. Then

$$u(t)\leqslant Ce^{Kt}$$

for all $t\in[0,\alpha]$.

Lemma 2:[14] Let $A\in C^{n\times n}$. Assume that $\lambda_1, \lambda_2,\ldots,\lambda_n$ are eigenvalues of $\overline{A'}A$, where $\overline{A'}$ denotes the complex conjugate and transpose of the matrix A, and $\mu_1, \mu_2,\ldots,\mu_n$ are eigenvalues of A. Then,

(i) $\lambda_i\geqslant 0$, $i=1,2,\ldots,n$.

(ii) $\min_i\lambda_i\leqslant\overline{\mu_j}\mu_j\leqslant\max_i\lambda_i$, $j=1,2,\ldots,n$.

Lemma 2 gives the bounds of $|\mu_j|$, $j=1,2,\ldots,n$. The following lemma determines the bounds of $\min_i\lambda_i$ and $\max_i\lambda_i$.

Lemma 3: Suppose that $A\in C^{n\times n}$ is invertible, $\max\{\|A\|,\|A^{-1}\|\}\leqslant M$, and all the eigenvalues of $\overline{A'}A$ are put in order $\lambda_1\leqslant\lambda_2\leqslant\cdots\leqslant\lambda_n$, then

(i) $M\geqslant 1$,

(ii) $1/M^2\leqslant\lambda_1$, $\lambda_n\leqslant M^2$.

Proof: (i) $M\geqslant 1$ can be seen from the fact that $M^2\geqslant\|A\|\cdot\|A^{-1}\|\geqslant\|AA^{-1}\|=1$.

(ii) Obviously, $\lambda_n\leqslant\|\overline{A'}A\|\leqslant\|A\|^2\leqslant M^2$. The first inequality is due to the fact that the spectral radius of a given matrix is not bigger than its arbitrary norm. On the other hand, $\|A^{-1}\|\leqslant M$ implies the largest eigenvalue of $(\overline{A^{-1}})'A^{-1}\leqslant\|A^{-1}\|^2\leqslant M^2$. So, the smallest eigenvalue of $A\overline{A'}\geqslant 1/M^2$. It is evident that the characteristic polynomial of $A\overline{A'}$ is the same as that of $\overline{A'}A$.[15] Therefore, $\lambda_1\geqslant 1/M^2$. The proof is complete.

Lemma 4:[15] Suppose that the matrices $A(t)$, $B(t)$ are differentiable with respect to t and both are multipliable, then

$$\frac{\mathrm{d}(A(t)B(t))}{\mathrm{d}t}=\frac{\mathrm{d}A(t)}{\mathrm{d}t}B(t)+A(t)\frac{\mathrm{d}B(t)}{\mathrm{d}t}.$$

Now we find the upper bound of the largest Lyapunov exponent ℓ_n of system (1). Integrating system (2) yields

$$\Phi(t)=I+\int_{t_0}^t f_x(x,\xi)\Phi(\xi)\mathrm{d}\xi. \tag{3}$$

Taking the matrix norm of both sides of (3) leads to

$$\|\Phi(t)\|\leqslant 1+M\int_{t_0}^t\|\Phi(\xi)\|\mathrm{d}\xi. \tag{4}$$

Applying Lemma 1 to (4) gives

$$\|\Phi(t)\|\leqslant e^{M(t-t_0)}. \tag{5}$$

By the fact that the spectral radius of a given matrix is not bigger than any norm of its, one has

$$|\mu_n(t)|\leqslant\|\Phi(t)\|\leqslant e^{M(t-t_0)}. \tag{6}$$

It immediately follows that the largest Lyapunov exponent of system (1) satisfies

$$\ell_n\leqslant M \tag{7}$$

from the definition of the Lyapunov exponent and (6).

In the following, we determine the lower bound of the smallest Lyapunov exponent ℓ_1 of system (1).

By assumption, $\Phi(t)$ is invertible. From Lemma 4, one can find

$$\mathbf{0}=\frac{\mathrm{d}(\Phi(t)\Phi^{-1}(t))}{\mathrm{d}t}=\frac{\mathrm{d}\Phi(t)}{\mathrm{d}t}\Phi^{-1}(t)+\Phi(t)\frac{\mathrm{d}\Phi^{-1}(t)}{\mathrm{d}t}. \tag{8}$$

Equations (8) and (2) imply

$$\frac{\mathrm{d}\Phi^{-1}}{\mathrm{d}t}=-\Phi^{-1}f_x(x,t),$$
$$\Phi^{-1}(t_0)=I. \tag{9}$$

Integration of the first equation of (9), then taking the matrix norm in both sides, one has

$$\|\Phi^{-1}(t)\|\leqslant 1+M\int_{t_0}^t\|\Phi^{-1}(\xi)\|\mathrm{d}\xi. \tag{10}$$

By almost the same reasoning of (5), one can obtain

$$\|\Phi^{-1}(t)\|\leqslant e^{M(t-t_0)}. \tag{11}$$

From (11), one can derive

$$\text{the largest eigenvalue of } [\Phi^{-1}(t)]^T\Phi^{-1}(t)\leqslant e^{2M(t-t_0)}. \tag{12}$$

Amongst the moduli of all the eigenvalues of $\Phi^{-1}(t)$, $|1/\mu_1(t)|$ is the largest one where $\mu_1(t)$ is the smallest eigenvalue of $\Phi(t)$ in absolute value. Utilizing Lemma 2 yields

$$\left|\frac{1}{\mu_1(t)}\right|\leqslant e^{M(t-t_0)},$$

i.e.,

$$|\mu_1(t)|\geqslant e^{-M(t-t_0)}. \tag{13}$$

It immediately follows that

$$\ell_1\geqslant -M. \tag{14}$$

So, the theorem below is established.

Theorem 1: The Lyapunov exponents ℓ_k, $k=1,2,\ldots,n$ of system (1) satisfy

$$-M\leqslant\ell_1\leqslant\ell_2\leqslant\cdots\leqslant\ell_n\leqslant M,$$

in which M is the upper bound of the norm of the Jacobian $f_x(x,t)$.

Remark 1

(i) This theorem is also suitable for the autonomous system provided that the norm of the related Jacobian is bounded.

(ii) The bound of the Lyapunov exponents is uniquely determined by the matrix norm of the Jacobian of $f(x,t)$ for (1) or that of $f(x)$ for the autonomous system.

(iii) The lower bound and the upper bound of the Lyapunov exponents are even optimal for some systems, which can be seen from the examples given later on. However, in general, this estimate is not the tightest. A tighter estimate is given below.

(iv) To compute chaotic attractors is not difficult if such attractors exist, but to compute the Lyapunov exponents of the considered trajectory is often quite difficult since the as-

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sociated numerical integration easily overflows. But from our theorem, the Lyapunov exponents are bounded since the norm of the Jacobian is bounded mainly due to the attractiveness of the chaotic attractor. So new efficient and robust algorithms for the Lyapunov exponents still need to be constructed though there are many numerical methods presently.

By Lemma 3, one can derive estimates for the Lyapunov exponents of a given discrete-time system in a similar manner.

Theorem 1′: Consider a discrete-time system

$$x_{k+1}=f(x_k), \quad x_k\in\Omega, \quad k=0,1,...,$$

$$x_0-\text{given},$$

where f is continuously differentiable. If $\max\{\|f(x)\|, \|f^{-1}(x)\|\}\leqslant M$, $x\in\Omega$, then for any $x_0\in\Omega$, all its Lyapunov exponents of the considered orbits satisfy

$$-\ln M\leqslant \ell_1\leqslant \ell_2\leqslant\cdots\leqslant \ell_n\leqslant \ln M.$$

The proof is easy, therefore is omitted here, or is referred to Ref. 12. As a matter of fact, a tighter estimation for lower bound of the Lyapunov exponents is given.[12] Here we introduce Theorem 1′ just to compare with Theorem 1.

In the following, a tighter estimate for a given continuous system is determined.

Integrating (2) from t to $t+\delta$ ($\delta>0$) yields

$$\Phi(t+\delta)=[I+\delta f_x(x,t)]\Phi(t)+o(\delta), \tag{15}$$

in which $\lim_{\delta\to 0^+}\|o(\delta)\|/\delta=0$. From (15), one has

$$\|\Phi(t+\delta)\|-\|\Phi(t)\|\leqslant(\|I+\delta f_x(x,t)\|-1)\cdot\|\Phi(t)\| + \|o(\delta)\|.$$

It follows that

$$\frac{\mathrm{d}\|\Phi(t)\|}{\mathrm{d}t}\leqslant\mathcal{M}(f_x(x,t))\cdot\|\Phi(t)\| \tag{16}$$

where $\mathrm{d}\|\Phi(t)\|/\mathrm{d}t=\lim_{\delta\to 0^+}(\|\Phi(t+\delta)\|-\|\Phi(t)\|)/\delta$, $\mathcal{M}(f_x(x,t))=\lim_{\delta\to 0^+}(\|I+\delta f_x(x,t)\|-1)/\delta$, which is so-called "marix measure," and which is a real number.[16]

Multiplying $e^{-\int_{t_0}^t \mathcal{M}(f_x(x,\tau))\mathrm{d}\tau}$ then integrating from t_0 to t in both sides of (16) gives

$$\|\Phi(t)\|\leqslant e^{\int_{t_0}^t \mathcal{M}(f_x(x,\tau))\mathrm{d}\tau}.$$

It follows that

$$\ell_n\leqslant\limsup_{t\to+\infty}\frac{1}{t}\int_{t_0}^t\mathcal{M}(f_x(x,\tau))\mathrm{d}\tau. \tag{17}$$

On the other hand, by almost the same reasoning as that of (17), from (9), one has

TABLE I. Some special matrix measures.

Norm in C^n	Matrix measure in $C^{n\times n}$
$\|x\|_1=\sum_{i=1}^n \|x_i\|$	$\mathcal{M}_1(A)=\max_j\left\{a_{jj}+\sum_{i\neq j}\|a_{ij}\|\right\}$
$\|x\|_2=\sqrt{\sum_{i=1}^n \|x_i\|^2}$	$\mathcal{M}_2(A)=\frac{1}{2}\lambda_{\max}(\overline{A'}+A)$
$\|x\|_\infty=\max_i\|x_i\|$	$\mathcal{M}_\infty(A)=\max_i\left\{a_{ii}+\sum_{j\neq i}\|a_{ij}\|\right\}$

$$\|\Phi^{-1}(t)\|\leqslant e^{\int_{t_0}^t \mathcal{M}(-f_x(x,\tau))\mathrm{d}\tau}.$$

So,

$$\ell_1\geqslant-\limsup_{t\to+\infty}\frac{1}{t}\int_{t_0}^t\mathcal{M}(-f_x(x,\tau))\mathrm{d}\tau.$$

So, all the Lyapunov exponents ℓ_k $(k=1,2,...,n)$ of system (1) satisfy

$$-\limsup_{t\to+\infty}\frac{1}{t}\int_{t_0}^t\mathcal{M}(-f_x(x,\tau))\mathrm{d}\tau \leqslant\ell_k\leqslant\limsup_{t\to+\infty}\frac{1}{t}\int_{t_0}^t\mathcal{M}(f_x(x,\tau))\mathrm{d}\tau. \tag{18}$$

In general, for a given norm, to determine the corresponding matrix measure is often difficult, so (18) is only of theoretical value. However, for 1-norm, 2-norm, and ∞-norm, the associate matrix measures can be accurately computed. See Table I.[16]

Here, $A=(a_{ij})_{n\times n}$, $\lambda_{\max}(\overline{A'}+A)$ indicates the largest eigenvalue of the matrix $\overline{A'}+A$. It should be noted that for a given matrix A, $\mathcal{M}_1(A)$ may be the smallest, but for another matrix B, $\mathcal{M}_2(B)$ (or $\mathcal{M}_\infty(B)$) may be the smallest.

Theorem 2: The Lyapunov exponents ℓ_k, $k=1,2,...,n$, of system (1) satisfy

$$\alpha\leqslant\ell_k\leqslant\beta,$$

in which

$$\alpha=\max\left\{-\limsup_{t\to+\infty}\frac{1}{t}\int_{t_0}^t\max_j\left\{-a_{jj}(\tau) + \sum_{i\neq j}|a_{ij}(\tau)|\right\}\mathrm{d}\tau, \frac{1}{2}\limsup_{t\to+\infty}\frac{1}{t}\int_{t_0}^t\lambda_{\min}([f_x(x,\tau)]' + f_x(x,\tau))\mathrm{d}\tau, -\limsup_{t\to+\infty}\frac{1}{t}\int_{t_0}^t\max_i\left\{-a_{ii}(\tau) + \sum_{j\neq i}|a_{ij}(\tau)|\right\}\mathrm{d}\tau\right\},$$

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$$\beta=\min\left\{\limsup_{t\to+\infty}\frac{1}{t}\int_{t_0}^{t}\max_j\left\{a_{jj}(\tau)+\sum_{i\neq j}|a_{ij}(\tau)|\right\}\mathrm{d}\tau,\frac{1}{2}\limsup_{t\to+\infty}\frac{1}{t}\int_{t_0}^{t}\lambda_{\max}([f_x(x,\tau)]'+f_x(x,\tau))\mathrm{d}\tau,\limsup_{t\to+\infty}\frac{1}{t}\int_{t_0}^{t}\max_i\left\{a_{ii}(\tau)+\sum_{j\neq i}|a_{ij}(\tau)|\right\}\mathrm{d}\tau\right\},$$

where $a_{ij}(t)=\partial f_i(x,t)/\partial x_j$, $i,j=1,2,...,n$, $'$ denotes the matrix transpose, $\lambda_{\min}([f_x(x,\tau)]'+f_x(x,\tau))$, $\lambda_{\max}([f_x(x,\tau)]'+f_x(x,\tau))$ stand for the smallest and largest eigenvalues of the matrix $[f_x(x,\tau)]'+f_x(x,\tau)$, respectively.

Remark 2

(i) From (2), one has $\int_t^{t+\delta}\mathrm{d}\Phi(t)/\mathrm{d}t=\int_t^{t+\delta}f_x(x,\tau)\Phi(\tau)\mathrm{d}\tau$. In (15), $\delta f_x(x,t)\Phi(t)$ is actually an approximation of $\int_t^{t+\delta}f_x(x,\tau)\Phi(\tau)\mathrm{d}\tau$ with order one. If its high-order approximation is used, then the estimation of $\|\Phi(t)\|$ will be theoretically more accurate but in fact difficult in applications, so is the estimation of ℓ_n. By the same explanation, the estimation of ℓ_1 will be theoretically more accurate if $\int_t^{t+\delta}\Phi^{-1}(\tau)f_x(x,\tau)\mathrm{d}\tau$ is approximated to high orders.

(ii) The bounds derived in Theorem 2 are nonsymmetric, and they are tighter than those derived in Theorem 1.

III. SEVERAL ILLUSTRATIVE EXAMPLES

Here, several examples are taken to show the validity of Theorems in Sec. II.

Example 1:[17]

$$\frac{\mathrm{d}x}{\mathrm{d}t}=-x,\quad \frac{\mathrm{d}y}{\mathrm{d}t}=\left(-1+\frac{1}{1+t}\right)y,\quad \frac{\mathrm{d}z}{\mathrm{d}t}=(-1+2e^{-t})z. \tag{19}$$

Here, the Jacobian is a diagonal matrix, namely, $\mathrm{diag}(-1,-1+1/(1+t),-1+2e^{-t})$, its spectral norm is 1, i.e., $M=1$. The solution of

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t}=\mathrm{diag}\left(-1,-1+\frac{1}{1+t},-1+2e^{-t}\right)\Phi,\quad \Phi(0)=I,$$

is

$$\Phi(t)=\mathrm{diag}(e^{-t},e^{-t+\ln(1+t)},e^{-t-2e^{-t}}).$$

It follows that the Lyapunov exponents of (19) are

$$\ell_1=\ell_2=\ell_3=-1\in[-M,M].$$

By using Theorem 2, one can also easily see that $-1=\ell_1\leqslant\ell_2\leqslant\ell_3=-1$.

Example 2:

$$\frac{\mathrm{d}x}{\mathrm{d}t}=-x,\quad \frac{\mathrm{d}y}{\mathrm{d}t}=\left(1-\frac{1}{1+t}\right)y,\quad \frac{\mathrm{d}z}{\mathrm{d}t}=(-1+2e^{-t})z. \tag{20}$$

By almost the same reasoning of Example 1, one has

$$M=1,\quad\text{and}\quad -M=\ell_1=\ell_2=-1<\ell_3=1=M.$$

From Theorem 2, one can derive the same result.

So one can see that the upper bound or/and the lower bound derived in Theorems 1 and 2 can be optimal for some differential systems.

Example 3:[18,19]

$$\frac{\mathrm{d}x}{\mathrm{d}t}=y,\quad \frac{\mathrm{d}y}{\mathrm{d}t}=z,\quad \frac{\mathrm{d}z}{\mathrm{d}t}=-0.6z-y+|x|-1. \tag{21}$$

(21) is the simplest chaotic nonpolynomial autonomous system in R^3, whose Lyapunov exponents were calculated below:[18,19]

$$\ell_1=-0.635,\quad \ell_2=0,\quad \ell_3=0.035.$$

The Jacobian of (21) is given as

$$J=\begin{pmatrix}0&1&0\\0&0&1\\\mathrm{sgn}(x)&-1&-0.6\end{pmatrix}.$$

So

$$J'J=\begin{pmatrix}1&-\mathrm{sgn}(x)&-0.6\,\mathrm{sgn}(x)\\-\mathrm{sgn}(x)&2&0.6\\-0.6\,\mathrm{sgn}(x)&0.6&1.36\end{pmatrix}.$$

Although J and $J'J$ are not continuous at $x=0$, the characteristic polynomial of $J'J$,

$$p(\lambda)=(\lambda-1)(\lambda-2)(\lambda-1.36)-0.36(\lambda-2)-(\lambda-1.36)-0.36(\lambda-1)-0.72,$$

does not depend upon (x,y,z), so is continuous in R^3. Its three roots are

$$\mu_1=0.330,\quad \mu_2=1,\quad \mu_3=3.023.$$

Obviously,

$$-M<\ell_1<\ell_2<\ell_3<M=\sqrt{3.023}=1.7387,$$

which conforms with Theorem 1 again.

On the other hand, the smallest eigenvalue and the largest one of $J'+J$ are -1.92 and 1.23, respectively. Obviously, $-0.96\leqslant\ell_1\leqslant\ell_2\leqslant\ell_3\leqslant0.615$ is also satisfied. (See Theorem 2.)

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The following example is somewhat complicated. We only compare the calculation results with Theorem 2 since it is known that the bounds determined in Theorem 2 are tighter than those derived in Theorem 1.

Example 4: Consider the following Rössler system[20]

$$\dot{x}=-y-z,$$
$$\dot{y}=x+ay, \tag{22}$$
$$\dot{z}=b+z(x-c).$$

When $a=b=0.2$, $c=5.7$, (22) has a chaotic attractor, and the corresponding exponents are $\ell_1=-5.391$, $\ell_2=0$, $\ell_3=0.0714$.[19,20]

The Jacobian of (22) reads as

$$J=\begin{pmatrix} 0 & -1 & -1 \\ 1 & 0.2 & 0 \\ z & 0 & x-5.7 \end{pmatrix}$$

in which $a=b=0.2$, $c=5.7$ are used.

The eigenvalues of $J'+J$ are listed below

$$\lambda_1=x-5.7-\sqrt{(x-5.7)^2+(z-1)^2}, \quad \lambda_2=0.4,$$
$$\lambda_3=x-5.7+\sqrt{(x-5.7)^2+(z-1)^2}.$$

By numerical integrations, we get

$$-\limsup_{t\to+\infty}\frac{1}{t}\int_{t_0}^{t}\max_j\left\{-a_{jj}(\tau)+\sum_{i\neq j}|a_{ij}(\tau)|\right\}d\tau$$
$$=-7.6267,$$

$$\frac{1}{2}\limsup_{t\to+\infty}\frac{1}{t}\int_{t_0}^{t}\lambda_{\min}([f_x(x,\tau)]'+f_x(x,\tau))d\tau$$
$$=-12.4419,$$

$$-\limsup_{t\to+\infty}\frac{1}{t}\int_{t_0}^{t}\max_i\left\{-a_{ii}(\tau)+\sum_{j\neq i}|a_{ij}(\tau)|\right\}d\tau$$
$$=-6.9871;$$

$$\limsup_{t\to+\infty}\frac{1}{t}\int_{t_0}^{t}\max_j\left\{a_{jj}(\tau)+\sum_{i\neq j}|a_{ij}(\tau)|\right\}d\tau=2.2446,$$

$$\frac{1}{2}\limsup_{t\to+\infty}\frac{1}{t}\int_{t_0}^{t}\lambda_{\max}([f_x(x,\tau)]'+f_x(x,\tau))d\tau=1.6812,$$

$$\limsup_{t\to+\infty}\frac{1}{t}\int_{t_0}^{t}\max_i\left\{a_{ii}(\tau)+\sum_{j\neq i}|a_{ij}(\tau)|\right\}d\tau=2.7623.$$

So $\alpha=-6.9871$, $\beta=1.6812$ (see Theorem 2). Obviously,

$$\alpha<\ell_1=-5.391<\ell_2=0<\ell_3=0.0714<\beta,$$

which conforms with Theorem 2.

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