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Periodic orbits arising from two-level quantized feedback control

Xiaohua Xia *

Department of Electrical, Electronic and Computer Engineering, University of Pretoria, Pretoria, 0002, South Africa

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Abstract

A quantized feedback gives rise to a system of the form $x^+ = f(x) = ax - q(x)$, in which $q(x)$ is the quantized feedback. Polynomials with "quantized" coefficients are introduced, and their properties are investigated. With the help of the roots of some interesting groups of polynomials derived from the polynomials with quantized coefficients, we characterize the value for a such that a periodic point of a certain order appears. It is shown that there are lower and upper bounds on a for the existence of a periodic point of a certain order. An exact (minimal) upper bound is also found for periodic points of any order.

1. Introduction

In [9], the periodicity introduced by Δ-modulated feedback control of a scalar system with a scaling parameter a is completely characterized, and it is shown that a periodic point of a certain order exists if and only if $|a|$ is bigger or equal to a real number determined as a unique root in $(1, \infty)$ of a polynomial. In particular, when a is big enough, there are periodic points of any order. Some results are also obtained for general high order systems subject to Δ-modulated feedback in [10].

There are some recent interests in *quantized feedback control* [2,1,4,6]. Δ-modulated feedback control is a two-level quantized feedback control without a deadzone, and a quantized feedback control is a cascade of Δ-modulated feedback control with a center deadzone. It has been well-known that both kinds of controls introduce periodicity into the system. The spectrum information of the digital output signals is practically important for in helping developing preventive measures if they are necessary. Yet it is difficult to characterize the periodicity due to the introduction of the discontinuities in Δ-modulation and quantization using established results and techniques for continuous maps [3,5,7,8]. It is remarked that there is no systematic study and rigorous results concerning periodicity due to quantized feedback.

In this paper, we study the periodicity of a scalar system under a feedback control with a two-level quantization. Our results will reveal that even for such simple cases, the appearance of periodicity of different orders with respect to the parameter a exhibits very interesting phenomenon. We will show that, similar to Δ-modulated control, there is a lower

* Tel.: +27 12 420 2165; fax: +27 12 362 5000.
E-mail address: xxia@postino.up.ac.za

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bound on a, and contrary to Δ-modulated control, there is an upper bound on a, so that a periodic point of a certain order exists. We also find the exact minimal upper bound for each periodic order with the help of interesting groups of polynomials. Discussions on the lower bounds are also given.

In Section 2, we introduce some groups of polynomials and study their properties. These are used in Section 3 to characterize the existence of periodic orbits arising from the system under a two-level quantized feedback control. Section 4 is devoted to conclusion.

2. Polynomials and their roots

We collect here some preliminaries that are needed in the later development.

Proposition 1. *Let $f(x) \in C^1[\alpha, \infty)$, where α is a real, satisfying*

$$f(\alpha) \leqslant 0, \quad \lim_{x\to\infty} f(x) = +\infty.$$

If $f'(\alpha) < 0$ and $f'(x)$ has only one real root in (α, ∞), then $f(x)$ has only one real root in (α, ∞). When $f(\alpha) < 0$, the condition $f'(\alpha) < 0$ can also be relaxed to $f'(\alpha) \leqslant 0$.

Proof. Denote $\bar{\alpha} \in (\alpha, \infty)$ be the only real root of $f'(x)$, then it is easy to conclude that for $x \in [\alpha, \bar{\alpha})$,

$$f'(x) < 0,$$

and for $x > \bar{\alpha}$,

$$f'(x) > 0.$$

Since $f(\alpha) \leqslant 0$, we have $f(\bar{\alpha}) < 0$. By the strictly monotonicity of $f(x)$ on $[\bar{\alpha}, \infty)$ and $\lim_{x\to\infty} f(x) = +\infty$, we conclude that $f(x)$ has only one root in (α, ∞) and it is in $(\bar{\alpha}, \infty)$. □

We define the following four sets of polynomials:

$$\begin{aligned}
p_1(a) &= p^1(a) = a - 1,\\
q_1(a) &= q^1(a) = a - 3,\\
p_2(a) &= p^2(a) = a^2 - 3,\\
q_2(a) &= q^2(a) = a^2 - 2a - 1,\\
p_3(a) &= a^3 - 2a - 1,\\
q_3(a) &= a^3 - 2a^2 - 1,\\
p^3(a) &= a^3 - 2a - 3,\\
q^3(a) &= a^3 - 2a^2 - 3,
\end{aligned}$$

and for $n \geqslant 4$,

$$\begin{aligned}
p_n(a) &= a^n - 2a^{n-2} - 1,\\
q_n(a) &= a^n - 2a^{n-1} - 1,\\
p^n(a) &= a^n - 2a^{n-2} - 2a^{n-3} - \cdots - 2a - 3,\\
q^n(a) &= a^n - 2a^{n-1} - 2a^{n-3} - \cdots - 2a - 3.
\end{aligned}$$

These polynomials have a very special property: the polynomials $p_1(a)$ and $p^1(a)$ have their only root at $a = 1$, and all other polynomials have only one real root in $(1, \infty)$.

Lemma 1

(i) *For $n \geqslant 2$, each of the polynomials $p_n(a)$, $q_n(a)$, $p^n(a)$ and $q^n(a)$ has only one real root in $(1, \infty)$.*

(ii) *Denote $\underline{p}_1$, $\underline{q}_1$, $\bar{p}_1$ and $\bar{q}_1$ the root of $p_1(a)$, $q_1(a)$, $p^1(a)$ and $q^1(a)$, respectively, and for $n \geqslant 2$, denote $\underline{p}_n$, $\underline{q}_n$, $\bar{p}_n$ and $\bar{q}_n$ the only root of $p_n(a)$, $q_n(a)$, $p^n(a)$ and $q^n(a)$ in $(1, \infty)$, respectively. Then for $n \geqslant 3$,*

(ii.1) $\underline{p}_{n+1} < \underline{p}_n$ *and* $\lim_{n\to\infty} \underline{p}_n = \sqrt{2}$;

(ii.2) $\underline{q}_{n+1} < \underline{q}_n$ *and* $\lim_{n\to\infty} \underline{q}_n = 2$;

(ii.3) $\bar{p}_{n+1} > \bar{p}_n$ *and* $\lim_{n\to\infty}\bar{p}_n = 2$;
(ii.4) $\bar{q}_{n+1} > \bar{q}_n$ *and*

$$\lim_{n\to\infty}\bar{q}_n = 1 + \frac{\sqrt[3]{27+11\sqrt{6}}}{3} + \frac{1}{\sqrt[3]{27+11\sqrt{6}}} \doteq 2.5241.$$

Proof. We give proofs for the results concerning $q^n(a)$. Similar (but simpler) proofs can be worked out for other cases.

(i) It is easy to verify that $q^2(a)$ and $q^3(a)$ have the their only real roots in $(1,\infty)$ as $\bar{q}_2 = 1+\sqrt{2}\doteq 2.4142$ and $\bar{q}_3\doteq 2.4556$, respectively. For $n \geqslant 4$, denote

$$\tilde{q}^n(a) = a^{n+1} - 3a^n + 2a^{n-1} - 2a^{n-2} - a + 3,$$

then we verify that

$$q^n(a) = \frac{\tilde{q}^n(a)}{a-1},$$

therefore, we only need to prove that $\tilde{q}^n(a)$ has only one real root in $(1,\infty)$.

Note that

$$\frac{\mathrm{d}\tilde{q}^n(a)}{\mathrm{d}a} = (n+1)a^n - 3na^{n-1} + 2(n-1)a^{n-2} - 2(n-2)a^{n-3} - 1,$$

$$\frac{\mathrm{d}^2\tilde{q}^n(a)}{\mathrm{d}a^2} = a^{n-4}((n+1)na^3 - 3n(n-1)a^2 + 2(n-1)(n-2)a - 2(n-2)(n-3))$$

$$\stackrel{\text{def}}{=} a^{n-4}\bar{q}^n(a),$$

$$\frac{\mathrm{d}\bar{q}^n(a)}{\mathrm{d}a} = 3(n+1)na^2 - 6n(n-1)a + 2(n-1)(n-2).$$

We verify that when $n \geqslant 4$, $\frac{\mathrm{d}\bar{q}^n(a)}{\mathrm{d}a}$ has the following only root in $(1,\infty)$,

$$\frac{3n(n-1) + \sqrt{3n^4 - 6n^3 + 15n^2 - 12n}}{3(n+1)n},$$

and

$$\frac{\mathrm{d}\bar{q}^n(1)}{\mathrm{d}a} = -(n-4)(n+1) \leqslant 0,$$

$$\bar{q}^n(1) = -2(n-2)^2 < 0,$$

$$\bar{q}^n(\infty) = \infty.$$

By Proposition 1, $\bar{q}^n(a)$ has only one real root on $(1,\infty)$. This implies that $\frac{\mathrm{d}^2\tilde{q}^n(a)}{\mathrm{d}a^2}$ has only one real root on $(1,\infty)$.

Similarly, we check that

$$\frac{\mathrm{d}^2\tilde{q}^n(1)}{\mathrm{d}a^2} = -2(n-2)^2 < 0,$$

$$\frac{\mathrm{d}\tilde{q}^n(1)}{\mathrm{d}a} = -2(n-1) < 0,$$

$$\tilde{q}^n(1) = 0,$$

$$\frac{\mathrm{d}\tilde{q}^n(\infty)}{\mathrm{d}a} = \infty,$$

$$\tilde{q}^n(\infty) = \infty.$$

By repeatedly revoking Proposition 1, we arrive at our conclusion that $\tilde{q}^n(a)$ has only one real root on $(1,\infty)$.

(ii) First note that since $q^n(2) = -2^{n-1} - 1 < 0$ and $q^n(3) = 2\times 3^{n-2} > 0$, so actually we have that for all $n \geqslant 2$,

$$2 < \bar{q}_n < 3. \tag{1}$$

Obviously,

$$\bar{q}_2 < \bar{q}_3 < \bar{q}_4.$$

For $n \geqslant 4$, since by definitions of $\bar{q}_n$ and $\tilde{q}^n(a)$,

$$\bar{q}_n^{n+1} - 3\bar{q}_n^n + 2\bar{q}_n^{n-1} - 2\bar{q}_n^{n-2} - \bar{q}_n + 3 = 0,$$

or

$$\bar{q}_n^{n+1} = 3\bar{q}_n^n - 2\bar{q}_n^{n-1} + 2\bar{q}_n^{n-2} + \bar{q}_n - 3,$$

we can verify that

$$\begin{aligned}\tilde{q}^{n+1}(\bar{q}_n) &= \bar{q}_n^{n+2} - 3\bar{q}_n^{n+1} + 2\bar{q}_n^n - 2\bar{q}_n^{n-1} - \bar{q}_n + 3 = \bar{q}_n(3\bar{q}_n^n - 2\bar{q}_n^{n-1} + 2\bar{q}_n^{n-2} + \bar{q}_n - 3) - 3\bar{q}_n^{n+1} + 2\bar{q}_n^n - 2\bar{q}_n^{n-1} - \bar{q}_n + 3 \\ &= (\bar{q}_n - 3)(\bar{q}_n - 1) < 0.\end{aligned}$$

Hence

$$q^{n+1}(\bar{q}_n) < 0.$$

So necessarily, the only real root $\bar{q}_{n+1}$ of $q^{n+1}(a)$ is greater than $\bar{q}_n$, i.e.,

$$\bar{q}_{n+1} > \bar{q}_n.$$

Denote the limit of $\bar{q}_n$ by $\bar{q}$, then since $\tilde{q}^n(\bar{q}_n) = 0$, we have

$$(\bar{q}_n^3 - 3\bar{q}_n^2 + 2\bar{q} - 2) - \frac{\bar{q}_n - 3}{\bar{q}_n^{n-2}} = 0. \tag{2}$$

When $n \to \infty$, the last term tends to zero, since

$$\left|\frac{\bar{q}_n - 3}{\bar{q}_n^{n-2}}\right| \leqslant \frac{1}{\bar{q}_n^{n-3}} + \frac{3}{\bar{q}_n^{n-2}} \overset{(1)}{<} \frac{1}{2^{n-3}} + \frac{3}{2^{n-2}}.$$

Taking the limit of two sides of (2), we have then

$$\bar{q}^3 - 3\bar{q}^2 + 2\bar{q} - 2 = 0.$$

This solves $\bar{q}$ in $(2,3)$ as

$$\bar{q} = 1 + \frac{\sqrt[3]{27 + 11\sqrt{6}}}{3} + \frac{1}{\sqrt[3]{27 + 11\sqrt{6}}} \doteq 2.5241. \qquad \square$$

3. Periodicity and bounds

Consider a first order discrete-time control system with a two-level quantized feedback

$$x^+ = f(x) \overset{\text{def}}{=} ax - q(x), \tag{3}$$

where *the scaling factor* $a > 0$ is a real number, the quantized feedback $q(x)$ is defined as

$$q(x) = \begin{cases} 1, & x \geqslant 0.5, \\ 0, & -0.5 < x < 0.5, \\ -1, & x \leqslant -0.5. \end{cases}$$

The following easy result is left to the readers to verify.

Proposition 2. *When* $0 < a \leqslant 1$, *there are only three periodic points of the system* (3) $\{0, \pm 1/(a+1)\}$, *and* 0 *is* 1-*periodic* (*fixed point*), *and* $\pm 1/(a+1)$ *are* 2-*periodic. The set* $\{0, \pm 1/(a+1)\}$ *are globally attracting.*

In the following, we only consider the case when $a > 1$.

Because of the symmetry of $f(x)$, we first have the following result.

Lemma 2. *If x is n-periodic, then $-x$ is n-periodic.*

Lemma 3

(i) *There is no periodic orbit entirely inside the interval* $(-1/2, 1/2)$.
(ii) *There is no periodic orbit entirely outside of the interval* $(-1/2, 1/2)$.

Proof. The proof of (i) is easy and omitted here. The proof of (ii) can be worked out to be contradiction. First of all the orbit cannot be entirely in either $(-\infty, -1/2]$ or $[1/2, \infty)$, because $f(x)$ is strictly increasing on the two intervals. So if there is such an orbit, it must contain a point $x \geqslant 1/2$ such that $ax - 1 \leqslant -1/2$, which is only possible when $a < 1$. □

The polynomials introduced in the last section are important to characterize the existence of periodic orbits of the system (3).

Note that 0 is a 1-periodic point (fixed point) of the system (3) for any a.

Theorem 1. *For any $n = 1, 2, \ldots$, the system (3) has non-zero n-periodic points if $\underline{p}_n < a \leqslant \bar{q}_n$.*

Proof. By Lemma 2, if the system has an n-periodic point, we can always assume that it has a positive n-periodic point.

For $n = 1$, if the system has a non-zero fixed point, it lies outside of $(-1/2, 1/2)$. This is because the map $f(x) = ax$, when $x \in (-1/2, 1/2)$, having no other fixed point than 0. Suppose $x \geqslant 1/2$ is a fixed point, then $ax - 1 = x$, and $x = 1/(a - 1)$. $x = 1/(a - 1)$ is both positive and greater than 1/2 if and only if

$$\underline{p}_1 = 1 < a \leqslant 3 = \bar{q}_1.$$

For $n = 2$, similar reasoning reveals that the system (3) has 2-periodic points if and only if there is an $x \in (0, 1/2)$ such that $ax \geqslant 1/2$ and $\{x, ax\}$ constitutes a 2-periodic orbit, i.e.,

$$a(ax) - 1 = a^2x - 1 = x$$

or

$$x = 1/(a^2 - 1).$$

Therefore, the following two inequalities must hold:

$$1/(a^2 - 1) < 1/2, \qquad a/(a^2 - 1) \geqslant 1/2.$$

It is easy to see that this is the case if and only if

$$\underline{p}_2 = \sqrt{3} < a \leqslant 1 + \sqrt{2} = \bar{q}_2.$$

For $n \geqslant 3$, our proof follows two steps:

(a) when $\underline{p}_n < a \leqslant \underline{q}_n$, the following n points constitute an n-periodic orbit:

$$g_i = \frac{a^i}{a^n - 1}$$

for $i = 0, 1, 2, \ldots, n - 1$.

(b) when $\underline{q}_n < a \leqslant \bar{q}_n$, the following n points constitute an n-periodic orbit:

$$h_0 = \frac{a^{n-2} + a^{n-3} + \cdots + a + 1}{a^n - 1},$$
$$h_1 = \frac{a^{n-1} + a^{n-2} + \cdots + a}{a^n - 1},$$
$$h_2 = \frac{a^{n-1} + a^{n-2} + \cdots + a^2 + 1}{a^n - 1},$$
$$h_i = \frac{a^{n-1} + a^{n-2} + \cdots + a^i + a^{i-2} + \cdots + a + 1}{a^n - 1}$$

for $i = 3, \ldots, n - 1$.

To prove (a), firstly note that for $a > 1$,

$$\frac{1}{a^n-1} < \frac{a}{a^n-1} < \cdots < \frac{a^{n-2}}{a^n-1} < \frac{a^{n-1}}{a^n-1}.$$

Secondly, when $\underline{p}_n < a \leqslant \underline{q}_n$, then

$$p_n(a) = a^n - 2a^{n-2} - 1 > 0,$$
$$q_n(a) = a^n - 2a^{n-1} - 1 \leqslant 0.$$

The first inequality implies

$$\frac{a^{n-2}}{a^n-1} < \frac{1}{2},$$

and the second inequality implies

$$\frac{a^{n-1}}{a^n-1} \geqslant \frac{1}{2}.$$

That is, we have

$$\frac{1}{a^n-1} < \frac{a}{a^n-1} < \cdots < \frac{a^{n-2}}{a^n-1} < \frac{1}{2} \leqslant \frac{a^{n-1}}{a^n-1}.$$

By the definition of $f(x)$, it is readily verified that

$$f(g_i) = g_{i+1}$$

for $i = 0, 1, \ldots, n-2$, and

$$f(g_{n-1}) = \frac{a^n}{a^n-1} - 1 = \frac{1}{a^n-1} = g_0.$$

Finally, since $g_i \neq g_j$, for all $i \neq j$, $\{g_0, g_1, \ldots, g_{n-1}\}$ constitutes a periodic orbit with prime period n.

To prove (b), we can similarly note the following inequalities:

$$h_0 < h_{n-1} < \cdots < h_2 < h_1,$$

and when $\underline{q}_n < a \leqslant \bar{q}_n$, then by Lemma 1 (ii.2) and (ii.3), $\bar{p}_n < \underline{q}_n < a \leqslant \bar{q}_n$. This then implies that

$$p^n(a) = a^n - 2a^{n-2} - 2a^{n-3} - \cdots - 2a - 3 > 0,$$
$$q^n(a) = a^n - 2a^{n-1} - 2a^{n-3} - \cdots - 2a - 3 \leqslant 0.$$

They consequently imply

$$h_0 = \frac{a^{n-2} + a^{n-3} + \cdots + a + 1}{a^n - 1} < \frac{1}{2},$$
$$h_{n-1} = \frac{a^{n-1} + a^{n-3} + \cdots + a + 1}{a^n - 1} \geqslant \frac{1}{2}.$$

That is, we have

$$h_0 < 1/2 \leqslant h_{n-1} < \cdots < h_2 < h_1,$$

and by definition of $f(x)$, it is routine to verify that

$$f(h_i) = h_{i+1}$$

for $i = 0, 1, \ldots, n-2$, and

$$f(h_{n-1}) = h_0.$$

Finally, since $h_i \neq h_j$, for all $i \neq j$, $\{h_0, h_1, \ldots, h_{n-1}\}$ constitutes a periodic orbit with prime period n.

We have noted in the proof that $1 < a \leqslant 3$ and $\sqrt{3} < a \leqslant 1 + \sqrt{2}$ are actually necessary and sufficient conditions for the existence of non-zero 1-periodic and 2-periodic points, respectively. □

3.1. Upper bounds

For a given positive integer $n \geqslant 2$, by an ordered set of n "quantized" parameters, we mean the set $\{\theta_0, \theta_1, \ldots, \theta_{n-1}\}$, in which, $\theta_i \in \{-1, 0, 1\}$, $i = 0, 1, \ldots, n-1$. An ordered set of polynomials with "quantized" coefficients $\mathscr{P}_{\{\theta_0, \theta_1, \ldots, \theta_{n-1}\}}$ corresponding to a given ordered set of "quantized" parameters $\{\theta_0, \theta_1, \ldots, \theta_{n-1}\}$ is defined as follows:

$$\left\{ P_i(a) \middle| P_i(a) = \sum_{j=0}^{n-1} \theta_{i+j} a^{k-j-1}, i = 0, 1, \ldots, n-1 \right\}, \tag{4}$$

where $\theta_{i+j} = \theta_{(i+j)\text{mod}(n)}$.

The ordered set of "quantized" parameters $\{\theta_0, \theta_1, \ldots, \theta_{n-1}\}$ is called shift-definite at a if, denoting the leading coefficient of $P_i(a)$ by $\bar{\theta}_i$, then

$$\begin{cases} \frac{\bar{\theta}_i P_i(a)}{a^n - 1} \geqslant 1/2, & \text{when } \bar{\theta}_i = \theta_i, \\ 0 < \frac{\bar{\theta}_i P_i(a)}{a^n - 1} < 1/2, & \text{when } \bar{\theta}_i \neq \theta_i = 0, \end{cases}$$

for all $P_i(a) \in \mathscr{P}_{\{\theta_0, \theta_1, \ldots, \theta_{n-1}\}}$, $i = 0, 1, \ldots, n-1$.

Theorem 2

(i) *A point* $x_0 \in R$ *is a periodic point with period* n *if and only if there is a set of* n *"quantized" parameters* $\{\theta_0, \theta_1, \ldots, \theta_{n-1}\}$, $\theta_i \in \{-1, 0, 1\}$, $i = 0, 1, \ldots, n-1$, *which is shift-definite, such that*

$$x_0 = \frac{1}{a^n - 1} \sum_{i=0}^{n-1} a^{n-i-1} \theta_i. \tag{5}$$

(ii) *A point* $x_0 \in R$ *is an* n*-periodic point* (*that is,* n *is the prime period*) *if and only if* n *is the smallest positive integer such that* (i) *holds.*

Proof. (i) (Necessity) For any point $x_0 \in R$, if it is a periodic point with period n, then, denoting $x_i = f^i(x_0)$, we have

$$x_0 = x_n = a^n x_0 - \sum_{i=0}^{n-1} a^{n-i-1} q(x_i),$$

and hence

$$x_0 = \frac{1}{a^n - 1} \sum_{i=0}^{n-1} a^{n-i-1} q(x_i).$$

Clearly, x_0 is in the form of (5) for $\theta_i = q(x_i)$, $i = 0, 1, \ldots, n-1$. Furthermore, if x_0 is a periodic point with period n, then it is easily verified that x_k, $k = 0, 1, \ldots, n-1$, also satisfies the following equalities:

$$x_k = f^k(x_0) = \frac{1}{a^n - 1} \sum_{i=0}^{n-1} a^{n-i-1} q(x_{k+i})$$

From the above n equalities and the fact that $x_{k+i} = x_{(k+i)\text{mod}(n)}$, we conclude that this set of parameters is shift-definite at a.

(Sufficiency) When the conditions in the theorem are satisfied by some point x_0, we have, first of all, from the definition of shift-definiteness, we can verify directly that

$$x_k = f^k(x_0) = \frac{1}{a^n - 1} \sum_{i=0}^{n-1} a^{n-i-1} \theta_{i+k}.$$

In particular,

$$x_n = f^n(x_0) = \frac{1}{a^n - 1} \sum_{i=0}^{n-1} a^{n-i-1} \theta_{i+n} = x_0.$$

(ii) The proof is direct. □

Theorem 3. *$a \leqslant \bar{q}_n$ is necessary for having an n-periodic point.*

Proof. To show this, let us assume that there is a set of n quantized parameters $\{\theta_0, \theta_1, \ldots, \theta_{n-1}\}$, $\theta_i \in \{-1, 0, 1\}$, $i = 0, 1, \ldots, n-1$, which is shift-definite, such that

$$x_j = \frac{1}{a^n - 1} \sum_{i=0}^{n-1} a^{n-i-1} \theta_{i+j}$$

for $j = 0, 1, \ldots, n-1$, are n-periodic points. Obviously, there is at least one x_j outside of $(-1/2, 1/2)$. Without loss of generality,

$$\theta_0 x_0 = \frac{1}{a^n - 1} \sum_{i=0}^{n-1} a^{n-i-1} \theta_i \theta_0 \geqslant \frac{1}{2},$$

and it is the smallest among $|x_j|$'s that are greater or equal to 1/2.

We claim that $|\theta_0| = 1$, because $|x_0| \geqslant 1/2$ and $\{\theta_0, \theta_1, \ldots, \theta_{n-1}\}$, $\theta_i \in \{-1, 0, 1\}$, $i = 0, 1, \ldots, n-1$ is shift-definite.

We also claim that $\theta_1 = 0$. This is because $|x_0| \geqslant 1/2$ and $|x_0|$ is the minimal $|x_i|$'s that are greater or equal to 1/2. This implies that $|x_1| = |ax_0 - \operatorname{sgn}(x_0)| < 1/2$. Also by the shift-definiteness of $\{\theta_0, \theta_1, \ldots, \theta_{n-1}\}$, $\theta_i \in \{-1, 0, 1\}$, $i = 0, 1, \ldots, n-1$, we know that $\theta_1 = 0$.

So we have

$$\theta_0 x_0 = \frac{a^{n-1} + \theta_0 \theta_2 a^{n-3} + \cdots + \theta_0 \theta_{n-1}}{a^n - 1},$$

and therefore,

$$\frac{a^{n-1} + a^{n-3} + \cdots + a + 1}{a^n - 1} \geqslant \frac{a^{n-1} + \theta_0 \theta_2 a^{n-3} + \cdots + \theta_0 \theta_{n-1}}{a^n - 1} = \theta_0 x_0 \geqslant \frac{1}{2}.$$

From here, we obtain that $a \leqslant \bar{q}_n$. □

3.2. Remarks on lower bounds

As can been seen that there are also lower bounds. However, they are more difficult to find.

We can easily check that $a > \underline{p}_1$, $a > \underline{p}_2$, $a > \underline{p}_3$, $a > \underline{p}_4$ and $a > \underline{p}_5$ are also necessary for the system to have 1-, 2-, 3-, 4- and 5-periodic points, respectively. However, in general, the lower bound $a > \underline{p}_n$ is not necessary for the existence of n-periodic points.

As a matter of fact, for all $1 < a \leqslant \bar{q}_6$, there is a 6-periodic orbit. For $1 < a \leqslant \underline{p}_6$, we can verify that an orbit starting from $1/(a^3 + 1)$ is

$$\left\{ \frac{1}{a^3+1}, \frac{a}{a^3+1}, \frac{a^2}{a^3+1}, -\frac{1}{a^3+1}, -\frac{a}{a^3+1}, -\frac{a^2}{a^3+1} \right\}.$$

Note that when $1 < a \leqslant \underline{p}_6$,

$$\frac{1}{a^3+1} < \frac{a}{a^3+1} < \frac{1}{2} < \frac{a^2}{a^3+1}.$$

In general, we can define the following two more sets of polynomials for $m \geqslant 3$:

$$p_m^e(a) = a^m - 2a^{m-2} + 1,$$
$$q_m^e(a) = a^m - 2a^{m-1} + 1.$$

It is also easy to prove that both $p_m^e(a)$ and $q_m^e(a)$ have a unique real root in $(1, \infty)$, denoted by $\underline{p}_m^e$, and $\bar{q}_m^e$ respectively, and

(i) $\underline{p}_m^e \bar{p} \uparrow \sqrt{2}$; (ii) $\bar{q}_m^e \bar{p} \uparrow 2$.

When $\underline{p}^e < a \leqslant \bar{q}_m^e$, then there exists $2m$-periodic points:

$$\left\{ \frac{1}{a^m+1}, \frac{a}{a^m+1}, \ldots, \frac{a^{m-1}}{a^m+1}, -\frac{1}{a^m+1}, -\frac{a}{a^m+1}, \ldots, -\frac{a^{m-1}}{a^m+1} \right\}.$$

It is easily verified that $\underline{p}_{2m} < \bar{q}_m^e$, so we know that when $\underline{p}_m^e < a \leqslant \bar{q}_{2m}$, there exist $2m$-periodic points.

Another example is a 13th order periodic point given by

$$\frac{a^{11} - a^7 + a^4 - a}{a^{13} - 1}$$

which exists for $a > 1.1593$.

4. Conclusion

We have investigated the periodicity of a scalar system introduced by a 2-level quantized feedback. It is found that there are lower and upper bounds of the scaling factor such that a periodic point of certain order exists. The exact upper bounds are characterized by using roots of interesting groups of polynomials. It is seen that the lower bounds are all smaller than $\sqrt{2}$, and their exact characterization is under current investigation.

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