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Periodic orbits arising from Delta-modulated feedback control

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Abstract

A Delta-modulated feedback gives rise to a system of the form $x^+ = f(x) = ax - \Delta\,\mathrm{sgn}(ax)$. In this paper, we will determine the a values, $1 < |a| < 2$, for which periodic orbits of each order exist. Polynomials with "sign" coefficients are introduced, and their properties are investigated. With the help of the roots of these polynomials, we characterize the minimal value for $|a|$ such that a periodic point of a certain order first appears. Our results show that even though the topological properties of the tent map and the map f are different, the mechanisms of giving rise to periodic orbits via parameter variations are exactly the same for $-2 < a < -1$, and only "slightly" different for $1 < a < 2$.

1. Introduction

Delta-modulation is a concept from telecommunication [11]. Delta-modulated feedback has been applied to the transmitting power regulation of a mobile unit in the Direct Sequence Code Division Multiple Access (DS-CDMA) cellular network [1]. An advantage of such a control method is that only one bit of datum is necessary for implementing the controller. This is the standard in IS-95 [11] for transmitting power control.

Delta-modulated control is bounded, bang-bang, and also a special kind of *quantized control*, which are topics of longstanding interests in the control community [2,3,5,8]. Delta-modulated feedback is a switching between two values. The resulting switching system is a special kind of piecewise linear systems [10,14,15].

The simplicity and speciality of Delta-modulation make it an attractive choice for control practitioners. Yet, the rich mathematical contents of this seemingly simple Delta-modulated control have yet to be discovered.

A Delta-modulated feedback of a one-dimensional discrete-time control system gives rise to a dynamical system of the following form [6,16]:

$$x^+ = f(x) \stackrel{\mathrm{def}}{=} ax - \Delta\,\mathrm{sgn}(ax), \tag{1}$$

where x^+ denotes the system state at the next discrete-time, a is a real number, and $\mathrm{sgn}(x)$ is defined as

$$\mathrm{sgn}(x) = \begin{cases} 1, & \text{when } x \geqslant 0, \\ -1, & \text{when } x < 0. \end{cases}$$

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We have shown in [16] that a dynamical system of such a simple nature exhibits very complex behaviors: (1) when $|a| < 1$, there is a minimal global attractor which consists of only two points. The two points form either one 2-periodic orbit or two 1-periodic orbits (fixed points); (2) when $1 \leqslant |a| < 2$, all the points in a maximal stabilizable region are driven to a closed interval $[-\Delta, \Delta]$, and f is invariant on $[-\Delta, \Delta]$; and (3) when $|a| \geqslant 2$, the maximal stabilizable set is a Cantor set, the Cantor set is a repeller of the system, and the system is chaotic on the Cantor set. In the development of the above results, we have also shown that in the third case, there are periodic points of any positive period, and in the second case and when $a > 0$, there are two 2-periodic points but no fixed points in the interval $[-\Delta, \Delta]$.

This last fact is interesting, because it is a departure from what the famous Sarkovskii theorem [13] claims for continuous dynamical systems. In this paper, we will vividly illustrate the conditions for the existence of all other periodic orbits. We will determine the a values for which periodic orbits of each order exist in the interval $[-\Delta, \Delta]$.

When $1 < a < 2$, the results are exactly the same for the corresponding results obtained in [7] (see also [4,12]) for the tent map, especially if we consider our map f on a bigger interval $[-\Delta/(|a|-1), \Delta/(|a|-1)]$. (The two fixed points of f are $\pm\Delta/(|a|-1)$.) We notice that even though the topological properties of the tent map and the map f are fundamentally different, the mechanisms of giving rise to periodic orbits via parameters are strikingly similar. Our approach is based on some interesting properties of the new map. We will firstly look at three groups of polynomials with "sign" coefficients in Section 2. These polynomials will be used to present results, in Section 3, for the case of $1 < a < 2$, of which we call the system (1) of type-I, and in Section 4, for the case of $-2 < a < -1$, of which we call the system (1) of type-II. Section 5 of the paper gives some concluding remarks.

2. Polynomials with "sign" coefficients

2.1. Definitions

For a given positive integer $k \geqslant 2$, by an ordered set of k "sign" parameters, we mean the set $\{\theta_0, \theta_1, \ldots, \theta_{k-1}\}$, in which, $\theta_i \in \{-1, 1\}$, $i = 0, 1, \ldots, k-1$. An ordered set of polynomials with "sign" coefficients $\mathscr{P}_{\{\theta_0,\theta_1,\ldots,\theta_{k-1}\}}$ corresponding to a given ordered set of "sign" parameters $\{\theta_0, \theta_1, \ldots, \theta_{k-1}\}$ is defined as follows:

$$\left\{P_i(a) | P_i(a) = \sum_{j=0}^{k-1} \theta_{i+j} a^{k-j-1},\ i = 0, 1, \ldots, k-1\right\}, \tag{2}$$

where $\theta_{i+j} = \theta_{(i+j)\bmod(k)}$.

The ordered set of "sign" parameters $\{\theta_0, \theta_1, \ldots, \theta_{k-1}\}$ is called (strictly) shift-definite at a if

$$\theta_i P_i(a) \geqslant (>) 0,$$

for all $P_i(a) \in \mathscr{P}_{\{\theta_0,\theta_1,\ldots,\theta_{k-1}\}}$, $i = 0, 1, \ldots, k-1$. It is called shift-languished at a if, for $i = 0, 1, \ldots, k-1$,

$$\theta_i (P_i(a) - P_{i+1}(a)) > 0,$$

in which we let $P_k(a) = P_0(a)$.

Proposition 1. *For any given positive integer $k > 1$, an ordered set of "sign" parameters $\{\theta_0, \theta_1, \ldots, \theta_{k-1}\}$ in which there are at least two different elements is shift-languished at any $a > 1$.*

Proof. Denote the ordered set of polynomials corresponding to $\{\theta_0, \theta_1, \ldots, \theta_{k-1}\}$ as

$$\mathscr{P}_{\{\theta_0,\theta_1,\ldots,\theta_{k-1}\}} = \{P_0(a), P_1(a), \ldots, P_{k-1}(a)\}.$$

We also denote the following polynomial:

$$\overline{P}_k(a) = a^{k-1} + a^{k-2} + \cdots + a + 1. \tag{3}$$

It is easily verified that the following equality

$$P_j(a) = a(P_{j-1}(a) - \theta_{j-1} a^{k-1}) + \theta_{j-1} \tag{4}$$

holds true for every $0 < j \leqslant k$. Hence, we have, for $j = 1, 2, \ldots, k$,

$$\theta_{j-1}(P_{j-1}(a) - P_j(a)) = \theta_{j-1}P_{j-1}(a) - \theta_{j-1}a(P_{j-1}(a) - \theta_{j-1}a^{k-1}) - 1 = -\theta_{j-1}(a-1)P_{j-1}(a) + a^k - 1$$
$$= (a-1)(\overline{P}_k(a) - \theta_{j-1}P_{j-1}(a)) > 0. \qquad \square$$

Define, on $(1,\infty)$, a function of a, by

$$\underline{\mathbf{P}}_{\{\theta_0,\dots,\theta_{k-1}\}}(a) = \min\{\theta_i P_i(a) | P_i \in \mathscr{P}_{\{\theta_0,\dots,\theta_{k-1}\}}\} \tag{5}$$

and we call it the minimal value function w.r.t. $\{\theta_0, \theta_1, \dots, \theta_{k-1}\}$. If there is a $P_i(a) \in \mathscr{P}_{\{\theta_0,\theta_1,\dots,\theta_{k-1}\}}$ such that

$$\underline{\mathbf{P}}_{\{\theta_0,\theta_1,\dots,\theta_{k-1}\}}(a) = \theta_i P_i(a),$$

and for all $j \neq i$, $P_j(a) \in \mathscr{P}_{\{\theta_0,\theta_1,\dots,\theta_{k-1}\}}$,

$$\underline{\mathbf{P}}_{\{\theta_0,\theta_1,\dots,\theta_{k-1}\}}(a) < \theta_j P_j(a),$$

then we call the minimal value function is strictly minimal at a. Implied by these definitions are the following results.

Lemma 1. *An ordered set of k "sign" parameters* $\{\theta_0, \theta_1, \dots, \theta_{k-1}\}$ *is shift-definite at a if and only if its minimal value function* $\underline{\mathbf{P}}_{\{\theta_0,\theta_1,\dots,\theta_{k-1}\}}(a) \geqslant 0$.

Lemma 2. *For every positive integer* $k \geqslant 2$, *the minimal value function* $\underline{\mathbf{P}}_{\{\theta_0,\theta_1,\dots,\theta_{k-1}\}}(a)$ *is continuous on* $(1,\infty)$.

Proof. For a given point $a_0 \in (1,\infty)$, if there exists a unique polynomial $P_i(a) \in \mathscr{P}_{\{\theta_0,\theta_1,\dots,\theta_{k-1}\}}$ such that $\underline{\mathbf{P}}_{\{\theta_0,\theta_1,\dots,\theta_{k-1}\}}(a_0) = \theta_i P_i(a_0)$, i.e., $\theta_i P_i(a_0) < \theta_j P_j(a_0)$ for all $j \neq i$, then by the continuity of a polynomial function, there must exist a neighborhood of the point a_0, denoted by $D(a_0, \delta)$, such that $\theta_i P_i(a) < \theta_j P_j(a)$ for $a \in D(a_0, \delta)$. By the definition of $\underline{\mathbf{P}}_{\{\theta_0,\theta_1,\dots,\theta_{k-1}\}}(a)$, one gets $\underline{\mathbf{P}}_{\{\theta_0,\theta_1,\dots,\theta_{k-1}\}}(a) = \theta_i P_i(a)$ for all $a \in D(a_0, \delta)$. Therefore the point a is a continuous point of $\underline{\mathbf{P}}_{\{\theta_0,\theta_1,\dots,\theta_{k-1}\}}(a)$.

If there exist l (>1) polynomials $\theta_{i_k} P_{i_k}(a)$ such that $\underline{\mathbf{P}}_{\{\theta_0,\theta_1,\dots,\theta_{k-1}\}}(a_0) = \theta_{i_k} P_{i_k}(a_0)$, $k = 1, 2, \dots, l$, then one can prove, by basic properties of polynomials, that there must exist a neighborhood of the point a_0, denoted by $(a_0 - \delta_0, a_0 + \delta_0)$, such that $\theta_i P_i(a) \neq \theta_j P_j(a)$ for all $a \in (a_0 - \delta_0, a_0 + \delta_0)$ and all $i \neq j$, $i, j \in \{i_k | k = 1, 2, \dots, l\}$. This implies that there are two polynomials $\theta_i P_i(a)$ and $\theta_j P_j(a)$ which satisfy the inequalities $\theta_i P_i(a) < \theta_s P_s(a)$ when $a \in (a_0 - \delta_0, a_0)$ and $\theta_j P_j(a) < \theta_s P_s(a)$ when $a \in (a_0, a_0 + \delta_0)$ for all $s \in \{i_k | k = 1, 2, \dots, l\}$, $s \neq i, j$. Hence, we get

$$\underline{\mathbf{P}}_{\{\theta_0,\theta_1,\dots,\theta_{k-1}\}}(a) = \begin{cases} \theta_i P_i(a), & a \in (a_0 - \delta_0, a_0], \\ \theta_j P_j(a), & a \in [a_0, a_0 + \delta_0). \end{cases}$$

We see that a_0 is also a continuous point. $\square$

The above proof shows that $\underline{\mathbf{P}}_{\{\theta_0,\theta_1,\dots,\theta_{k-1}\}}(a)$ is a polynomial or a piecewise polynomial, since the number of intersection points of k polynomials is finite.

We will also need another concept in what follows. An ordered set of k "sign" parameters $\{\theta_0, \theta_1, \dots, \theta_{k-1}\}$ is said to be in a primary ordering at a if

$$\underline{\mathbf{P}}_{\{\theta_0,\theta_1,\dots,\theta_{k-1}\}}(a) = \theta_0 P_0(a).$$

We prove a useful lemma before we introduce the three groups of polynomials.

Lemma 3. *Suppose for all* $a \in (1,\infty)$, *for an ordered set of k "sign" parameters* $\{\theta_0, \theta_1, \dots, \theta_{k-1}\}$, *its* (*strictly*) *minimal value function is given by the polynomial*

$$P(a) = \theta_0 a^{k-1} + \theta_1 a^{k-2} + \cdots + \theta_{k-2} a + \theta_{k-1}.$$

Then, for all $a \in (1,\infty)$, *the polynomial defined by*

$$P^*(a) = (a-1)P(a^2)$$

is the (*strictly*) *minimal value function w.r.t. the ordered set of* $2k$ *"sign" parameters given by the coefficients of* $P^*(a)$, *with decreasing order of powers.*

Proof. Denote the set of coefficients of $P^*(a)$, with decreasing order of powers, as $\{\theta_0^*, \theta_1^*, \ldots, \theta_{2k-1}^*\}$. Then by the definition of $P^*(a)$, we have, for $j = 0, 1, \ldots, k-1$, $\theta_{2j}^* = \theta_j$, $\theta_{2j+1}^* = -\theta_j$. Denote

$$\mathscr{P}_{\{\theta_0,\theta_1,\ldots,\theta_{k-1}\}} = \{P_0(a), P_1(a), \ldots, P_{k-1}(a)\},$$
$$\mathscr{P}^*_{\{\theta_0^*,\theta_1^*,\ldots,\theta_{2k-1}^*\}} = \{P_0^*(a), P_1^*(a), \ldots, P_{2k-1}^*(a)\}.$$

Then it is easily verified that, for $j = 0, 1, \ldots, k-1$,

$$P_{2j}^*(a) = (a-1)P_j(a^2),$$
$$P_{2j+1}^*(a) = -aP_j(a^2) + P_{j+1}(a^2).$$

To see that $P^*(a) = \theta_0 P_0^*(a)$ is (strictly) minimal for $a > 1$, firstly we have

$$\theta_{2j}^* P_{2j}^*(a) - \theta_0^* P_0^*(a) = \theta_j(a-1)P_j(a^2) - \theta_0(a-1)P_0(a^2) = (a-1)(\theta_j P_j(a^2) - \theta_0 P_0(a^2)) \geqslant 0 \text{ (or } > 0).$$

Similarly,

$$\begin{aligned}\theta_{2j+1}^* P_{2j+1}^*(a) - \theta_0^* P_0^*(a) &= -\theta_j(-aP_j(a^2) + P_{j+1}(a^2)) - \theta_0(a-1)P_0(a^2)\\ &= (a-1)(\theta_j P_j(a^2) - \theta_0 P_0(a^2)) + \theta_j(P_j(a^2) - P_{j+1}(a^2)) \geqslant 0 \text{ (or } > 0).\end{aligned}$$

In the last step, we have used the fact that $\{\theta_0, \theta_1, \ldots, \theta_{k-1}\}$ is shift-languished. Hence, we have proved that $P^*(a)$ is (strictly) minimal. □

2.2. Polynomials *P*

Lemma 4. *The system of polynomials defined as* $\mathbf{P}_2(a) = a - 1$, *and for positive integers* $m \geqslant 1$,

$$\mathbf{P}_{2^{m+1}}(a) = (a^{2^m} - 1)\mathbf{P}_{2^m}(a) \tag{6}$$

have the following properties:

(i) *For* $m \geqslant 1$,

$$\mathbf{P}_{2^{m+1}}(a) = (a-1)\mathbf{P}_{2^m}(a^2). \tag{7}$$

(ii) *For every* $m \geqslant 1$, *denote the ordered set of parameters corresponding to the coefficients*, *with decreasing order of powers*, *of the polynomials* $\mathbf{P}_{2^m}(a)$ *as* $\{\theta_0', \theta_1', \ldots, \theta_{2^m-1}'\}$. *Then this ordered set of parameters is in a primary ordering and shift-definite at any* $a \in [1, \infty)$. *The polynomial* $\mathbf{P}_{2^m}(a)$ *itself is the minimal value function on the interval* $(1, \infty)$, *and it is strictly minimal at all* $a \in (1, \infty)$.

Proof

(i) By the definition (6), denoting $\bar{a} = a^2$, then we have

$$\mathbf{P}_{2^{m+1}}(a) = (a^{2^m} - 1)\mathbf{P}_{2^m}(a) = \prod_{i=0}^{m}(a^{2^i} - 1) = (a-1)\prod_{i=0}^{m-1}(\bar{a}^{2^i} - 1) = (a-1)\mathbf{P}_{2^m}(a^2).$$

(ii) We prove by mathematical induction that, at $a \in [1, \infty)$, the ordered set of parameters corresponding to the coefficients of $\mathbf{P}_{2^m}(a)$ with decreasing order of powers, is shift-definite (strictly at $a \in (1, \infty)$), and $\mathbf{P}_{2^m}(a)$ is the (strictly at $a \in (1, \infty)$) minimal value function.

This is easily verified for $m = 1$. Assume this is true for some $m > 1$. Then from (7) and Lemma 3, we can verify all the assertions for $\mathbf{P}_{2^{m+1}}(a)$. □

2.3. Polynomials *Q*

This group of polynomials were studied in [7].

Lemma 5. *The system of polynomials defined as* $\mathbf{Q}_3(a) = a^2 - a - 1$, *and for* $k > 1$,

$$\mathbf{Q}_{2k+3}(a) = a^2\mathbf{Q}_{2k+1}(a) + a - 1 \tag{8}$$

have the following properties: for each $k \geqslant 1$, the coefficients of the polynomial $\mathbf{Q}_{2k+1}(a)$ with decreasing order of powers are $\{1, -1, -1, 1, -1, \dots, 1, -1\}$. This ordered set of $2k+1$ parameters is in a primary ordering, and

$$\underline{\mathbf{P}}_{\{1,-1,-1,1,-1,\dots,1,-1\}}(a) = \mathbf{Q}_{2k+1}(a), \tag{9}$$

and it is also strictly minimal at all $a \in (1, \infty)$.

Proof. We need only to show (9).

Let $\{\theta_0, \theta_1, \dots, \theta_{2k}\} = \{1, -1, -1, 1, -1, \dots, 1, -1\}$ be the coefficients of $\mathbf{Q}_{2k+1}(a)$, with decreasing order of powers, and denote

$$\mathscr{P}_{\{\theta_0,\theta_1,\dots,\theta_{2k}\}} = \{P_0(a), P_1(a), \dots, P_{2k}(a)\}.$$

According to (2), we have

$$P_0(a) = \mathbf{Q}_{2k+1}(a) = a^{2k} - a^{2k-1} + \sum_{i=0}^{2k-2}(-1)^{i+1}a^i,$$

$$P_1(a) = -a^{2k} - a^{2k-1} + \sum_{i=1}^{2k-2}(-1)^{i+1}a^i + 1.$$

We can easily verify that

$$\theta_1 P_1(a) - P_0(a) = 2a^{2k-1} > 0.$$

Generally, we have, for $l \geqslant 1$, that

$$P_{2l}(a) = \sum_{i=2l}^{2k}(-1)^{i+1}a^i + a^{2l-1} - a^{2l-2} - a^{2l-3} + \sum_{i=0}^{2l-4}(-1)^i a^i,$$

$$P_{2l+1}(a) = \sum_{i=2l+1}^{2k}(-1)^i a^i + a^{2l} - a^{2l-1} - a^{2l-2} + \sum_{i=0}^{2l-3}(-1)^{i+1}a^i,$$

and

$$\theta_{2l}P_{2l}(a) - P_0(a) = \sum_{i=2l-2}^{2k-2}(-1)^i a^i > 0,$$

$$\theta_{2l+1}P_{2l+1}(a) - P_0(a) = \sum_{i=2l-1}^{2k-2}(-1)^i a^i > 0.$$

Therefore, (9) holds true at all a in the interval $(1, \infty)$. □

Proposition 2

(i) *For every $k \geqslant 1$, the polynomial $\mathbf{Q}_{2k+1}(a)$ has a unique positive real root in $(1, \infty)$.*
(ii) *Denote the root of $\mathbf{Q}_{2k+1}(a)$ as a_{2k+1} for $k = 1, 2, \dots$ Then $\sqrt{2} < a_{2k+3} < a_{2k+1}$.*
(iii) $\lim_{k\to+\infty} a_{2k+1} = \sqrt{2}$.

Proof

(i) For the first polynomial $\mathbf{Q}_3 = a^2 - a - 1$, one can directly verify that the only root in $(1, \infty)$ is $a_3 = (1 + \sqrt{5})/2$. From the definition of $\mathbf{Q}_{2k+3}(a)$, we calculate that

$$\mathbf{Q}_{2k+3}(a) = \frac{a^{2k+3} - 2a^{2k-1} - 1}{a + 1}.$$

We only need to prove that the polynomial defined by $\overline{\mathbf{Q}}_{2k+3}(a) = a^{2k+3} - 2a^{2k+1} - 1$ has a unique root in $(1, \infty)$. Since

$$\frac{\mathrm{d}\overline{\mathbf{Q}}_{2k+3}(a)}{\mathrm{d}a} = a^{2k}((2k + 3)a^2 - 2(2k + 1)),$$

we see that $\mathrm{d}\overline{\mathbf{Q}}_{2k+3}(a)/\mathrm{d}a$ is zero in $(1, \infty)$ only when $a = a^* = \sqrt{2(2k + 1)/(2k + 3)}$, and

$$\begin{cases} \dfrac{\mathrm{d}\overline{\mathbf{Q}}_{2k+3}(a)}{\mathrm{d}a} < 0, & \text{when } 1 \leqslant a < a^*, \\ \dfrac{\mathrm{d}\overline{\mathbf{Q}}_{2k+3}(a)}{\mathrm{d}a} > 0, & \text{when } a > a^*. \end{cases}$$

Therefore, we have, for $a \in (1, a^*]$,

$$\overline{\mathbf{Q}}_{2k+3}(a) < \overline{\mathbf{Q}}_{2k+3}(1) = -2.$$

$\overline{\mathbf{Q}}_{2k+3}(a)$ is strictly monotonically increasing in the interval $[a^*, \infty)$. Since $\overline{\mathbf{Q}}_{2k+3}(2) = 3 \times 2^{2k+1} - 1 > 0$, we know that $\overline{\mathbf{Q}}_{2k+3}(a)$ has a unique root in (a^*, ∞).

(ii) From the above proof, we see that $\overline{\mathbf{Q}}_{2k+3}(\sqrt{2}) = -1$, and we can actually conclude that $a_{2k+3} > \sqrt{2}$. To prove $a_{2k+3} < a_{2k+1}$, we note that

$$\mathbf{Q}_{2k+3}(a) = a^2\mathbf{Q}_{2k+1}(a) + a - 1,$$

so that

$$\mathbf{Q}_{2k+3}(a_{2k+1}) = a_{2k+1} - 1 > 0,$$

therefore

$$a_{2k+3} < a_{2k+1}.$$

(iii) The conclusion in (ii) guarantees the existence of a limit, denoted by a_∞, when k tends to infinity, of the sequence $\{a_{2k+1}\}$, and $a_\infty \geqslant \sqrt{2}$. Note that a_{2k+1} is also the unique root of $\overline{\mathbf{Q}}_{2k+1}(a)$, therefore we have

$$a_{2k+1}^{2k+1} - 2a_{2k+1}^{2k-1} - 1 = 0,$$

$$a_{2k+1}^2 - 2 - \frac{1}{a_{2k+1}^{2k-1}} = 0,$$

$$a_\infty^2 - 2 - \lim_{k\to\infty} \frac{1}{a_{2k+1}^{2k-1}} = 0.$$

Since $a_\infty \geqslant \sqrt{2}$, the third term of the left hand side of the last equation is zero, thus we have $a_\infty^2 - 2 = 0$. That is, $a_\infty = \sqrt{2}$. □

Lemma 6. *When $a \geqslant a_{2k+1}$, the ordered set of parameters consisting of coefficients of $\mathbf{Q}_{2k+1}(a)$, with decreasing power order, is shift-definite.*

Lemma 7. *Define, for $k \geqslant 1$, $m \geqslant 1$,*

$$\mathbf{Q}_{(2k+1)2^m}(a) = \mathbf{Q}_{2k+1}(a^{2^m})\mathbf{P}_{2^m}(a).$$

Then

(i) $\mathbf{Q}_{(2k+1)2^{m+1}}(a) = (a-1)\mathbf{Q}_{(2k+1)2^m}(a^2)$.
(ii) $\mathbf{Q}_{(2k+1)2^m}(a)$ *has a unique root* $a_{(2k+1)2^m}$ *in the interval* $(1,\infty)$, *and* $a_{(2k+1)2^m} = (a_{2k+1})^{1/2^m}$.
(iii) $\mathbf{Q}_{(2k+1)2^n}(a)$ *is the strictly minimal value function, w.r.t. the ordered set of parameters consisting of coefficients of* $\mathbf{Q}_{(2k+1)2^n}(a)$, *with decreasing order powers. When* $a \geqslant a_{(2k+1)2^n}$, *this ordered set of parameters is shift-definite.*

Proof. (i) Can be directly verified. (ii) Is implied by Proposition 2(i). (iii) Is proved by using mathematical induction on n, with the help of Proposition 2(i), Lemma 3 and Lemma 6. □

*2.4. Polynomials **H***

Lemma 8. *The system of polynomials defined as $\mathbf{H}_2(a) = \mathbf{P}_2(a)$, and for $k \geqslant 1$, as*

$$\mathbf{H}_{2k+2}(a) = a^2\mathbf{H}_{2k}(a) - \mathbf{H}_2(a), \tag{10}$$

have the following properties:

(i) *For all $k > 1$,*

$$\mathbf{H}_{2k}(a) = a\mathbf{Q}_{2k-1}(a) + 1. \tag{11}$$

(ii) *For every $k > 2$, the polynomial $\mathbf{H}_{2k}(a)$ has a unique real root in the interval $(1,\infty)$.*

(iii) *The sequence of real roots of polynomials $\mathbf{H}_{2k}(a)$ in the interval $(1,\infty)$, denoted by $\bar{a}_{2k}$, is strictly monotonically increasing when $k \geqslant 3$. In particular, $\bar{a}_6 = a_6$, and $\lim_{k\to+\infty} \bar{a}_{2k} = \sqrt{2}$.*

Proof

(i) For $k = 2$, by definition (10) and with simple calculation, it is easily verified that

$$\mathbf{H}_4(a) = a^2\mathbf{H}_2(a) - \mathbf{H}_2(a) = a\mathbf{Q}_3(a) + 1.$$

Assume (11) holds true for some $k > 2$. For $k + 1$, we have

$$\mathbf{H}_{2k+2}(a) = a^2\mathbf{H}_{2k}(a) - \mathbf{H}_2(a) = a^2(a\mathbf{Q}_{2k-1}(a) + 1) - \mathbf{H}_2(a) = a(a^2\mathbf{Q}_{2k-1}(a) + a - 1) + 1 = a\mathbf{Q}_{2k+1}(a) + 1.$$

(ii) By definition, we can obtain that

$$\mathbf{H}_{2k}(a) = (a-1)\left(a^{2(k-1)} - \sum_{i=0}^{k-2} a^{2i}\right) = \frac{a^{2k} - 2a^{2(k-1)} + 1}{a+1}.$$

From this, we see that $\mathbf{H}_{2k}(a)$ has the same root in $(1,\infty)$ as the polynomial defined by $\overline{\mathbf{H}}_{2k}(a) = a^{2k} - 2a^{2(k-1)} + 1$. Note that

$$\frac{\mathrm{d}\overline{\mathbf{H}}_{2k}(a)}{\mathrm{d}a} = a^{2k-3}(2ka^2 - 4(k-1)),$$

which is zero only when $a = a^{**} = \sqrt{2(1-1/k)}$, and

$$\begin{cases} \dfrac{\mathrm{d}\overline{\mathbf{H}}_{2k}(a)}{\mathrm{d}a} < 0, & \text{when } 1 < a < a^{**}, \\ \dfrac{\mathrm{d}\overline{\mathbf{H}}_{2k}(a)}{\mathrm{d}a} > 0, & \text{when } a > a^{**}. \end{cases}$$

Therefore, we have, for $a \in (1, a^{**}]$,

$$\overline{\mathbf{H}}_{2k}(a) < \overline{\mathbf{H}}_{2k}(1) = 0.$$

$\overline{\mathbf{H}}_{2k}(a)$ is monotonically increasing in the interval $[a^{**},\infty)$. Since $\overline{\mathbf{H}}_{2k}(\sqrt{2}) = 1$, we know that $\overline{\mathbf{H}}_{2k}(a)$ has a unique root in (a^{**},∞). Note that when $k \geqslant 3$, $a^{**} \geqslant \sqrt{4/3}$. So, from the above from the above proof, we have

$$\sqrt{4/3} < \bar{a}_{2k} < \sqrt{2}. \tag{12}$$

(iii) Note that

$$\mathbf{H}_{2k+2}(\bar{a}_{2k}) = \bar{a}_{2k}^2\mathbf{H}_{2k}(\bar{a}_{2k}) - \mathbf{H}_2(\bar{a}_{2k}) < 0.$$

This shows that the $\bar{a}_{2k} < \bar{a}_{2k+2}$ when $k \geqslant 3$. From (12), the limit, denoted as $\bar{a}_\infty$, of the sequence $\{\bar{a}_{2k}\}$ exists and satisfies $\sqrt{4/3} \geqslant \bar{a}_\infty \leqslant \sqrt{2}$.
Therefore, we have

$$\bar{a}_{2k}^{2k} - 2\bar{a}_{2k-2}^{2k} + 1 = 0,$$

$$\bar{a}_{2k}^2 - 2 - \frac{1}{\bar{a}_{2k}^{2k-2}} = 0,$$

$$\bar{a}_\infty^2 - 2 - \lim_{k\to\infty} \frac{1}{\bar{a}_{2k}^{2k-2}} = 0.$$

Since $\bar{a}_\infty \geqslant \sqrt{4/3} > 1$, the third term of the left hand side of the last equation is zero, thus we have $\bar{a}_\infty^2 - 2 = 0$. That is, $\bar{a}_\infty = \sqrt{2}$. □

Note that

$$\frac{1}{\sqrt{a}-1}\mathbf{H}_{2k+2}(\sqrt{a}) = a^k - a^{k-1} - \cdots - a^2 - a - 1.$$

The group of polynomials at the right-side of the above equation were studied in [9].

3. Periodic orbits of type-I systems

In this section, we study periodic points of systems of type-I.

3.1. Relationship of periodic points and ‘‘sign’’ polynomials

Theorem 1

(i) *A point $x_0 \in R$ is a periodic point with period n if and only if there is a set of n* ‘‘*sign*’’ *parameters* $\{\theta_0, \theta_1, \ldots, \theta_{n-1}\}$, $\theta_i \in \{-1, 1\}$, $i = 0, 1, \ldots, n-1$, *which is shift-definite, such that*

$$x_0 = \frac{\Delta}{a^n - 1}\sum_{i=0}^{n-1} a^{n-i-1}\theta_i. \tag{13}$$

(ii) *A point $x_0 \in R$ is an n-periodic point* (*that is, n is the prime period*) *if and only if n is the smallest positive integer such that* (*i*) *holds.*

Proof (*Necessity*). (i) For any point $x_0 \in R$, if it is a periodic point with period n, then, denoting $x_i = f^i(x_0)$, we have

$$x_0 = x_n = a^n x_0 - \Delta\sum_{i=0}^{n-1} a^{n-i-1}\operatorname{sgn}(x_i),$$

and hence

$$x_0 = \frac{\Delta}{a^n - 1}\sum_{i=0}^{n-1} a^{n-i-1}\operatorname{sgn}(x_i).$$

Clearly, x_0 is in the form of (13) for $\theta_i = \operatorname{sgn}(x_i)$, $i = 0, 1, \ldots, n-1$. Furthermore, if x_0 is a periodic point with period n, then it is easily verified that x_k, $k = 0, 1, \ldots, n-1$, also satisfies the following equalities:

$$x_k = f^k(x_0) = \frac{\Delta}{a^n - 1}\sum_{i=0}^{n-1} a^{n-i-1}\operatorname{sgn}(x_{k+i}).$$

From the above n equalities and the fact that $\Delta/(a^n - 1) > 0$, we have

$$\operatorname{sgn}(x_k) = \operatorname{sgn}\left(\sum_{i=0}^{n-1} a^{n-i-1}\operatorname{sgn}(x_{k+i})\right)$$

Note that $x_{k+i} = x_{k+i \bmod (n)}$, therefore this set of parameters is shift-definite at a.

(Sufficiency) When the conditions in the theorem are satisfied by some point x_0, we have, first of all, from the definition of shift-definiteness,

$$\operatorname{sgn}(x_0) = \operatorname{sgn}\left(\sum_{i=0}^{n-1} a^{n-i-1}\theta_i\right) = \theta_0.$$

Note also that

$$x_1 = f^1(x_0) = \frac{\Delta a}{a^n - 1}\sum_{i=0}^{n-1} a^{n-i-1}\theta_i - \Delta\operatorname{sgn}(x_0) = \frac{\Delta}{a^n - 1}\sum_{i=0}^{n-1} a^{n-i}\theta_i - \Delta\theta_0 = \frac{\Delta}{a^n - 1}\sum_{i=0}^{n-1} a^{n-i-1}\theta_{i+1}.$$

So, we have

$$\operatorname{sgn}(x_1) = \operatorname{sgn}\left(\sum_{i=0}^{n-1} a^{n-i-1}\theta_{i+1}\right) = \theta_1.$$

Similar to the above discussion, we can get

$$x_k = f^k(x_0) = \frac{\Delta}{a^n - 1}\sum_{i=0}^{n-1} a^{n-i-1}\theta_{i+k},$$

and

$$\operatorname{sgn}(x_k) = \operatorname{sgn}\left(\sum_{i=0}^{n-1} a^{n-i-1}\theta_{i+k}\right) = \theta_k.$$

In particular,

$$x_n = f^n(x_0) = \frac{\Delta}{a^n - 1}\sum_{i=0}^{n-1} a^{n-i-1}\theta_{i+n} = x_0.$$

(ii) The proof is direct. □

Remark 1. The two sets of parameters, $(1, 1, \dots, 1)$ and $(-1, -1, \dots, -1)$, are shift-definite at any $a > 1$, therefore, according to Theorem 1, they give rise to periodic points. By invoking (13), we see that they give rise to two fixed points, $\Delta/(a-1)$ and $-\Delta/(a-1)$, respectively. From [16], we know that these two fixed points are the only periodic points outside $[-\Delta, \Delta]$. Therefore, any shift-definite set of parameters with at least two different signs is associated with a periodic point in $[-\Delta, \Delta]$. Conversely, any periodic point in $[-\Delta, \Delta]$ is associated with a shift-definite set of parameters with at least two different signs.

Theorem 2. *If the minimal value function w.r.t. a shift-definite set of "sign" parameters* $\{\theta_0, \theta_1, \dots, \theta_{n-1}\}$ *is strictly minimal at a, then the periodic point given by* (13) *has a prime period n.*

Proof. Without loss of generality, assume that $\{\theta_0, \theta_1, \dots, \theta_{n-1}\}$ is in primary ordering. Denote the ordered set of polynomials corresponding to $\{\theta_0, \theta_1, \dots, \theta_{n-1}\}$ as

$$\mathscr{P}_{\{\theta_0,\theta_1,\dots,\theta_{k-1}\}} = \{P_0(a), P_1(a), \dots, P_{n-1}(a)\}.$$

Then, for $i = 1, 2, \dots, n-1$,

$$\underline{\mathbf{P}}_{\{\theta_0,\theta_1,\dots,\theta_{n-1}\}}(a) = \theta_0 P_0(a) < \theta_i P_i(a). \tag{14}$$

The point x_0 as defined in (13) is periodic with a period n, according to Theorem 1(i).

Denote the periodic orbit starting from x_0 as $\{x_0, x_1, \dots, x_{n-1}\}$. The following the proof of Theorem 1, we have

$$x_i = \frac{\Delta}{a^n - 1} P_i(a).$$

Obviously, x_0 cannot be a fixed point. We also see that x_0 do not have a period k, $1 < k < n$. Otherwise, we have $x_k = x_0$, that is,

$$\frac{\Delta}{a^n - 1} P_k(a) = \frac{\Delta}{a^n - 1} P_0(a),$$

which implies

$$P_k(a) = P_0(a).$$

Since $\{\theta_0, \theta_1, \dots, \theta_{n-1}\}$ is shift-definite, $P_k(a)$ and $P_0(a)$ have the same sign as their leading coefficients. That is, $\theta_k = \theta_0$. We then have

$$\theta_k P_k(a) = \theta_0 P_0(a),$$

contradicting (14). □

3.2. 2^m-periodic points

We are ready to present our results for three distinct cases: 2^m-periodic points, odd order periodic points and other even order periodic points.

Theorem 3. *If $a > 1$, then there exists a 2^m-periodic point in $[-\Delta, \Delta]$ for all $m > 0$.*

Proof. From Lemma 4(ii) and Theorem 1(i), the point defined by

$$x_{2^m} = \frac{\Delta}{a^{2^m} - 1}\mathbf{P}_{2^m}(a)$$

with $m \geqslant 1$, is a periodic point in $[-\Delta, \Delta]$ with period 2^m. Lemma 4(ii) and Theorem 2 together imply that x_{2^m} is a 2^m-periodic point. □

3.3. Odd-order periodic points

Now, we turn to odd periods. We need three propositions.

Proposition 3. *If there is a periodic point with an odd period $2k + 1 \geqslant 3$ for system (1) with some $a^* > 1$, then there is an a^{**}, $1 < a^{**} \leqslant a^*$, such that 0 is a periodic point with period $2k + 1$ of system (1) with a^{**}.*

Proof. From Theorem 1 and Lemma 1, there is an ordered set of $2k + 1$ "sign" parameters $\{\theta_0, \theta_1, \ldots, \theta_{2k}\}$ such that its minimal value function is non-negative at a^*, i.e.,

$$\underline{\mathbf{P}}_{\{\theta_0,\theta_1,\ldots,\theta_{2k}\}}(a^*) \geqslant 0.$$

Since $2k + 1 \geqslant 3$, there are at least two elements with different signs in $\{\theta_0, \theta_1, \ldots, \theta_{2k}\}$, as explained in Remark 1. Therefore, from the definition of a minimal value function,

$$\underline{\mathbf{P}}_{\{\theta_0,\theta_1,\ldots,\theta_{2k}\}}(1) < 0.$$

From Lemma 2, there is an a^{**}, $1 < a^{**} \leqslant a^*$, such that

$$\underline{\mathbf{P}}_{\{\theta_0,\theta_1,\ldots,\theta_{2k}\}}(a^{**}) = 0.$$

This means, from Lemma 1, that $\{\theta_0, \theta_1, \ldots, \theta_{2k}\}$ is shift-definite at a^{**}. From Theorem 1, it gives rise to a periodic orbit with period $2k + 1$ for system (1) with a^{**}. Clearly, 0 is in this orbit. □

Proposition 4. *For all $a \in [\sqrt{2}, a_{2k+1})$, suppose $\{x_i, i = 0, 1, \ldots\}$ is the orbit of system (1) starting from the initial state $x_0 = 0$. Then, $x_1 = -\Delta$, and for $i = 1, 2, \ldots, k + 1$,*

$$x_{2i} = -\mathbf{H}_{2i}(a)\Delta, \tag{15}$$

$$x_{2i+1} = -\mathbf{Q}_{2i+1}(a)\Delta. \quad □ \tag{16}$$

Proof. We can directly verify that

$$x_2 = -\mathbf{H}_2(a)\Delta = -(a - 1)\Delta,$$
$$x_3 = -\mathbf{Q}_3(a)\Delta = -(a^2 - a - 1)\Delta.$$

Suppose the equations in (15) and (16) hold for $i \leqslant k$. We will prove that they also hold for $i + 1$.
Since $a < a_{2k+1} \leqslant a_{2i+1}$, we have $\mathbf{Q}_{2i+1}(a) < 0$, and hence $x_{2i+1} > 0$. So,

$$x_{2i+2} = ax_{2i+1} - \Delta = -(a\mathbf{Q}_{2i+1}(a) + 1)\Delta \overset{(11)}{=} -\mathbf{H}_{2i+2}(a)\Delta.$$

Since $a > \sqrt{2} > \bar{a}_{2i+2}$, we have $\mathbf{H}_{2i+2}(a) > 0$, and hence $x_{2i+2} < 0$. So,

$$x_{2i+3} = ax_{2i+2} + \Delta = -(a^2\mathbf{Q}_{2i+1}(a) + a - 1)\Delta \overset{(8)}{=} -\mathbf{Q}_{2i+3}(a)\Delta. \quad □$$

Proposition 5. *If $a < \sqrt{2}$, then*

(i) $0 \notin f([-\mathbf{P}_2(a)\Delta, \mathbf{P}_2(a)\Delta))$;
(ii) $f^2([-\mathbf{P}_2(a)\Delta, \mathbf{P}_2(a)\Delta)) = [-\mathbf{P}_2(a)\Delta, \mathbf{P}_2(a)\Delta)$.

Proof. The conclusion follows from the following calculations:

$$\begin{aligned} f([-\mathbf{P}_2(a)\Delta, \mathbf{P}_2(a)\Delta)) &= [-\Delta, \mathbf{Q}_3(a)\Delta \cup [-\mathbf{Q}_3(a)\Delta, \Delta)) \\ f^2([-\mathbf{P}_2(a)\Delta, \mathbf{P}_2(a)\Delta)) &= f([-\Delta, \mathbf{Q}_3(a)\Delta) \cup [-\mathbf{Q}_3(a)\Delta, \Delta)) \\ &= [-\mathbf{P}_2(a)\Delta, 0) \cup [0, \mathbf{P}_4(a)\Delta) \cup [-\mathbf{P}_4(a)\Delta, 0) \cup [0, \mathbf{P}_2(a)\Delta) \\ &\overset{a<\sqrt{2}}{=} [-\mathbf{P}_2(a)\Delta, \mathbf{P}_2(a)\Delta). \qquad \square \end{aligned}$$

Now we are ready to prove the following theorem.

Theorem 4. *For every positive integer $k \geqslant 1$, system* (1) *has a $(2k+1)$-periodic point if and only if $a \geqslant a_{2k+1}$.*

Proof. The sufficiency is given by combining Lemma 6, Theorem 1(i), Theorem 2 and Lemma 5.

To prove the necessity, we show that when $a < a_{2k+1}$, there is no periodic points of order $2k+1$. By Lemma 2, we only need to show that 0 is not a periodic point of period $2k+1$.

Firstly, when $\sqrt{2} \leqslant a < a_{2k+1}$, Proposition 4 says that 0 is not of periodic $2k+1$. Secondly, when $1 < a < \sqrt{2}$, Proposition 5(ii) says $f^{2k}(0) \in [-(a-1)\Delta, (a-1)\Delta]$. Proposition 5(i) then says that $f^{2k+1}(0) \neq 0$. $\square$

3.4. Other even-order periodic points

Lastly, we turn to all remaining even-order periodic orbits (other than 2^n-periodic orbits discussed earlier).

We need two propositions.

Proposition 6. *If $a < a_3$, then $f_a^2(x)$ restricted to $[-(a-1)\Delta, (a-1)\Delta]$ and $f_{a^2}(x)$ restricted to $[-\Delta, \Delta]$ are topologically conjugate.*

Proof. Let $h(x) = x/(a-1)$. We can verify that

$$h(f_a^2(h^{-1}(x))) \equiv f_{a^2}(x) \tag{17}$$

for $x \in [-\Delta, \Delta]$ when $a < a_3$.

Firstly, we have

$$f_a^2(x) = \begin{cases} a^2x + \Delta(a-1), & x \in [-\Delta/a, 0) \\ a^2x - \Delta(a-1), & x \in [0, \Delta/a) \end{cases} \tag{18}$$

and, when $a < a_3$, for each $x \in [-\Delta, \Delta]$,

$$h^{-1}(x) = (a-1)x \in [-(a-1)\Delta, (a-1)\Delta] \subset [-\Delta/a, \Delta/a).$$

Hence,

$$\begin{aligned} h(f_a^2(h^{-1}(x))) = \frac{1}{a-1} f_a^2((a-1)x) &= \begin{cases} \dfrac{1}{a-1}(a^2(a-1)x + \Delta(a-1)), & x \in [-\Delta, 0) \\ \dfrac{1}{a-1}(a^2(a-1)x - \Delta(a-1)), & x \in [0, \Delta] \end{cases} = \begin{cases} a^2x + \Delta, & x \in [-\Delta, 0) \\ a^2x - \Delta, & x \in [0, \Delta] \end{cases} \\ &= f_{a^2}(x). \qquad \square \end{aligned}$$

Proposition 7. *Any periodic orbit of a prime even period $2k > 2$ contains at least one point in the interval $(-(a-1)\Delta, (a-1)\Delta) = (-\mathbf{P}_2(a)\Delta, \mathbf{P}_2(a)\Delta)$.*

Proof. If this is not the case, then there is at least one point in the periodic orbit, say $z_0 \in (-\Delta, -(a-1)\Delta]$, satisfying $f(z_0) = az_0 + \Delta > -a\Delta + \Delta = -(a-1)\Delta$, so it must belong to $[(a-1)\Delta, \Delta)$. We thus have

$$f^2(z_0) = a^2 z_0 + a\Delta - \Delta$$

which, by similar reasoning, is in $(-\Delta, -(a-1)\Delta]$. In general, we have

$$f^{2i}(z_0) = a^{2i} z_0 + (a^{2i-1} - a^{2i} + \cdots + a - 1)\Delta.$$

Since z_0 is a $2k$ periodic point, we should have $f^{2k}(z_0) = z_0$. We can solve z_0 from this and obtain

$$z_0 = -\frac{a^{2k-1} - a^{2k} + \cdots + a - 1}{a^{2k} - 1}\Delta = -\frac{1}{a+1}\Delta.$$

Since $-1/(a+1)\Delta$ is 2-periodic, this is a contradiction. □

Theorem 5. *System* (1) *has a periodic point of a prime period* $2^n k$, *k odd*, $k \geqslant 3$, *if and only if* $a \geqslant a_{2^n k}$.

Proof. Sufficiency is given by Lemma 7(iii), Theorem 1(i) and Theorem 2. The necessity is given by combining the two proceeding propositions. □

Some more conclusions can be drawn.

Corollary 1

(i) *System* (1) *has a periodic point for all natural numbers* $n = 1, 2, \ldots,$ *if and only if* $a \geqslant a_3 = (1+\sqrt{5})/2$.

(ii) *For a given positive integer n, every point of the following* 2^n *points*

$$x_0 = \frac{\Delta}{a^n - 1}\sum_{i=0}^{n-1}\theta_i a^{n-i-1} \quad \forall \theta_i \in \{1, -1\}, \tag{19}$$

is a periodic point if and only if $a \geqslant \bar{a}^2_{2n}$.

(iii) *When* $a \geqslant 2$, *for every point in the form* (19) *is a periodic point with period n.*

Proof. (i) Is clear from Theorems 3–5.

To prove (ii) note that

$$\theta_0 \sum_{i=0}^{n-1}\theta_i a^{n-i-1} \geqslant a^{n-1} - \sum_{i=0}^{n-2} a^i = \frac{(a^n - 2a^{n-1} + 1)}{a-1}.$$

So, for all $\theta_i \in \{-1, 1\}$, $i = 0, 1, \ldots, n-1$, the corresponding point $\frac{\Delta}{a^n-1}\sum_{i=0}^{n-1}\theta_i a^{n-i-1}$ is a periodic point of period n if and only if the point $\frac{\Delta}{(a^n-1)(a-1)}(a^n - 2a^{n-1} + 1)$ is so. This is the case if and only if $\overline{\overline{H}}(a) \stackrel{\text{def}}{=} a^n - 2a^{n-1} + 1 \geqslant 0$. Note that

$$\mathbf{H}_{2n}(a) = (a-1)\overline{\overline{H}}(a^2).$$

The conclusion follows.

(iii) Is a special case of (ii). □

4. Periodic solutions of type-II systems

In this section, we consider systems of type-II:

$$x^+ = f_{-a}(x) \stackrel{\text{def}}{=} -ax + \Delta\,\mathrm{sgn}(x), \tag{20}$$

in which $1 < a < 2$. We call the type-I system with the same a value as the dual system of (20).

Proposition 8. *If* $\{x_i, i = 0, 1, \ldots\}$ *is an orbit of* (20), *then* $\{(-1)^i x_i, i = 0, 1, \ldots\}$ *is an orbit of the dual system of* (20)

Proof. Denote $y_i = (-1)^i x_i$. Then

$$y_{i+1} = (-1)^{i+1} x_{i+1} = (-1)^{i+1}(-ax_i + \Delta\,\mathrm{sgn}(x_i)) = a(-1)^i x_i - \Delta\,\mathrm{sgn}((-1)^i x_i) = ay_i - \Delta\,\mathrm{sgn}(y_i).$$

From here, we see that $\{y_i, i = 0, 1, \ldots\}$ is an orbit of the dual system. □

Theorem 6

(i) *For type-II systems, a point* $x_0 \in R$ *is a periodic point with period n if and only if* x_0 *has the form*

$$x_0 = \frac{\Delta}{a^n + (-1)^{n+1}} \sum_{i=0}^{n-1} (-1)^i a^{n-i-1} \theta_i, \tag{21}$$

where the parameters $\theta_i \in \{-1, 1\}$, $i = 0, 1, \ldots, n-1$, *and for* $k = 0, 1, \ldots, n-1$,

$$\theta_k = \mathrm{sgn}\left(\sum_{i=0}^{n-1} (-1)^i a^{n-i-1} \theta_{k+i} \right), \tag{22}$$

i.e., the sign of the right-side term in (22) *is the same as the sign of its leading term. Here,* $k + i = (k+i)\mathrm{mod}(n)$ *for the subscript indices of the parameters* θ_i.

(ii) *A point* $x_0 \in R$ *is an n-periodic point of type-II systems* (20) *if and only if n is the smallest positive integer such that* (21) *and* (22) *hold.*

Proof. The proof of the theorem can be carried out by procedures similar to the proof of Theorem 1. □

Proposition 9. $x_0 \in [\Delta, \Delta]$ *is a* $(2n+1)$*-periodic point of* (20) *if and only if*

$$y_0 = \frac{a^{2n+1} + 1}{a^{2n+1} - 1} x_0$$

is a $(2n+1)$*-periodic point of its dual system.*

Proof. By Theorem 6, there are $2n+1$ "sign" parameters, $\{\theta_0, \theta_1, \ldots, \theta_{2n}\}$, satisfying (22) for $k = 0, 1, \ldots, 2n$, such that

$$x_0 = \frac{\Delta}{a^{2n+1} + 1} \sum_{i=0}^{2n-1} (-1)^i a^{2n-i-1} \theta_i.$$

Note that

$$y_0 = \frac{\Delta}{a^{2n+1} - 1} \sum_{i=0}^{2n-1} (-1)^i a^{2n-i-1} \theta_i.$$

If we denote $\theta'_i = (-1)^i \theta_i$, then by multiplying both sides of (22) with $(-1)^k$, we obtain, for $k = 0, 1, \ldots, 2n$,

$$(-1)^k \theta_k = \mathrm{sgn}\left(\sum_{i=0}^{2n-1} a^{2n-i-1} (-1)^{k+i} \theta_{k+i} \right),$$

i.e.,

$$\theta'_k = \mathrm{sgn}\left(\sum_{i=0}^{2n-1} a^{2n-i-1} \theta'_{k+i} \right),$$

so, the parameters $\{\theta'_0, \theta'_1, \ldots, \theta'_{2n-1}\}$ is shift-definite at a.

By Theorem 1, y_0 is of $2n+1$ periodic for system (1). We use exactly the same argument to show that conversely if y_0 is of $2n+1$ periodic for (1) then

$$x_0 = \frac{a^{2n+1} - 1}{a^{2n+1} + 1} y_0$$

is of $2n+1$ periodic for (20).

The conclusion holds of course for prime odd periods. □

Finally, we have the following result.

Theorem 7

(i) *System* (20) *has a periodic point of a prime period* 2^n *for any* $a \in (1, \infty)$.

(ii) *System* (20) *has a periodic point of a prime period* $2^n k$, $n = 0, 1, 2, \ldots, k \geqslant 3$, k *odd, if and only if* $a \geqslant a_{2^n k}$.

Proof

(i) For any $a \in (1,\infty)$, system (20) has a fixed point at $\Delta/(a+1)$. From Theorem 3,

$$x_0 = \frac{\Delta}{a^{2^m}-1}\mathbf{P}_{2^m}(a)$$

is a 2^m-periodic point for (1). From Proposition 8, x_0 is a periodic point for (20) with period 2^m. 2^m is clearly the prime period according to Proposition 8 and Theorem 2.

(ii) For $n=0$, we know from Proposition 9 that system (20) has a k-periodic point ($k \geqslant 3$ odd) if and only if system (1) has one of such periodic point. By Theorem 4, this is the case if and only if $a \geqslant a_k$.

When $a \geqslant a_{2(2k+1)}$, denote the periodic orbits of systems (1) and (20), starting from

$$x_0 = \frac{\Delta}{a^{2(2k+1)}-1}\mathbf{Q}_{2(2k+1)}(a),$$

as $\{x_i, i=0,1,\ldots,2(2k+1)-1\}$ and $\{y_i,\ i=0,1,\ldots,2(2k+1)-1\}$, respectively. By Theorem 5, $\{x_i, i=0,1,\ldots,2(2k+1)-1\}$ is a $2(2k+1)$-periodic orbit for (1). From Proposition 8, $\{y_i,\ i=0,1,\ldots,2(2k+1)-1\}$ is a periodic orbit for (20) with period $2(2k+1)$. We will show that $2(2k+1)$ is the prime period for (20).

Assume that there is a positive integer $n' < 2(2k+1)$, such that $y_0 = y_{n'}$. Then n' must be a factor of $2(2k+1)$.

Firstly, n' could not be even. Since if it is even, then, the equalities $x_{n'} = (-1)^{n'} y_{n'} = y_0 = x_0$, contradicting the fact that $2(2k+1)$ is the prime period of x_0.

Secondly, if n' is a positive odd integer, then $x_{n'} = -y_{n'} = -x_0, x_0 \neq 0$, since otherwise it is again contracting the fact that $2(2k+1)$ is the prime period of x_0.

Denote the ordered set of polynomials, defined in (2) corresponding to the coefficients of $\mathbf{Q}_{2(2k+1)}(a)$, with decreasing order of powers, as

$$\mathscr{P}_{\{\theta_0,\theta_1,\ldots,\theta_{2(2k+1)-1}\}} = \{P_0(a), P_1(a), \ldots, P_{2(2k+1)-1}(a)\}.$$

Then, from the previous development, we have, in particular,

$$x_0 = \frac{\Delta}{a^{2(2k+1)}-1}P_0(a),$$
$$x_{n'} = \frac{\Delta}{a^{2(2k+1)}-1}P_{n'}(a).$$

Since $x_0 > 0$, $x_{n'} < 0$. Therefore, the leading coefficient of $P_{n'}(a)$ is $\theta_{n'} = -1$. We thus have

$$\theta_{n'}P_{n'}(a) = P_0(a).$$

This is in contradiction with the conclusion of Lemma 7 that $P_0(a) = \mathbf{Q}_{2(2k+1)}(a)$ is strictly minimal at all $a_{2(2k+1)} \leqslant a < 2$.

When $a < a_{2(2k+1)}$, it is clear that there is no $2(2k+1)$-periodic point for (20).

For $k \geqslant 1$ and $m \geqslant 2$, suppose $\{y_i,\ i=0,1,\ldots,(2k+1)2^m-1\}$ is a $(2k+1)2^m$-periodic orbit of system (20). Then, from Proposition 8, we know that $(2k+1)2^m$ is a period of a periodic orbit defined by $\{x_i = (-1)^i y_i, i = 0,1,\ldots,(2k+1)2^m-1\}$ for system (1). $(2k+1)2^m$ is actually the prime period for $\{x_i, i=0,1,\ldots,(2k+1)2^m-1\}$. Since otherwise there is a positive integer, $n' < (2k+1)2^m$, such that $x_0 = x_{n'}$, so that

(1) when n' is even, it follows from the relation $y_0 = y_{n'} = x_{n'} = x_0$ that $(2k+1)2^m$ is not a prime period of the orbit $\{y_i,\ i=0,1,\ldots\}$ (a contradiction);
(2) when n' is odd, it is easily verified that n' is a factor of $2k+1$, therefore, $2n' < (2k+1)2^m$, but $2n'$ is a period of the orbit $\{x_i,\ i=0,1,\ldots\}$, contradicting to the assumption that $(2k+1)2^m$ is a prime period.

From this, we conclude that system (20) has a $(2k+1)2^m$-periodic point if and only if $a \geqslant a_{(2k+1)2^m}$. □

5. Concluding remarks

For a Delta-modulated feedback system, we have determined the a values for which periodic orbits of each order exist for the case when $1 < |a| < 2$. A vivid illustration of the results are given with the help of the roots of the polynomials with "sign" coefficients. We have presented the differences and similarities between the tent map and the map f discussed in this paper in terms of their topological properties and the mechanisms of giving rise to periodic orbits via parameter variations.

Combining with the results that we obtained in [16] for the cases of $|a| \leqslant 1$ and $|a| \geqslant 2$, the relationship between the existence of periodic points and the parameter a is now completely characterized.

Just to complete the list, we summarize the results of [16] concerning periodic points as follows. When $a = -1$, there are both 1-periodic and 2-periodic points. For each a, $-1 < a < 0$, there are two 1-periodic (fixed) points. For each a, $0 \leqslant a < 1$, there are only two 2-periodic points. When $a = 1$, all points in $[-\Delta, \Delta)$ are 2-periodic points.

Note that when $a < 0$, f is an endomorphism on $[-\Delta, \Delta]$, and when $1 < a < 2$, f is an endomorphism on $[-\Delta/(a-1), \Delta/(a-1)]$. The results obtained in this paper comply with Sarkovskii's results on the ordering of periods of an endomorphism. When $a > 0$, f is also an endomorphism on $[-\Delta, \Delta]$. In this case, the Sarkovskii's ordering is changed only "slightly" in the sense that the ordering is kept to the second from the last one which is 1.

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