achieve exact tracking) with approximation approaches (that modify the internal dynamics) to remove the nonhyperbolicity of the internal dynamics. It was shown that, by giving up some of the precision in tracking, it is possible to achieve stable inversion of nonlinear nonminimum phase systems with nonhyperbolic internal dynamics.

REFERENCES

[1] A. Isidori, *Nonlinear Control Systems: An Introduction.* New York: Springer-Verlag, 1989.

[2] L. Qui and E. J. Davison, "Performance limitations of nonminimum phase systems in the servomechanism problem," *Automatica,* vol. 29, pp. 337–349, Mar. 1993.

[3] S. Devasia, D. Chen, and B. Paden, "Nonlinear inversion-based output tracking," *IEEE Trans. Automat. Contr.,* vol. 41, pp. 930–943, July 1996.

[4] G. Meyer, L. R. Hunt, and R. Su, "Nonlinear system guidance in the presence of transmission zero dynamics," NASA Tech. Memo. 4661, Jan. 1995.

[5] S. Devasia, "Output tracking with nonhyperbolic and near nonhyperbolic internal dynamics: Helicopter hover control," *AIAA J. Guidance, Contr., and Dynam.,* vol. 20, no. 3, pp. 573–580, 1997.

[6] B. A. Francis and W. M. Wonham, "The internal model principle of control theory," *Automatica,* vol. 12, no. 5, pp. 457–465, 1976.

[7] A. Isidori and C. I. Byrnes, "Output regulation of nonlinear systems," *IEEE Trans. Automat. Contr.,* vol. 35, pp. 131–140, Feb. 1990.

[8] J. Huang, "Output regulation of nonlinear systems with nonhyperbolic zero dynamics," *IEEE Trans. Automat. Contr.,* vol. 40, pp. 1497–1500, Aug. 1995.

[9] J. Huang and W. J. Rugh, "An approximation method for the nonlinear servomechanism problem," *IEEE Trans. Automat. Contr.,* vol. 37, pp. 1395–1398, Sept. 1992.

[10] A. J. Krener, *The Construction of Optimal Linear and Nonlinear Regulators in Systems, Models and Feedback: Theory and Application,* A. Isidori and T. J. Tarn, Eds. Boston, MA: Birkhauser, 1992.

[11] S. Devasia, B. Paden, and C. Rossi, "Exact-output tracking theory for systems with parameter jumps," *Int. J. Contr.,* vol. 67, no. 1, pp. 117–131, 1997.

[12] L. M. Silverman, "Inversion of multivariable linear systems," *IEEE Trans. Automat. Contr.,* vol. 14, pp. 270–276, Mar. 1969.

[13] R. M. Hirschorn, "Invertibility of multivariable nonlinear control systems," *IEEE Trans. Automat. Contr.,* vol. 24, pp. 855–865, June 1979.

[14] M. D. Di Benedetto and P. Lucibello, "Inversion of nonlinear time-varying systems," *IEEE Trans. Automat. Contr.,* vol. 38, pp. 1259–1264, Aug. 1993.

[15] S. Devasia and B. Paden, "Exact output tracking for nonlinear time-varying systems," *IEEE Trans. Automat. Contr.,* vol. 43, pp. 283–288, Feb. 1998.

[16] B. Paden, D. Chen, R. Ledesma, and E. Bayo, "Exponentially stable tracking control for multi-joint flexible manipulators," *ASME J. Dynamic Syst., Measurement and Contr.,* vol. 115, no. 1, pp. 53–59, 1993.

[17] S. Devasia and E. Bayo, "Redundant actuators to achieve minimal vibration trajectory tracking of flexible multibodies: Theory and application," *J. Nonlinear Dynam.,* vol. 6, no. 4, pp. 419–431, 1994.

[18] P. Martin, S. Devasia, and B. Paden, "A different look at output tracking: Control of a VTOL aircraft," *Automatica,* vol. 32, no. 1, pp. 101–107, 1996.

[19] R. Gurumoorthy and S. R. Sanders, "Controlling nonminimum phase nonlinear systems—The inverted pendulum on a cart example," in *Proc. American Control Conf.,* San Francisco, CA, 1993, pp. 680–685.

[20] Z. Xu and J. Hauser, "Higher order approximate feedback linearization about a manifold for multi-input systems," *IEEE Trans. Automat. Contr.,* vol. 40, pp. 833–840, May 1995.

[21] S. Gopalswamy and J. K. Hedrick, "Tracking nonlinear nonminimum phase systems using sliding control," *Int. J. Contr.,* vol. 57, no. 5, pp. 1141–1158, 1993.

[22] H. K. Khalil, *Nonlinear Systems.* New York: MacMillan, 1991.

[23] G. Meyer, L. R. Hunt, and R. Su, "Nonlinear system guidance," in *Proc. IEEE Conf. Decision and Control,* New Orleans, LA, Dec. 13–15, 1995, pp. 590–595.

[24] J. K. Hale, *Ordinary Differential Equations.* New York: Wiley, 1969.

[25] H. Amann, *Ordinary Differential Equations—An Introduction to Nonlinear Analysis.* New York: W. de Gruyter, 1990.

Disturbance Decoupling by Measurement Feedback for SISO Nonlinear Systems

X. Xia and C. H. Moog

***Abstract*—The measurement feedback disturbance decoupling problem of nonlinear systems with single-input/single-output and single measurement is considered in this paper. Necessary and sufficient conditions are given for disturbance decoupling by static measurement feedback. New necessary conditions and sufficient conditions are presented for disturbance decoupling by dynamic measurement feedback.**

***Index Terms*—Disturbance decoupling, measurement feedback, nonlinear systems.**

I. INTRODUCTION

Many contributions have already been brought for the disturbance decoupling control of nonlinear systems using state feedback; see [8], [11], and [15] for the historical accounts and some of the major developments. Additional contributions on dynamic and quasistatic state feedbacks are found in [5] and [12]. It is a general fact that the number of results on measurement feedback control problems available is definitely limited, although for practical applications state information may not be fully available, and measurement feedback is then the most realistic. In this paper, we are concerned with the problem of disturbance decoupling of nonlinear systems by measurement feedback. The latter problem got its first contribution in [9] in which the disturbance decoupling problem by static and dynamic measurement feedback is characterized in such geometric terms as (f, g) invariance (or controlled invariance), (h, f) invariance (or conditioned invariance), and (h, f, g) invariance. Note, however, that the dynamic measurement feedback considered therein is restricted to the so-called *pure dynamic measurement feedback*. A wider class of regular dynamic measurement feedbacks was considered in [2] and the above-mentioned geometric approach was generalized to obtain a necessary and sufficient condition, and a necessary condition, respectively, for the solvability of disturbance decoupling by so-called quasistatic measurement feedback, and regular dynamic measurement feedback, respectively. All these results are hardly constructive since they depend on the existence of some (h, f, g) invariant distribution/subspace. More generally, an effective method is notably missing for the design of dynamic measurement feedbacks for nonlinear control problems. Another approach to the dynamic output feedback linearization problem, using certain geometric results on output injection linearization, can be found in [10]. Although we consider single-input/single-output (SISO) nonlinear systems with a single measurement, some of the results may get an obvious generalization as sketched in the concluding section. All results in the paper are local in nature and its content of the paper features:

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- necessary and sufficient conditions for the static measurement feedback disturbance decoupling problem (DDPO, following the linear tradition) in Section II;
- necessary conditions for the dynamic measurement feedback disturbance decoupling problem (DDDPO) in Section III, as well as sufficient conditions which are based on a new formulation and solution to the linearization problem by output injection.

For the latter problem, we refer to [6] and [13] for a survey and some recent new results. Our new formulation yields the same conclusion as in [6] but does not require the input–output differential equation of the system which is difficult if not totally impossible to obtain due to the application of the implicit function theorem. It is worthwhile to note that the connection of DDPO/DDDPO with the observer form of linear systems was discovered in [14]: the latter gives a natural and intrinsic way of constructing a dynamic measurement feedback realizing (C, A, B) invariance for linear systems. For nonlinear systems, the relation between output feedback linearization and output injection linearization was also discovered in [10].

The input–output relation of the closed-loop system remains in general nonlinear, and any measurement feedback technique available for unperturbed systems may be applied afterwards.

The approach of the paper is independent from the *geometric* approach in [2] and may be used for other measurement feedback control problems which received a small amount of contributions and which constitute a challenge for future research.

II. Disturbance Decoupling by Static Measurement Feedbacks

Consider Σ

$$\begin{aligned}\dot{x} &= f(x, u, q)\\ y &= h(x)\end{aligned}\tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the input, $y \in \mathbb{R}$ is the controlled output, and $q \in \mathbb{R}$ is the disturbance; f and h are meromorphic functions of their arguments. Consider also the measured output of the system (1)

$$z = h_m(x)$$

where $z \in \mathbb{R}$ and h_m is a meromorphic function. Let $\mathcal{K}$ denote the field of meromorphic functions of x, u, q, and a finite number of derivatives of u and q. Also define $E = \text{span}_{\mathcal{K}}\{d\xi | \xi \in \mathcal{K}\}$, $\mathcal{X} = \text{span}_{\mathcal{K}}\{dx\}$, and $\mathcal{U} = \text{span}_{\mathcal{K}}\{du, d\dot{u}, \cdots, du^{(k)}, \cdots\}$.

Define

$$\begin{aligned}\Omega = \{\omega \in \mathcal{X} | \forall k \in \mathbb{N}:\\ \omega^{(k)} \in \text{span}_{\mathcal{K}}\{dx, dy^{(r)}, \cdots, dy^{(r+k-1)}\}\}.\end{aligned}\tag{2}$$

For affine nonlinear systems, Ω is shown in [7] to be the annihilator of some controllability distribution contained in $\ker dh$. What is important now is the fact that Ω is finitely computable by the following algorithm:

$$\begin{aligned}\Omega^0 &= \text{span}_{\mathcal{K}}\{dx\},\\ \Omega^{k+1} &= \{\omega \in \Omega^k | \dot{\omega} \in \Omega^k + \text{span}_{\mathcal{K}}\{dy^{(r)}\}\} \qquad (k \in \mathbb{N})\end{aligned}$$

i.e., there exists an integer $k^* \in \mathbb{N}$ such that: 1) $\forall k \geq k^*+1$, $\Omega^k = \Omega^{k^*}$ and 2) $\Omega = \Omega^{k^*}$.

A proof of these facts can be given following the same line as in [7]. Before proceeding to the feedback disturbance decoupling problem, let us summarize, without proof, some of the equivalent formulations of disturbance decoupling.

Lemma 1: Assume the relative degree r of y is finite, then the following statements are equivalent:

1) the system Σ is disturbance decoupled;
2) $\forall k \in \mathbb{N}$, $\text{span}_{\mathcal{K}}\{dy, d\dot{y}, \cdots, dy^{(k)}, \cdots,\} \subset \mathcal{X} + \mathcal{U}$;
3) $\forall k \in \mathbb{N}$, $dy^{(r+k)} \in \text{span}_{\mathcal{K}}\{dx, du, \cdots, du^{(k)}\}$;
4) $\forall k \in \mathbb{N}$, $\omega_0^{(k)} \in \text{span}_{\mathcal{K}}\{dx, du, \cdots, du^{(k-1)}\}$, where $\omega_0 \in \mathcal{X}$, s.t. $\omega_0 = dy^{(r)} - \xi du$ for some nonzero ξ;
5) $dy^{(r)} \in \Omega + \text{span}_{\mathcal{K}}\{du\}$;

from the induction assumption.

Suppose $u = \alpha(x, v)$ is a regular static state feedback, i.e., $\partial\alpha/\partial v \neq 0$. Let us investigate how Ω changes under the action of this feedback. Denote $\overline{\mathcal{K}}$ the field of meromorphic functions of x, v, q and a finite number of derivatives of v and q, and $\overline{\mathcal{E}} = \text{span}_{\overline{\mathcal{K}}}\{d\bar{\xi} | \bar{\xi} \in \overline{\mathcal{K}}\}$. Note that a regular feedback defines in a natural way a *field isomorphism* ϕ from $\mathcal{K}$ to $\overline{\mathcal{K}}$, and we have

$$\begin{cases} du = \frac{\partial\alpha}{\partial x}\,dx + \frac{\partial\alpha}{\partial v}\,dv\\ \quad\vdots\\ du^{(k)} = *\,dx + *\,dq + \cdots + *\,dq^{(k-1)} + *\,dv + \cdots\\ \quad + *\,dv^{(k-1)} + \frac{\partial\alpha}{\partial v}\,dv^{(k)}\\ \quad\vdots\end{cases}\tag{3}$$

where $*$'s are meromorphic functions in $\overline{\mathcal{K}}$. Since $\partial\alpha/\partial v \neq 0$, (3) defines an isomorphism Φ from $\mathcal{E}$ to $\overline{\mathcal{E}}$, which is *compatible* with the field isomorphism ϕ, i.e., for any $\omega_1, \omega_2 \in \mathcal{E}$, and any $\theta_1, \theta_2 \in \mathcal{K}$, Φ has the property

$$\Phi(\theta_1\omega_1 + \theta_2\omega_2) = \phi(\theta_1)\Phi(\omega_1) + \phi(\theta_2)\Phi(\omega_2).$$

Immediately, we have the following.

Proposition 1: Let $\overline{\Omega}$ be the subspace defined as in (2) for the composite system of (1) and the feedback $u = \alpha(x, v)$, then $\overline{\Omega} = \Phi(\Omega)$.

A regular feedback is called a measurement feedback if it can be written in the form $u = \alpha(z, v)$. DDPO of (1) is said to be solvable if the closed-loop system under a measurement feedback is disturbance decoupled. When $dz \in \Omega$, then the system is already disturbance decoupled. Thus, in the rest of the paper we shall assume that $dz \notin \Omega$, and now we are able to give the following result.

Theorem 1: Suppose the output y has a finite relative degree r. Then DDPO of the system (1) is solvable if and only if: 1) $dy^{(r)} \in \Omega + \text{span}_{\mathcal{K}}\{dz, du\}$ and 2) $d\omega \wedge \omega = 0$, where $\omega \in \text{span}_{\mathcal{K}}\{dz, du\}$, is such that $dy^{(r)} - \omega \in \Omega$.

Proof—Necessity: Suppose $u = \alpha(z, v)$ is the decoupling regular static measurement feedback. Thus, we can write

$$v = \alpha^{-1}(z, u).\tag{4}$$

Now that the closed-loop system is disturbance decoupled, by Lemma 1

$$dy_{cl}^{(r)} \in \Omega_{cl} + \text{span}_{\overline{\mathcal{K}}}\{dv\}\tag{5}$$

in which y_{cl} is the controlled output corresponding to the closed loop system, Ω_{cl} is similarly defined for the closed loop system, and $\overline{\mathcal{K}}$ is the field consisting of meromorphic functions of x, v, q and a finite number of derivatives of v and q. By (4), $dv \in \text{span}_{\mathcal{K}}\{dz, du\}$, thus from (5), $dy_{cl}^{(r)} \in \Omega_{cl} + \text{span}_{\overline{\mathcal{K}}}\{dz, du\}$. Lemma 1 then implies that

$$dy^{(r)} \in \Omega + \text{span}_{\mathcal{K}}\{dz, du\}.$$

If $dz \notin \Omega$, then $\Omega \cap \text{span}_{\mathcal{K}}\{dz, du\} = 0$ and ω is uniquely defined by

$$\omega = \xi\left(\frac{\partial\alpha^{-1}}{\partial z}\,dz + \frac{\partial\alpha^{-1}}{\partial u}\,du\right)$$

for some $\xi \in \mathcal{K}$. This implies condition 2) in Theorem 1.

Sufficiency: Since $dz \notin \Omega$, 1) and 2) guarantee the existence of a function $\phi(z, u)$ such that

$$dy^{(r)} = \omega_0 + \xi \cdot d\phi \tag{6}$$

in which $\omega_0 \in \Omega$, $\xi \in \mathcal{K}$. By the definition of relative degree and (5), we have $\partial\phi/\partial u \neq 0$, so we can define a regular static measurement feedback by $u = \phi^{-1}(z, v)$. From (6), $dy^{(r)} \in \Omega + \text{span}_{\mathcal{K}}\{dv\}$, or for the closed-loop system

$$dy_{cl}^{(r)} \in \Omega_{cl} + \text{span}_{\overline{\mathcal{K}}}\{dv\}.$$

By Lemma 1, the closed-loop system is disturbance decoupled. ■

III. Disturbance Decoupling by Dynamic Measurement Feedbacks

In this section, we first derive some necessary conditions and then sufficient conditions for the solvability of the disturbance decoupling problem via dynamic measurement feedback. Necessary conditions obtained in [2] also hold here for our case of nonaffine systems, though the assumption of SISO and single measurement permits more straightforward derivations. We now only focus on developing new necessary conditions that are of interest for our later use. From Theorem 1, one sees that when $dz \in \Omega$, (1) is decouplable by a regular static measurement feedback if and only if the system (1) is already disturbance decoupled. This result holds true also in the dynamic case.

Proposition 2: Suppose the controlled output y of the system (1) has a finite relative degree r and $dz \in \Omega$. Then (1) is decouplable by a regular dynamic measurement feedback if and only if the system is already disturbance decoupled.

Proof: We need only to prove the necessity, i.e., condition 3) in Lemma 1. Clearly, it holds for $k = 0$. Suppose it holds for $k = 0, 1, \cdots, l-1$. As in [2], we have

$$dy^{(r+l)} \in \text{span}\{dx, dz, \cdots, dz^{(l)}, du, \cdots, du^{(l)}\}.$$

Since $dz \in \Omega$

$$dy^{(r+l)} \in \text{span}\{dx, dy^{(r)}, \cdots, dy^{(r+l-1)}, du, \cdots, du^{(l)}\}.$$

From the induction assumption, $dy^{(r+l)} \in \text{span}\{dx, du, \cdots, du^{(l)}\}$. ■

As pointed out, the necessary condition in [2] is not sufficient; some other integrability conditions have to be imposed for sufficiency. The next result deals with a special case.

Proposition 3: Suppose the controlled output y of the system (1) has a finite relative degree r. Also suppose

$$dy^{(r)} \in \Omega + \text{span}\{dz, du\}.$$

Then DDDPO is solvable if and only if DDPO is solvable.

Proof: We need only to show that DDDPO has no solution if DDPO has none. Let

$$dy^{(r)} = \omega_0 + \xi_1 dz + \xi_2\, du \tag{7}$$

in which $\omega_0 \in \Omega$, $\xi_1, \xi_2 \in \mathcal{K}$. DDPO has no solution means, by Theorem 1, that, defining $\omega = \xi_1\, dz + \xi_2\, du$, $d\omega \wedge \omega \neq 0$, i.e.,

$$(\xi_1\, d\xi_2 - \xi_2\, d\xi_1) \wedge dz \wedge du \neq 0. \tag{8}$$

This implies, in particular, that $\xi_1\xi_2 \neq 0$. Since by assumption $dz \notin \Omega$, one proves easily that the relative degree σ_q of z with respect to q is finite. For convenience, denote

$$dz^{(\sigma_q)} = \xi_3\, dq \bmod \text{span}_{\mathcal{K}}\{dx, du, \cdots, du^{(\sigma_q-1)}\} \tag{9}$$

in which $0 \neq \xi_3 \in \mathcal{K}$. And then by (7), it is a routine matter to prove that the relative degree of y with respect to q is $r + \sigma_q$, and

$$dy^{(r+\sigma_q)} = \xi_1\xi_3\, dq + \xi_2\, du^{(\sigma_q)} \cdot (\text{mod span}\{dx, du, \cdots, du^{(\sigma_q-1)}\}). \tag{10}$$

Suppose there is a dynamic measurement feedback which decouples, then since

$$du^{(k)} \in \text{span}\{dx, d\eta, dv, \cdots, dv^{(k)}\}$$

for $k = 0, 1, \cdots, \sigma_q - 1$, in which η is the state of the dynamic feedback, and $du^{(\sigma_q)} = (\partial\alpha/\partial z)\, dz^{(\sigma_q)}(\text{mod span}\{dx, d\eta, dv, \cdots, dv^{(\sigma_q)}\})$, from (9)

$$du^{(\sigma_q)} = \frac{\partial\alpha}{\partial z}\xi_3\, dq(\text{mod span}\{dx, d\eta, dv, \cdots, dv^{(\sigma_q)}\}). \tag{11}$$

Thus from (10) and (11) and the fact that the closed-loop system is disturbance decoupled, we have, $\xi_1\xi_3 + \xi_2\xi_3(\partial\alpha/\partial z) = 0$. Since $\xi_3 \neq 0$,

$$\frac{\partial\alpha}{\partial z} = -\frac{\xi_1}{\xi_2}. \tag{12}$$

Note that α is a function of z, η, v, so $\partial\alpha/\partial z$ is also a function of z, η, v. If $\partial\alpha/\partial v = 0$, $\partial\alpha/\partial z$ is a function of z, η. And if $\partial\alpha/\partial v \neq 0$, then $v = \alpha^{-1}(z, \eta, u)$. So in either case, we can view $\partial\alpha/\partial z$ at the left-hand side of (12) a function $\theta(z, u, \eta)$, whereas the right-hand side is a function of x, u, q and a finite number of derivatives of u and q. Since by construction, $d\eta$, η, being the state of the dynamic measurement feedback, is independent from $dx, du, dq, \cdots, du^{(k)}, dq^{(k)}, \cdots$. Thus, by differentiating both sides of (12) and simple algebraic arguments, we derive that the function $\theta(z, u, \eta)$ actually does not depend on η, explicitly. This means that the fraction ξ_1/ξ_2 is a function of z, u, i.e., there is a function $\overline{\theta}(z, u)$ such that $\xi_1/\xi_2 = \overline{\theta}(z, u)$. This implies that $d(\xi_1/\xi_2) \wedge dz \wedge du = 0$, or equivalently, $(\xi_2\, d\xi_1 - \xi_1\, d\xi_2) \wedge dz \wedge du = 0$, contradicting (8). This proves that DDDPO has no solution. ■

To give sufficient conditions, we first give a new formulation of the problem of linearization by output injection. Define $E^0 = 0$

$$E^k = \text{span}_{\mathcal{K}}\{dz, \cdots, dz^{(k-1)}, du, d\dot{u}, \cdots, du^{(k-1)}\}.$$

Given $\omega \in E$, if there exist functions $\phi_1(z, u), \cdots, \phi_s(z, u)$ such that

$$\omega = d\phi_1^{(s-1)} + \cdots + d\phi_s, \tag{13}$$

then we say that ω is linearizable by s z-injections $\phi_1, \cdots, \phi_s$. Clearly, a necessary condition for the linearization of $\omega \in E$ by s z-injections is $\omega \in E^s$. In order to give sufficient conditions, assume

$$\dim E^s = 2s. \tag{14}$$

We propose the following algorithm first; solvability conditions are then stated in terms of integrability of some differential one forms defined in the algorithm.

Basic Algorithm: Initial check: $\omega \in E^s$. If no, stop! Otherwise, denote $\omega_1 := \omega$.

Step 1: Pick functions $\xi_1^1, \xi_2^1 \in \mathcal{K}$, such that

$$\omega_1 - \xi_1^1\, dz^{(s-1)} - \xi_2^1\, du^{(s-1)} \in E^{s-1}. \tag{15}$$

Define a differential one form $\overline{\omega}_1$ as $\overline{\omega}_1 = \xi_1^1\, dz + \xi_2^1\, du$. Check: $d\overline{\omega}_1 = 0$. If no, stop!

Step j—($j = 2, \cdots, s$): Let $\phi_{j-1}(z, u)$ be such that $d\phi_{j-1} = \overline{\omega}_{j-1}$. Denote ω_j as $\omega_j = \omega_{j-1} - d\phi_{j-1}^{(s-j+1)}$. Choose $\xi_1^j, \xi_2^j \in K$ such that $\omega_j - \xi_1^j\, dz^{(s-j)} - \xi_2^j\, du^{(s-j)} \in E^{s-j}$. Define a differential one form $\overline{\omega}_j$ as $\overline{\omega}_j = \xi_1^j\, dz + \xi_2^j\, du$. Check: $d\overline{\omega}_j = 0$. If no, stop!

Theorem 2: Under assumption (14), $\omega \in E$ is linearizable by s z-injections $\phi_1, \cdots, \phi_s$ if and only if $\omega \in E^s$ and

$$d\overline{\omega}_i = 0, \tag{16}$$

for $i = 1, \cdots, s$. In this case, the functions $\phi_i(z, u)$ are unique up to constants.

Proof—Necessity: Suppose (13) holds for $\phi_1(z, u), \cdots, \phi_s(z, u)$. Since

$$d\phi_i^{(k)} = \frac{\partial \phi_i}{\partial z}\, dz^{(k)} + \frac{\partial \phi_i}{\partial u}\, du^{(k)} (\text{mod } E^k) \tag{17}$$

one easily concludes that $\overline{\omega}_1$ defined in Step 1 is

$$\overline{\omega}_1 = \frac{\partial \phi_1}{\partial z}\, dz + \frac{\partial \phi_1}{\partial u}\, du.$$

Thus, (16) holds for $i = 1$. Repeat the reasoning for ω_2, etc., we get $d\overline{\omega}_2 = 0$, etc.

Sufficiency: First, note that the existence of ξ_1^1 and ξ_2^1 is guaranteed by the condition $\omega \in E^s$, and they are unique because of (14). Let ϕ_1 be such that $d\phi_1 = \overline{\omega}_1$. We show that $\omega - d\phi_1^{(s-1)} \in E^{s-1}$. By (17)

$$d\phi_1^{(s-1)} = \frac{\partial \phi_1}{\partial z}\, dz^{(s-1)} + \frac{\partial \phi_1}{\partial u}\, du^{(s-1)} (\text{mod } E^{s-1}).$$

By $d\phi_1 = \overline{\omega}_1 = \xi_1^1\, dz + \xi_2^1\, du$, we have, $\partial \phi_1/\partial z = \xi_1^1$, $\partial \phi_1/\partial u = \xi_2^1$, so

$$d\phi_1^{(s-1)} = k_1^1\, dz^{(s-1)} + k_2^1\, du^{(s-1)} (\text{mod } E^{s-1}). \tag{18}$$

Equations (15) and (18) imply that $\omega - d\phi_1^{(s-1)} \in E^{s-1}$. Similarly, we can show that $\omega - d\phi_1^{(s-1)} - \cdots - d\phi_k^{(s-k)} \in E^{s-k}$, for $k = 1, 2, \cdots, s$. ■

To give sufficient conditions for DDDPO, let us assume k^* is the maximal integer $k \in \mathbb{N}$ such that

$$\dim \text{span}\{dz, \cdots, dz^{(k-1)}, du, \cdots, du^{(k-1)}\} = 2k. \tag{19}$$

Theorem 3: Suppose the relative degree r of the controlled output y of the system (1) is finite. If:

1) there exists finite $s \le k^*$ such that
$$dy^{(r+s-1)} \in (\Omega + \dot{\Omega} + \cdots + \Omega^{(s-1)}) + \text{span}\{dz, \cdots, dz^{(s-1)}, du, \cdots, du^{(s-1)}\};$$
2) there exists an $\omega \in \text{span}\{dz, \cdots, dz^{(s-1)}, du, \cdots, du^{(s-1)}\}$ such that $dy^{(r+s-1)} - \omega \in \Omega + \cdots + \Omega^{(s-1)}$, and ω is linearizable by s z-injections $\phi_1, \cdots, \phi_s$;
3) for $i = 0, 1, \cdots, s-2$,
$$dy^{(r+i)} \in \mathcal{X} + \text{span}\{d\phi_1, d(\dot{\phi}_1 + \phi_2), \cdots, d(\phi_1^{(i-1)} + \cdots + \phi_i)\};$$

then DDDPO is solvable.

Proof: Let

$$\eta_1 := \phi_1(z, u). \tag{20}$$

From the definition of the relative degree r, $\partial \phi_1/\partial u \neq 0$. Solve (20) in u and set, with an abuse of notation, $u = \phi_1^{-1}(z, \eta_1)$. Design the following dynamic measurement feedback:

$$\begin{aligned} u &= \phi_1^{-1}(z, \eta_1) \\ \dot{\eta}_i &= \eta_{i+1} - \phi_{i+1}(z, \phi_1^{-1}(z, \eta_1)), \qquad i = 1, \cdots, s-2 \\ \dot{\eta}_{s-1} &= v - \phi_s(z, \phi_1^{-1}(z, \eta_1)). \end{aligned}$$

From this construction, we note that

$$\eta_i = \phi_1^{(i)} + \cdots + \phi_{i+1}, \qquad i = 1, \cdots, s-1 \tag{21}$$
$$v = \phi_1^{(s)} + \cdots + \phi_s. \tag{22}$$

Consider now the closed-loop system. By 3) and (21), we have

$$dy^{(r+i)} \in \mathcal{X} + \text{span}\{d\eta\} \tag{23}$$

for $i = 0, 1, \cdots, s-2$. And by 1)

$$dy^{(r+s-1)} - \omega \in \Omega + \dot{\Omega} + \cdots + \Omega^{(s-1)}. \tag{24}$$

Item 2) and (22) imply that

$$dy^{(r+s-1)} - \omega = dy^{(r+s-1)} - dv. \tag{25}$$

By definition of Ω, $\Omega + \dot{\Omega} + \cdots + \Omega^{(s-1)} \subseteq \text{span}\{dx, dy^{(r)}, \cdots, dy^{(r+s-2)}\}$, and from (23), $\Omega + \dot{\Omega} + \cdots + \Omega^{(s-1)} \subseteq \text{span}\{dx, d\eta\}$. So from (24)

$$dy^{(r+s-1)} - dv \in \text{span}\{dx, d\eta\}. \tag{26}$$

This and (23) imply that the new relative degree of y is $r + s - 1$. Also from (24)

$$\begin{aligned} dy^{(r+s-1+k)} - dv^{(k)} &\in \Omega + \cdots + \Omega^{(r+s+k-1)} \\ &\subseteq \text{span}\{dx, dy^{(r)}, \cdots, dy^{(r+s+k-2)}\} \end{aligned} \tag{27}$$

and using again (23)

$$dy^{(r+s-1+k)} - dv^{(k)} \in \text{span}\{dx, d\eta, dy^{(r+s-1)}, \cdots, dy^{(r+s+k-2)}\}$$

for all $k \in \mathbb{N}$. A simple mathematical induction argument shows that

$$dy^{(r+s-1+k)} \in \text{span}\{dx, d\eta, dv, d\dot{v}, \cdots, dv^{(k)}\}.$$

Lemma 1 says then the closed-loop system is already disturbance decoupled. ■

Remark 1: Note that condition 1) of Theorem 3 comes naturally from our study on the necessity of DDDPO and Proposition 2. Condition 2) of Theorem 3 is a set of integrability conditions. Concerning the existence of ω in 2), we would like to point out that ω is unique when $(\Omega + \cdots + \Omega^{(s-1)}) \oplus \text{span}\{dz, \cdots, dz^{(s-1)}, du, \cdots, du^{(s-1)}\}$ is a direct sum.

Theorem 3 does not contain Theorem 1 as a special case. At the expense of more flexibilities in constructing items that are in need, we are able to make the above sufficient condition general enough to cover the case of DDPO.

Theorem 4: Suppose the relative degree r of the controlled output y of the system (1) is finite. If:

1) there exists $s \le k^*$ such that
$$dy^{(r+s-1)} \in (\Omega + \dot{\Omega} + \cdots + \Omega^{(s-1)}) + \text{span}\{dz, \cdots, dz^{(s-1)}, du, \cdots, du^{(s-1)}\};$$
2) there exists a function $\xi(z, \cdots, z^{(s-1)}, u, \cdots, u^{(s-1)})$ and an integrating factor λ satisfying
$$dy^{(r+s-1)} - \lambda d\xi \in \Omega + \dot{\Omega} + \cdots + \Omega^{(s-1)}$$
and $d\xi$ is linearizable by s z-injections $\phi_1, \cdots, \phi_s$;
3) for $i = 0, 1, \cdots, s-2$,
$$dy^{(r+i)} \in \mathcal{X} + \text{span}\{d\phi_1, d(\dot{\phi}_1 + \phi_2), \cdots, d(\phi_1^{(i-1)} + \cdots + \phi_i)\};$$

then DDDPO is solvable.

Proof: The proof can follow what has been described for proving Theorem 3. One only needs to replace dv in (25) and (26) by θdv, in which $\theta \in \mathcal{K}$ satisfies $\omega = \theta d\xi$, and to replace $dv^{(k)}$ in (27) by $d(\theta dv)^{(k)}$. All these replacements will not affect the proof. ■

Whether the conditions of Theorem 4 are necessary or not remains an open problem.

IV. EXAMPLES

Examples are worked out to show the application and limitation of the above results.

Example 1: The system described by

$$\dot{x}_1 = x_2, \qquad y = x_1$$
$$\dot{x}_2 = x_3 \sin x_2 + u \cos x_2, \qquad z = x_3$$
$$\dot{x}_3 = q$$

cannot be disturbance decoupled by any regular dynamic measurement feedback. Since the relative degree of y is two

$$\Omega = \text{span}\{dx_1, dx_2\}$$
$$d\ddot{y} = (x_3 \cos x_2 - u \sin x_2)\, dx_2 + \sin x_2\, dz + \cos x_2\, du \in \Omega + \text{span}\{dz, du\}.$$

Define $\omega = \sin x_2\, dz + \cos x_2\, du$, then obviously $d\omega \wedge \omega \neq 0$. By Proposition 3, DDDPO is not solvable.

Example 2: Consider the system

$$\dot{x}_1 = x_2 + \sin x_1, \qquad y = x_1$$
$$\dot{x}_2 = x_1 x_2 x_6 + x_3 + x_5 + u, \qquad z = x_5$$
$$\dot{x}_3 = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + (1 + x_5)u$$
$$\dot{x}_4 = -x_3 + \cos x_5 + x_6 \sin(x_1 x_2)$$
$$\dot{x}_5 = -x_4 + q$$
$$\dot{x}_6 = x_1.$$

It is easily computed that

$$\Omega = \text{span}\{dx_1, dx_2, dx_6\},$$
$$\Omega + \dot{\Omega} + \ddot{\Omega} = \text{span}\{dx_1, dx_2, dx_6, d(x_3 + x_5 + u)\ d(x_4 + x_5 u + \dot{u})\}$$
$$\text{span}\{dz, d\dot{z}, d\ddot{z}, du, d\dot{u}, d\ddot{u}\} = \text{span}\{dx_5, dq - dx_4, d\dot{q} - dx_3 + d[x_6 \sin(x_1 x_2)], du, d\dot{u}, d\ddot{u}\}.$$

One checks that the conditions of Theorem 3 are satisfied for $s = 3$. Note that

$$(\Omega + \dot{\Omega} + \ddot{\Omega}) \cap \text{span}\{dz, d\dot{z}, d\ddot{z}, du, d\dot{u}, d\ddot{u}\} = 0.$$

ω in 3) is uniquely defined by

$$\omega = (1 - \sin z + \dot{u})\, dz + (1 + \dot{z})\, du + u\, d\dot{z} + z\, d\dot{u} + d\ddot{z} + d\ddot{u}.$$

By the basic algorithm and Theorem 2, we verify that ω can be linearized by 3 z-injections $z + u$, zu and $z + \cos z + u$. Thus by the construction of the proof of Theorem 3, the following dynamic measurement feedback is a decoupling feedback:

$$u = -z + \eta_1$$
$$\dot{\eta}_1 = \eta_2 - \eta_1 z + z^2$$
$$\dot{\eta}_2 = \eta_1 - \cos z + v.$$

V. CONCLUSIONS

In this paper a necessary and sufficient condition for *DDPO* was obtained as well as a necessary condition and a sufficient condition for *DDDPO*. The NSC for DDPO is completely new, and it turns out that the sufficient conditions for DDDPO are less restrictive than the existing ones, and most importantly they provide specific procedures to construct a dynamic output feedback. Some of the above results can be rewritten in a straightforward manner for the class of multivariable systems which have vector relative degrees.

Note that our approach was to incorporate the output injection technique with the problems under consideration, a technique well known for linear systems [14]. Compared to the geometric approach taken in [2], this paper provides an approach in quite an independent manner and brings about appealing sufficient conditions for DDDPO. While comparing with the study of the dynamic output feedback linearization problem performed in [10], the algebraic treatment of the output injection problem fits well with structural design of nonlinear systems. This, hopefully, is expected to open ways for studies of other dynamic output feedback design problems, which, together with some open questions left in this paper, are the topic for further research.

REFERENCES

[1] R. Andiarti, “Invariances géeralisées pour les systémes nonlinéaireset applications,” Ph.D. dissertation, Ecole Centrale de Nantes, Univ. de Nantes, 1995.

[2] R Andiarti and C. H. Moog, “Output feedback disturbance decoupling in nonlinear systems,” *IEEE Trans. Automat. Contr.*, vol. 41, pp. 1683–1689, 1996.

[3] S. Battilotti, “Noninteracting control with stability for nonlinear systems,” *Lect. Notes in Contr. Inf. Sci.*, vol. 196. Berlin, Germany: Springer-Verlag, 1994.

[4] ——, “Noninteraction via measurement feedback for nonlinear systems with relative degree,” in *IEEE Trans. Automat. Contr.*, vol. 44, pp. 774–778, 1999.

[5] E. Delaleau and M. Fliess, “Nonlinear disturbance rejection by quasistatic feedback,” in *Proc. MTNS’93*, Regensburg, Germany.

[6] A. Glumineau, C. H. Moog, and F. Plestan, “New algebro-geometric conditions for the linearization by input–output injection,” *IEEE Trans. Automat. Contr.*, vol. 41, pp. 598–603, 1996.

[7] H. J. C. Huijberts, L. Colpier, and P. Moreau, “Nonlinear input–output decoupling by static output feedback,” in *Proc. European Contr. Conf.*, Rome, Italy, 1995.

[8] A. Isidori, *Nonlinear Control Systems*, 3rd ed. Berlin, Germany: Springer-Verlag, 1995.

[9] A. Isidori, A. J. Krener, C. Gori-Giorgi, and S. Monaco, “Nonlinear decoupling via feedback: A differential geometric approach,” *IEEE Trans. Automat. Contr.*, vol. 26, pp. 331–345, 1981.

[10] R. Marino and P. Tomei, “Dynamic output feedback linearization and global stabilization,” *Syst. Contr. Lett.*, vol. 17, pp. 115–121, 1991.

[11] H. Nijmeijer and A. J. van der Schaft, *Nonlinear Dynamical Control Systems.* New York: Springer-Verlag, 1990.

[12] A. M. Perdon, Y. F. Zheng, C. H. Moog, and G. Conte, “Disturbance decoupling for nonlinear systems: A unified approach,” *Kybernetika*, vol. 29, pp. 479–484, 1993.

[13] F. Plestan, “Linéarisation par injection d’entrée-sortie généralisée et synthèse d’observateurs,” Ph.D. dissertation, Ecole Centrale de Nantes, Univ. de Nantes, 1995.

[14] J. M. Schumacher, “Compensator synthesis using (C, A, B)-pairs,” *IEEE Trans. Automat. Contr.*, vol. 25, pp. 1133–1138, 1980.

[15] L. L. M. Van der Wegen, “Local disturbance decoupling with stability for nonlinear systems,” in *Lect. Notes in Contr. Inf. Sci.*, vol. 166. Berlin, Germany: Springer-Verlag, 1991.

[16] X. Xia, “Parameterization of decoupling control laws for nonlinear affine systems,” *IEEE Trans. Automat. Contr.*, vol. 38, pp. 916–928, 1993.