Well Posedness of Piecewise-Linear Systems With Multiple Modes and Multiple Criteria

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***Abstract*—In this note, the results of a previous paper are generalized to obtain necessary and sufficient conditions for the well posedness of piecewise-linear systems with multiple modes and multiple criteria. To check the necessary and sufficient conditions, we present new algorithmic procedures by making use of the famous Fourier–Motzkin elimination technique.**

***Index Terms*—Hybrid systems, lexicographic inequalities, piecewise-linear systems, well posedness.**

I. Introduction

An important class of hybrid systems is the one described by piecewise-linear systems [11], [2], [8], [10], [7]. A fundamental issue of piecewise-linear systems is the problem of existence and uniqueness of solutions, or the well-posedness problem [12].

In [7] and [6], the problems of well posedness and feedback well posedness are investigated under the definition of Carathéodory solutions. For systems with two modes, or bimodal systems, both the problems of well posedness and feedback well posedness are completely characterized for the case of a single criterion. For bimodal systems with multiple criteria, simple necessary and sufficient conditions are also found for the well posedness problem in [7]. For a special case of multiple modes, a set of necessary and sufficient conditions are given for the well-posedness problem under the assumption that each mode is observable in [7]. To check the conditions, [7] proposed an algorithm based on a linear programming argument. However, it is noted that the linear programming (LP) problem as formulated there is degenerate.

In this note, we generalize the above results about well-posedness of piecewise-linear systems to the more general case. Necessary and sufficient conditions are derived for the well-posedness problem of piecewise-linear systems with multiple modes and multiple criteria. We also define a concept of mode well posedness: the existence of solution in a unique mode, and show that the characterization of [7] in the multiple cases is for mode well posedness. Another feature of the note is the approach to check these necessary and sufficient conditions. Instead of a linear programming approach, we present an approach by making use of the technique of the Fourier–Motzkin elimination, which is well known in linear optimization [3]. Finally, algorithmic procedures are worked out.

The organization of the note is as follows. In Section II, the Fourier–Motzkin elimination is introduced to study some important properties of polyhedral cones. Section III is devoted to developing necessary and sufficient conditions for well posedness. An algorithm are given in Section IV. Section V is the conclusion.

We will make extensive use of the notations and results of [7]. In particular, we will use the following notation: for lexicographic inequalities of $x \in \mathcal{R}^n$, if for some i, $x_j = 0$ $(j = 1, 2, \ldots, i-1)$, while $x_i > 0 (<0)$, we denote it by $x \succeq (\preceq) 0$.

Manuscript received April 24, 2001; revised December 6, 2001. Recommended by Associate Editor M. M. Polycarpou.

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Digital Object Identifier 10.1109/TAC.2002.803544.

II. Polyhedral Cones

First, two properties of polyhedral cones are investigated.

Problem One: For a given subspace $\mathcal{N} \subset \mathcal{R}^n$, is a *polyhedral cone* $P = P(T) = \{x | x \in \mathcal{R}^n, Tx \geq 0\}$ for some $m \times n$ matrix T contained in $\mathcal{N}$?

Problem Two: Given an $m \times n$ matrix T, is the *polyhedral cone* $P^o = P^o(T) = \{x | x \in \mathcal{R}^n, Tx > 0\} = \emptyset$?

To solve these problems, we make use of the Fourier–Motzkin elimination procedures [4], [9], [3].

For the first problem, denote $r = \mathrm{codim}\mathcal{N} = n - \dim \mathcal{N}$, and find a matrix $C_1 \in \mathcal{R}^{r \times n}$ such that $\mathcal{N} = \ker C_1$, and let $C_2 \in \mathcal{R}^{(n-r) \times n}$ be such that $(C_1^T, C_2^T)^T$ is a nonsingular matrix, and define a coordinate transformation by

$$z = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} x \stackrel{\mathrm{def}}{=} Cx$$

then $P(T) \subset \mathcal{N}$ is equivalent to

$$\{z \in \mathcal{R}^n | Az \geq 0\} \subset \{z \in \mathcal{R}^n | z_1 = 0, \ldots, z_r = 0\} \tag{1}$$

in which $A = TC^{-1}$.

Denote A^i the ith column of A, and A_i the ith row of A.

Lemma 1: If $P(T) \subset \mathcal{N}$ then

1) for each $i \in \{1, \ldots, r\}$, at least one element of A^i is positive, and one element of A^i is negative;
2) for each $i \in \{r+1, \ldots, n\}$, elements of A^i cannot be all positive or negative.

Proof: 1) By contradiction. Without loss of generality, assume that all elements of A^1 are positive, then $(1, 0, \ldots, 0)^T$ belongs to the set at the left-hand side of the inclusion (1), but not the set at the right-hand side. So, $P(T) \not\subset \mathcal{N}$.

2) Also by contradiction. Assume, without loss of generality, all elements of A^n are positive, then choose a number λ such that

$$\lambda \geq -\max_{\{1 \leq i \leq m\}} \frac{a_{i1}}{a_{in}}$$

then it is easy to verify that $(1, 0, \cdots, 0, \lambda)^T \in P(A)$, but does not belong to the right-hand side of (1).

Fourier–Motzkin Elimination: Denote

$$Q = \{k | a_{kn} < 0\} \quad P = \{k | a_{kn} > 0\} \quad Z = \{k | a_{kn} = 0\}$$

and

$$\tilde{m} = |Z| + |Q \times P|$$

in which, denoting $|\cdot|$ for the size of a finite set $\cdot$.

Define matrix B, called a Fourier–Motzkin elimination of A, as

$$B = (b_{ij}) \in \mathcal{R}^{\tilde{m} \times (n-1)}$$

in the following way.

- For the first $|Z|$ rows

$$b_{kj} = a_{k'j}$$

for $j = 1, \ldots, n-1$, and some (one and only one) $k' \in Z$.

- For the last $|P \times Q|$ rows

$$b_{ij} = a_{i'j} - \frac{a_{i'n}}{a_{k'n}} a_{k'j}$$

for $j = 1, \ldots, n-1$, and some (one and only one) index set $(i', k') \in P \times Q$.

It is noted that the Fourier–Motzkin elimination of a matrix A does not exist if and only if $Z = P = \emptyset$ or $Z = Q = \emptyset$, i.e., (2) in Lemma 1 is violated.

Theorem 1:

i) If $r < n$, (1) holds if and only if the Fourier–Motzkin elimination B of A exists and

$$\{z \in \mathcal{R}^{n-1} | Bz \geq 0\} \subset \{z \in \mathcal{R}^{n-1} | z_1 = 0, \ldots, z_r = 0\}. \quad (2)$$

ii) If $r = n$, (1) holds if and only if A^n has both positive and negative elements and

$$\{z \in \mathcal{R}^{n-1} | Bz \geq 0\} \subset \{z \in \mathcal{R}^{n-1} | z_1 = 0, \ldots, z_{n-1} = 0\}. \quad (3)$$

Proof: (Necessity): Case $r < n$: The existence of the Fourier–Motzkin elimination B of A is implied by Lemma 1.

For any $(z_1, \ldots, z_{n-1})^T \in \{z \in \mathcal{R}^{n-1} | Bz \geq 0\}$, by the construction of the matrix B, one has

$$\max_{i \in P} \frac{a_{i1}z_1 + \cdots + a_{in-1}z_{n-1}}{-a_{in}} \leq \min_{i \in Q} \frac{a_{i1}z_1 + \cdots + a_{in-1}z_{n-1}}{-a_{in}}.$$

Choose a z_n such that

$$\max_{i \in P} \frac{a_{i1}z_1 + \cdots + a_{in-1}z_{n-1}}{-a_{in}} \leq z_n \leq \min_{i \in Q} \frac{a_{i1}z_1 + \cdots + a_{in-1}z_{n-1}}{-a_{in}}$$

then it is easy to see that $(z_1, \ldots, z_n)^T \in \{z \in \mathcal{R}^n | Az \geq 0\}$. So by (1), $(z_1, \ldots, z_r)^T = 0$, i.e., $(z_1, \ldots, z_{n-1})^T \in \{z \in \mathcal{R}^{n-1} | z_1 = 0, \ldots, z_r = 0\}$.

That is, (2) holds.

Case $r = n$: By Lemma 1, A^n has both positive and negative elements.

Similar to the proof of (2), we can prove that (3) holds.

(Sufficiency): Case $r = n$: By contradiction. Assume that $z = (z_1, \ldots, z_n)^T \in \{z | Az \geq 0\}$, but $(z_1, \ldots, z_n)^T \neq 0$. Then, for all $k \in Q$,

$$a_{k1}z_1 + \cdots + a_{kn-1}z_{n-1} + a_{kn}z_n \geq 0. \quad (4)$$

Define

$$\overline{z}_n = \min_{k \in Q} \frac{a_{k1}z_1 + \cdots + a_{kn-1}z_{n-1}}{-a_{kn}} \quad (5)$$

then by (4) and the fact that a_{kn}s are negative, one has $\overline{z}_n \geq 0$.

Define $\overline{z} = (z_1, \ldots, z_{n-1}, \overline{z}_n)^T$, then by (4) and (5), it is verified that $A_k \overline{z} \geq 0$, for $k \in Q$, and there is an index $\overline{k} \in Q$ such that $A_{\overline{k}} \overline{z} = 0$. That is, $\overline{k} \in Q$ is chosen such that

$$a_{\overline{k}1}z_1 + \cdots + a_{\overline{k}n-1}z_{n-1} + a_{\overline{k}n}\overline{z}_n = 0 \quad (6)$$

and for all other $k \in Q$

$$\frac{a_{k1}z_1 + \cdots + a_{kn-1}z_{n-1}}{-a_{kn}} \geq \frac{a_{\overline{k}1}z_1 + \cdots + a_{\overline{k}n-1}z_{n-1}}{-a_{\overline{k}n}}. \quad (7)$$

One claims that

$$(z_1, \ldots, z_{n-1})^T \in \{z \in \mathcal{R}^{n-1} | Bz \geq 0\}.$$

As a matter of fact, for the first $|Z|$ rows of B, obviously

$$b_{i1}z_1 + \cdots + b_{in-1}z_{n-1} \geq 0. \quad (8)$$

From (4) and (6), one knows that $a_{kn}(z_n - \overline{z}_n) \geq 0$, or

$$\overline{z}_n \geq z_n. \quad (9)$$

Now, for each of the last $|P \times Q|$ rows of B, since there is a (unique) index set $(i', k') \in P \times Q$ such that

$$\begin{aligned} &b_{i1}z_1 + \cdots + b_{in-1}z_{n-1} \\ &= \left(a_{i'1} - \frac{a_{i'n}}{a_{k'n}} a_{k'1}\right) z_1 + \cdots + \left(a_{i'n-1} - \frac{a_{i'n}}{a_{k'n}} a_{k'n-1}\right) z_{n-1} \\ &= a_{i'1}z_1 + \cdots + a_{i'n-1}z_{n-1} - \frac{a_{i'n}}{a_{k'n}} (a_{k'1}z_1 + \cdots + a_{k'n-1}z_{n-1}) \\ &\overset{(7)}{\geq} a_{i'1}z_1 + \cdots + a_{i'n-1}z_{n-1} - \frac{a_{i'n}}{a_{\overline{k}n}} (a_{\overline{k}1}z_1 + \cdots + a_{\overline{k}n-1}z_{n-1}) \\ &= a_{i'1}z_1 + \cdots + a_{i'n-1}z_{n-1} + a_{i'n}\overline{z}_n \\ &\overset{(9)}{\geq} a_{i'1}z_1 + \cdots + a_{i'n-1}z_{n-1} + a_{i'n}z_n \geq 0. \end{aligned} \quad (10)$$

Combining (8) and (10), one has $(z_1, \ldots, z_{n-1})^T \in \{z \in \mathcal{R}^{n-1} | Bz \geq 0\}$.

Finally, from (4), it is easy to see that $(z_1, \ldots, z_{n-1})^T$ is not zero; a contradiction.

Case $r < n$: For any $(z_1, \ldots, z_n)^T \in P(A)$, from the construction of B, it is concluded that $(z_1, \ldots, z_{n-1})^T \in P(B)$. So from (2), $z_1 = 0, \ldots, z_r = 0$. That is, (1) holds.

Now, it is ready to give an algorithm to check whether $P(T) \subset \mathcal{N}$ for a matrix $T \in \mathcal{R}^{m \times n}$ and subspace of $\mathcal{R}^n$ of co-dimension r.

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Algorithm SubCone
Step 1: Find a nonsingular matrix C = (C_1^T, C_2^T)^T ∈ R^{n×n} such that N = ker C_1. Denote A^1 = TC^{-1}, and r = n - dim N
Check: all elements of the last column of A^1 are positive or negative. If yes, stop.
Step k (2 ≤ k ≤ n - r): Find the Fourier–Motzkin elimination of A^{k-1} as A^k.
Check: all elements of the last column of A^k are positive or negative. If yes, stop!
Step k (n - r + 1 ≤ k ≤ n): Find the Fourier–Motzkin elimination of A^{k-1} as A^k.
Check: the last column of A^k has both negative element and positive elements. If not, stop!
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If it passes all the steps of the algorithm *SubCone,* then one concludes that $P(T) \subset \mathcal{N}$.

We now turn to the second problem. Denote the Fourier–Motzkin elimination of T by $F(T)$.

Theorem 2: $P^o(T) = \emptyset$ if and only if the elements of the last column of T are not all positive nor all negative, and $P^o(F(T)) = \emptyset$.

Proof: (Necessity) If all elements of the last column of T are positive (negative), then $(0, \ldots, 0, 1)^T$ ($(0, \ldots, 0, -1)^T$) belongs to $P^o(T)$. If we denote in the Fourier–Motzkin procedure as P, Q, and Z the sets of row indices of the elements of the last column with positive, negative and zero elements, respectively, then there are only the following three cases:

I) $P \neq \emptyset$ and $Q \neq \emptyset$;
II) $P = \emptyset$ and $Z \neq \emptyset$;
III) $Q = \emptyset$ and $Z \neq \emptyset$.

In case II), the Fourier–Motzkin elimination $F(T)$ is defined by rows in Z of the matrix T. If $(x_1, \ldots, x_{n-1})^T \in P^o(F(T))$, choose

$$x_n < \min_{i \in Q} \frac{a_{i1}x_1 + \cdots + a_{in-1}x_{n-1}}{-a_{in}}$$

then it is easy to see that $(x_1, \ldots, x_n)^T \in P(T)$. A contradiction.

Similarly, in case III), $P^o(F(T)) = \emptyset$.

In case I), If $(x_1, \ldots, x_{n-1})^T \in P^o(F(T))$, choose x_n satisfying

$$\max_{i \in P} \frac{a_{i1}x_1 + \cdots + a_{in-1}x_{n-1}}{-a_{in}} < x_n < \min_{i \in Q} \frac{a_{i1}x_1 + \cdots + a_{in-1}x_{n-1}}{-a_{in}}$$

which exists, by the construction of $F(T)$. Then, it is easy to see that $(x_1, \ldots, x_n)^T \in P(T)$. A contradiction.

(Sufficiency) Again, there are only the aforementioned three cases. In cases II) and III), $P^o(F(T)) = \emptyset$ implies $P^o(T) = \emptyset$.

In case I), if $(x_1, \ldots, x_n)^T \in P^o(T)$, then by the construction of $F(T)$, $(x_1, \ldots, x_{n-1})^T \in P^o(F(T))$. So again, $P^o(T) = \emptyset$.

To check $P^o(T) = \emptyset$, we propose the following algorithm.

```
Algorithm EmptyCone
Step k (1 ≤ k ≤ n): Check whether the
  elements of the last column of T are all
  positive or all negative, if yes, then
  stop.
T := F(T) and goto Step k+1
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If it passes all steps of the algorithm *EmptyCone*, then $P^o(T) = \emptyset$.

Remark 1: Both algorithms can be improved by arguments of extreme rays, as done in the dual algorithms of double description [5].

III. Necessary and Sufficient Conditions

Consider piecewise-linear systems described by m modes, and each mode is defined by

$$\dot{x} = A_i x, \qquad \text{when } C_i^1 x \succeq 0, \ldots, C_i^{p_i} x \succeq 0 \tag{11}$$

for $i = 1, \ldots, m$, in which $x \in \mathcal{R}^n$, A_i's are $n \times n$ matrices, C_i^j is a $q_i \times n$ matrix, for $i = 1, \ldots, m$ and $j = 1, \ldots, p_i$.

When $m = 2$, the system (11) is called bimodal. For a bimodal system, if $p_1 = 1$, $p_2 = 1$, $q_1 = 1$, $q_2 = 1$, and $C = C_1^1 = -C_2^1$, then the system is called a bimodal system with a single criterion.

Note that the form of (11) already appeared in the discussions of the note [7]. However, necessary and sufficient conditions were presented for the case of bimodal case and a special case with multiple modes and multiple criteria. In (11), the creiteria are also defined by *lexicographic inequalities,* and no observability condition is imposed for (A_i, C_i^j), etc. We will extend the results of [7] to obtain necessary and sufficient conditions for the well posedness of (11). To see the relationship between the new conditions and the conditions obtained in [7], we also define a concept of mode well posedness: the existence of solution in a unique mode. We will show that the conditions of [7] in the multiple cases correspond to a characterization for mode well posedness.

Let us first recall the following definitions.

Definition 1: If, for a given initial state $x(t_0)$, $x(t)$ satisfies on $[t_0, t_0 + \delta)$ for some $\delta > 0$

$$x(t) = x(t_0) + \int_{t_0}^{t_1} f(x(\tau))\, d\tau \tag{12}$$

where $f(x)$ is the vector field given by the right-hand side of (11), and there is no left-accumulation point of event times [7] on $[t_0, t_0 + \delta)$, then $x(t)$ is said to be a continuous-state solution of the system (11) on $[t_0, t_0 + \delta)$ in the sense of Carathéodory or simply a C solution.

Definition 2: The system (11) is said to be C-well posed if there exists a unique solution of (11) on $[0, \infty)$ in the sense of Carathéodory for every initial state $x_0 \in \mathcal{R}^n$.

Definition 3: Let $\mathcal{S}$ be a subset of $\mathcal{R}^n$. If for the initial state x_0 there exists an $\epsilon > 0$ such that $x(t) \in \mathcal{S}$ for all $t \in [0, \epsilon]$, then we say that the system has the smooth continuation property at x_0 with respect to $\mathcal{S}$. Moreover, if from all $x_0 \in \mathcal{S}$ smooth continuation is possible with respect to $\mathcal{S}$, then the system is said to have the smooth continuation property with respect to $\mathcal{S}$.

[7, Lemma 2.1] is easily generalized.

Lemma 2: The following statements are equivalent.

i) The system (11) is C-well posed.

ii) For the system (11), from every initial state $x_0 \in \mathcal{R}^n$, smooth continuation is possible in one of the m modes, in other words, with respect to

$$\{x \in \mathcal{R}^n | C_i^1 x \succeq 0, \ldots, C_i^{p_i} x \succeq 0\}$$

except for the cases that solutions in any two modes are the same in some time interval.

iii) For every initial state $x_0 \in \mathcal{R}^n$, the following hold.

a) There is an $i \in \{1, \ldots, m\}$ and $\delta > 0$ such that

$$C_i^1 e^{A_i t} x_0 \succeq 0, \ldots, C_i^{p_i} e^{A_i t} x_0 \succeq 0 \tag{13}$$

for $t \in [0, \delta)$.

b) For any $i \neq j$, if

$$C_i^1 e^{A_i t} x_0 \succeq 0, \ldots, C_i^{p_i} e^{A_i t} x_0 \succeq 0$$
$$C_j^1 e^{A_j t} x_0 \succeq 0, \ldots, C_j^{p_j} e^{A_j t} x_0 \succeq 0$$

for $t \in [0, \delta)$, then

$$e^{A_i t} x_0 = e^{A_j t} x_0 \tag{14}$$

for $t \in [0, \delta)$.

Proof: The equivalence of i) and ii) can be worked out along the same lines as in [7, proof of Lemma 2.1]. Condition iii) is just a restatement of ii).

Denote, for $i = 1, \ldots, m; j = 1, \ldots, p_i$

$$T_i^j = \begin{bmatrix} T_{i1}^j \\ \vdots \\ T_{iq_i}^j \end{bmatrix}$$

where, for $k = 1, \ldots, q_i$

$$T_{ik}^j = \begin{bmatrix} C_{ik}^j \\ C_{ik}^j A_i \\ \vdots \\ C_{ik}^j A_i^{h_{ik}^j - 1} \end{bmatrix}$$

in which C_{ik}^j is the kth row of C_i^j, and each $h_{ik}^j (k = 1, \ldots, q_i)$ is the maximal value of the rank such that $[T_{i1}^{jT} \ldots T_{ik}^{jT}]^T$ has a row-full rank.

Denote

$$\mathcal{S}_i^j = \{x | T_i^j x \succeq 0\} \quad \mathcal{S}_i = \bigcap_{j=1}^{p_i} \mathcal{S}_i^j.$$

For any $i \neq j$, define

$$\mathcal{K}_{ij} = \bigcap_k \ker(A_i^k - A_j^k).$$

Theorem 3: The system (11) is C-well posed on $\mathcal{R}^n$ if and only if

$$\bigcup_{i=1}^m \mathcal{S}_i = \mathcal{R}^n \tag{15}$$

$$\mathcal{S}_i \cap \mathcal{S}_j \subset \mathcal{K}_{ij}, \qquad \text{for all } i \neq j. \tag{16}$$

Proof: Follow similar lines in [7, proof of Lemma 2.3], (13) holds if and only if

$$T_i^1 x_0 \succeq 0, \cdots, T_i^{p_i} x_0 \succeq 0$$

or, equivalently

$$x_0 \in \mathcal{S}_i.$$

So iii.a) of Lemma 2 holds if and only if (15) holds.

Note also that (14) holds if and only if

$$A_i x_0 = A_j x_0, \, A_i^2 x_0 = A_j^2 x_0, \, \ldots, \, A_i^k x_0 = A_j^k x_0, \ldots$$

or, equivalently

$$x_0 \in \mathcal{K}_{ij}.$$

So iii.b) of Lemma 2 holds if and only if (16) holds.

For a bimodal case, the aformentioned result reduces to [7, Th. 5.1].

Note that conditions in Theorem 3 differ from those of [7, Th. 6.1] in (16). There it was required that the intersection is zero. We will show that this type of condition corresponds to another weak concept.

Definition 4: A C solution $x(t)$ is said to be in mode i at τ, if there exists a $\delta > 0$ such that

$$C_i^1 x(t) \succeq 0, \ldots, C_i^{p_i} x(t) \succeq 0$$

for $t \in [\tau, \tau + \delta)$.

From this definition, a solution is in mode i at an instant τ, then it is in mode i at all instants in a time interval $[\tau, \tau + \delta)$.

Definition 5: The system (11) is called M-well posed if any nonzero C solution is in one and only one mode.

To characterize M-well posedness, we have the following result.

Lemma 3: If (11) is C-well posed, then a C solution starting from x_0 is in mode i if and only if $x_0 \in \mathcal{S}_i$.

Proof: (Sufficiency) If $x_0 \in \mathcal{S}_i$, then (13) holds for $t \in [0, \delta)$. When the system (11) is C-well posed, $e^{A_i t} x_0$ is a solution to (11) and in mode i.

(Necessity) If the system is C-well posed and a solution starting from x_0 is in mode i, then smooth continuation is possible in the ith mode, that is, (13) holds for $t \in [0, \delta)$. Along the same lines as in [7, proof of Lemma 2.3], $x_0 \in \mathcal{S}_i$.

Immediately, we have the following characterization.

Theorem 4: The system (11) is M-well posed on $\mathcal{R}^n$ if and only if

$$\bigcup_{i=1}^{m} \mathcal{S}_i = \mathcal{R}^n \tag{17}$$

$$\mathcal{S}_i \cap \mathcal{S}_j = \{0\}, \qquad \text{for all } i \neq j. \tag{18}$$

For bimodal systems with a single criterion, the above result can be simplified. As in [7], we define, for $i = 1, 2$

$$\mathcal{S}_i^+ = \{x | T_{A_i} x \succeq 0\}$$
$$\mathcal{S}_i^- = \{x | T_{A_i} x \preceq 0\}$$

in which T_{A_i} is the observability matrix of (C, A_i).

Theorem 5: For a bimodal system with a single criterion, the following statements are equivalent.

i) The system (11) is M-well posed.
ii) $\mathcal{S}_1^+ \cup \mathcal{S}_2^- = \mathcal{R}^n$ and $\mathcal{S}_1^+ \cap \mathcal{S}_2^- = 0$.
iii) Both the pairs (C, A_1) and (C, A_2) are observable, and $\mathcal{S}_1^+ \cup \mathcal{S}_2^- = \mathcal{R}^n$.
iv) Both the pairs (C, A_1) and (C, A_2) are observable, and $\mathcal{S}_1^+ \cap \mathcal{S}_1^- = 0$.
v) Both the pairs (C, A_1) and (C, A_2) are observable, and the system (11) is C-well posed.

Proof: Following the same line as [7, proof of Lemma 2.3], the equivalence of i) and ii) is easily proved. The equivalence of iii), iv). and v) is contained in [7, Th. 4.1].

Clearly, iii) implies ii). We only need to show that ii) implies that both the pairs (C, A_1) and (C, A_2) are observable.

Denote

$$m_i = \operatorname{rank} T_{A_i}$$

for $i = 1, 2$. Without loss of generality, assume $m_1 \geq m_2$, and denote T_{A_1} as $T_{A_1} = [T_{A_1 1}^T, T_{A_1 2}^T]^T$ with the $m_2 \times n$-dimensional matrix $T_{A_1 1}$ and $\operatorname{rank} T_{A_1 1} = m_2$.

From [7, Lemma 3.4], $\mathcal{S}_1^+ \cup \mathcal{S}_2^- = \mathcal{R}^n$ is equivalent to

$$\mathcal{S}_1^+ \cap \mathcal{S}_2^- = \{x \in \mathcal{R}^n | T_{A_1 1} x = 0, \, T_{A_1 2} x \succeq 0\}.$$

Therefore, i) implies that

$$\{x \in \mathcal{R}^n | T_{A_1 1} x = 0, \, T_{A_1 2} x \succeq 0\} = \{0\}. \tag{19}$$

This is the case only if $\operatorname{rank} T_{A_1 1} = n$, since otherwise, there is always a nonzero solution to

$$T_{A_1} x = 0$$

violating (19).

Thus, necessarily, $m_2 = m_1 = n$, both the pairs (C, A_1) and (C, A_2) are observable.

IV. Algorithm

To check conditions (15) and (16), if we denote as C_{i1} the matrix consisting of the first rows of the matrices $C_i^1, \ldots, C_i^{p_i}$, then necessarily,

$$\bigcup_{i=1}^{m} P(C_{i1}) = \mathcal{R}^n, \; P(C_{i1}) \cap P(C_{j1}) \subset \mathcal{K}_{ij}, \text{ or } \subset \ker C^{(1)} \tag{20}$$

for some nonzero vector $C^{(1)}$.

We note that $\bigcup_{i=1}^{m} P(C_{i1}) = \mathcal{R}^n$ holds if and only if for each row $C_{i1}^j, j = 1, \ldots, p_i$

$$\bigcup_{i=1}^{m} P(C_{i1}^j) = \mathcal{R}^n$$

or, equivalently

$$\bigcap_{i=1}^{m} P^o(-C_{i1}^j) = \emptyset. \tag{21}$$

We can then use *SubCone* to check $P(C_{i1}) \cap P(C_{j1}) \subset \mathcal{K}_{ij}$ and *EmptyCone* to check (21). We will call these the primary checks of the conditions (15) and (16).

If $P(C_{i1}) \cap P(C_{j1}) \subset \ker C^{(1)}$, we define

$$\mathcal{S}_{ijk}^{(1)} = \mathcal{S}_k \cap \ker C^{(1)}$$

then, (15) and (16) imply that

$$\bigcup_{k=1}^{m} \mathcal{S}_{ijk}^{(1)} = \ker C^{(1)} \quad \mathcal{S}_{ijk}^{(1)} \cap \mathcal{S}_{ijl}^{(1)} \subset \mathcal{K}_{kl} \cap \ker C^{(1)}.$$

We can identify the subspace $\ker C^{(1)}$ with $\mathcal{R}^{n^{(1)}}$ with a reduced dimension $n^{(1)} < n$. It is easy to verify that $\mathcal{S}_{ijk}$ is lexicographic cone on the subspace $\mathcal{R}^{n^{(1)}}$. Denote

$$\mathcal{K}_{ijkl}^{(1)} = \mathcal{K}_{kl} \cap, \ker C^{(1)}$$

then, we see that for each $i = 1, \ldots, m, j = 1, \ldots, m$

$$\bigcup_{k=1}^{m} \mathcal{S}_{ijk}^{(1)} = \mathcal{R}^{n^{(1)}} \tag{22}$$

$$\mathcal{S}_{ijk}^{(1)} \cap \mathcal{S}_{ijl}^{(1)} \subset \mathcal{K}_{ijkl}^{(1)}. \tag{23}$$

Equations (22) and (23) take the same form as (15) and (16), but on a subspace of a reduced dimension. So, we can repeat the whole process to give an algorithm to check (15) and (16).

For each $i = 1, \ldots, m$, $j = 1, \ldots, m$, the set of equations (22) and (23) is called a Fourier–Motzkin reduction of the set of equations (15) and (16).

We remark that the general procedure outlined above is similar in spirit to the one given in [7]. There are two differences however. The first difference is that we have a subroutine *SubCone* to check, e.g., $P(C_{i1}) \cap P(C_{j1}) \subset \mathcal{K}_{ij}$, even when $\mathcal{K}_{ij} \neq 0$. In [7], a degenerate LP problem was formulated for the case when $\mathcal{K}_{ij} = \{0\}$. The second difference is exactly the avoidance of the degenerate LP problem. The Fourier–Motzkin elimination is noticeably more efficient in the highly degenerate cases [5].

```
Algorithm
Step 1: Perform the primary checks for
  (15) and (16). If one of these checks
  can not pass, then stop.
Denote as FM_1 the collection of all
  Fourier–Motzkin reductions of (15) and
  (16).
Step k: For each member of FM_{k-1}, per-
  form the primary checks. If one of these
  checks can not pass, then stop.
Denote as FM_k the collection of all
  Fourier–Motzkin reductions of all mem-
  bers of FM_{k-1}.
```

If the algorithm proceeds to step $\max(q_i)$, then the conditions of Theorem 3 hold, and (11) is C-well posed.

Example: Consider the following three-modal system:

$$\begin{aligned} \dot{x} &= A_1 x, \qquad (x_1 - x_4,\, x_3)^T \succeq 0, \qquad x_4 \geq 0 \\ \dot{x} &= A_2 x, \qquad -x_1 + x_4 \geq 0, \qquad x_4 \geq 0 \\ \dot{x} &= A_3 x, \qquad -x_1 + x_4 \geq 0, \qquad -x_4 \geq 0 \end{aligned}$$

in which

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$A_2 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 2 & 0 & -1 \\ 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$A_3 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 2 & 0 & -1 \\ 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

It can be calculated that

$$\begin{aligned} \mathcal{S}_1 &= \{x | (x_1 - x_4,\, x_2,\, x_3)^T \succeq 0\} \cap \{x | x_4 \geq 0\} \\ \mathcal{S}_2 &= \{x | -(x_1 - x_4,\, x_2)^T \succeq 0\} \cap \{x | x_4 \geq 0\} \\ \mathcal{S}_3 &= \{x | -(x_1 - x_4,\, x_2)^T \succeq 0\} \cap \{x | -x_4 \geq 0\} \end{aligned}$$

and

$$\begin{aligned} \mathcal{K}_{12} &= \{x | x_1 - x_4 = 0,\, x_2 = 0\} \\ \mathcal{K}_{13} &= \{x | x_1 - x_4 = 0,\, x_2 = 0,\, x_4 = 0\} \\ \mathcal{K}_{23} &= \{x | x_4 = 0\}. \end{aligned}$$

It can be verified that the system is C-well posed. To check this via the algorithm, we illustrate the first step in the following, further steps can be carried out in the similar way.

We have

$$C_{11} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad C_{21} = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$C_{31} = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

and we need to check (21). Equation (21) is equivalent to

$$\{-x_1 + x_4 > 0,\, -x_4 > 0,\, x_1 - x_4 > 0,\, x_4 > 0\} = \emptyset.$$

Performing *EmptyCone*, one sees that it is indeed the case.

Since $P(C_{11}) \cap P(C_{21}) \subset \{x | x_1 - x_4 = 0\}$. Now, assuming $x_1 - x_4 = 0$, construct $\mathcal{S}_{kij}^{(1)}$ and moving to the next step, etc.

V. Conclusion

In this note, the results of [7] were generalized to piecewise-linear systems with multiple modes and multiple ceiteria. It was also shown that the conditions as presented in [7] for the multiple case correspond to a weak concept of the so-called "mode well posedness" defined in the note. To check these conditions, we presented new algorithmic procedures by making use of the famous Fourier–Motzkin elimination technique.

It should be pointed out that the algorithms *SubCone* and *EmptyCone* and, thus, the algorithm to check the necessary and sufficient conditions grow very fast in terms of the number of matrices of reduced dimensions. Though it is not the purpose of this note, to improve the efficiency of the algorithms should be a topic of future research. Existing literature on the numerical efficiency of the Fourier–Motzkin procedures and the double description method can be found in, e.g., [1] and [5].

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