Periodic Orbits From Δ-Modulation of Stable Linear Systems

Xiaohua Xia and Alan S. I. Zinober

Abstract—**The Δ-modulated control of a single input, discrete time, linear stable system is investigated. The modulation direction is given by $c^T x$ where $c \in R^n \backslash \{0\}$ is a given, otherwise arbitrary, vector. We obtain necessary and sufficient conditions for the existence of periodic points of a finite order. Some concrete results about the existence of a certain order of periodic points are also derived. We also study the relationship between certain polyhedra and the periodicity of the Δ-modulated orbit.**

Index Terms—**Delta-modulation, periodic points, sliding-mode control, polyhedra.**

I. Introduction

The determination of self-excited oscillations or limit cycles, originated in the work of Poincaré and Lyapunov, is an old and difficult problem in the classic qualitative theory of dynamical systems [7]. For discrete-time systems, the problem has been tackled from different angles, from counting the number of types of periodic orbits [6], the arithmetic of the number of periodic points [4], existence [10], and calculation [13] of the periodic points.

Hybrid systems resulting from the switching of controllers constitute a special class of nonlinear dynamical systems [12]. Though stability properties around a specific limit cycle/periodic orbits have been discussed [8], there are very few results on the existence and characterization of periodic points introduced by switchings. In [11], the existence of a globally attractive periodic behavior is proved for some switched flow networks. Periodic points arising from Δ-modulation have been characterized for scalar systems in [14], [15] and for a special class of higher order systems in [5]. Periodic orbits of different order have also been shown in [16] and [17] to exist when discretizing the equivalent control based sliding-mode controllers.

In this note, we investigate the Δ-modulated control of a single input, discrete time, linear stable system. The Δ-modulation is designed along a direction given by $c^T x$ where c is a given, otherwise arbitrary, vector in $R^n \backslash \{0\}$. We define a modulated orbit corresponding to an orbit of the feedback system. We prove that the periodicity of the system orbit is related intrinsically to the periodicity of the corresponding modulated orbit. Necessary and sufficient conditions are stated for the existence of periodic points of a finite order. Some concrete results about the existence of a certain order of periodic points are also obtained. The relationship between certain polyhedra and the periodicity of the Δ-modulated orbit is explored.

II. General Results

We consider a discrete-time control system of order n

$$x^+ = Ax + bu \tag{1}$$

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where $x \in R^n$ is the state, x^+ denotes the system state at the next discrete time step, $u \in R$ is the scalar input, A is an $n \times n$ matrix of real numbers, and b is a column vector of n real numbers. In this note, we assume that A is a stable matrix, i.e., the eigenvalues of A lie within the unit circle.

For any input sequence $\{u_0, u_1, \ldots\}$ and an initial state x_0, there is a corresponding orbit of system (1) $\{x^{(0)}, x^{(1)}, \ldots\}$, in which

$$x^{(0)} = x_0$$
$$x^{(i)} = Ax^{(i-1)} + bu_{i-1}$$

for $i = 1, 2, \ldots$. As usual, the orbit $\{x^{(0)}, x^{(1)}, \ldots\}$ is called periodic with period L, if there is a positive integer L such that $x^{(L-1)} = x^{(0)}$. The smallest such integer L is called a prime period, and we say that the orbit is L-periodic. Any point in a periodic orbit is called a periodic point.

The following result concerning periodic orbits from external periodic excitation is well known.

Theorem 1:

i) For a periodic input sequence of period L, there is a periodic orbit of period L for (1).
ii) This periodic orbit is globally attracting.

Now, we turn to the situation of Δ-modulated control of system (1). In this case, the control u is Δ-modulated feedback defined by

$$u = \operatorname{sgn}(c^T x) \tag{2}$$

in which $c \in R^n \backslash \{0\}$ is a fixed, otherwise arbitrary, modulation direction. Δ-modulation is a very robust scheme of modulation, a concept borrowed from communication. A great advantage of Δ-modulated feedback is that it needs only a bit of datum to implement the controller [1]. To avoid confusion, we define

$$\operatorname{sgn}(c^T x) = \begin{cases} 1, & \text{when } c^T x \geq 0 \\ -1, & \text{when } c^T x < 0. \end{cases}$$

Suppose $\{x_0, x_1, \ldots,\}$ is an orbit of the closed-loop system (1) and (2) starting from x_0. The sequence defined by $\{s_0, s_1, \ldots,\}$, where $s_i = \operatorname{sgn}(c^T x_i)$, for $i = 0, 1, \ldots$, is a *binary* sequence of 1's and -1's. We will call it a modulated orbit of the closed-loop system (1) and (2) corresponding to the orbit $\{x_0, x_1, \ldots\}$.

Obviously, the modulated orbit of a periodic orbit of the closed-loop system (1) and (2) is periodic. Therefore, to determine the periodicity of an orbit of a Δ-modulated system, from Theorem 1, it is decisive to see whether the Δ-modulation in (2) introduces a periodic *binary* sequence. This is addressed by the following theorem.

Theorem 2: The Δ-modulated system (1) and (2) has a periodic orbit of period L if and only if there are $\sigma_0, \sigma_1, \ldots, \sigma_{L-1} \in \{-1, 1\}$ such that

$$\begin{cases} c^T(I - A^L)^{-1} \sum_{j=0}^{L-1} A^{L-j-1} b\sigma_{i+j} \geq 0, & \text{when } \sigma_i = 1 \\ c^T(I - A^L)^{-1} \sum_{j=0}^{L-1} A^{L-j-1} b\sigma_{i+j} < 0, & \text{when } \sigma_i = -1 \end{cases} \tag{3}$$

for $i = 0, 1, \ldots, L-1$, in which $\sigma_{i+j} = \sigma_{(i+j) \bmod L}$.

Proof: (Necessity) If $\{x_0, x_1, \ldots\}$ is a periodic orbit with period L, then denote

$$\sigma_i = s_i = \operatorname{sgn}(c^T x_i)$$

for $i = 0, 1, \ldots, L-1$. Since $\{x_0, x_1, \ldots\}$ is periodic with a period L, we can obtain

$$x_i = (I - A^L)^{-1} \sum_{j=0}^{L-1} A^{L-j-1} b\sigma_{i+j}$$

for $i = 0, 1, \ldots, L-1$. Hence

$$c^T (I - A^L)^{-1} \sum_{j=0}^{L-1} A^{L-j-1} b\sigma_{i+j} = c^T x_i$$

which implies the conditions of the theorem. □

(Sufficiency) Denote

$$x^{(i)} = (I - A^L)^{-1} \sum_{j=0}^{L-1} A^{L-j-1} b\sigma_{j+i} \tag{4}$$

and

$$f(x) = Ax + b \,\mathrm{sgn}\,(c^T x).$$

It is straightforward to verify that under the conditions of the theorem

$$f^{(i)}(x^{(0)}) = x^{(i)}$$

for $i = 0, 1, \ldots, L-1$, and

$$f^{(L)}(x^{(0)}) = x^{(0)}$$

i.e., the orbit starting at $x^{(0)}$ has period L.

A Δ-modulated system can have many periodic points. The first interesting result is the following.

Corollary 1:

i) If (A, b) is controllable, then there is a $c \in R^n$ such that the closed-loop system (1) and (2) has n-periodic orbits.
ii) If (c^T, A) is observable, then there is a $b \in R^n$ such that the closed-loop system (1) and (2) has n-periodic orbits.

Proof: We prove i) only. Interested readers can work out the proof for ii) similarly.

The controllability of (A, b) implies [9] the existence of the inverse of

$$(A^{n-1}b \ldots Ab\, b)^{-1}.$$

For $n \geq 1$, we can therefore choose

$$c^T = (1, 0, \ldots, 0)(A^{n-1}b \ldots Ab\, b)^{-1}(I - A^n)$$

then for any *binary* sequence $\{s_0, s_1, \ldots, s_{n-1}\}$

$$c^T (I - A^n)^{-1}(A^{n-1}bs_0 + \cdots + Abs_{n-2} + bs_{n-1}) = s_0.$$

The inequalities in (3) automatically hold. By Theorem 2, for this choice of c, any n binary sequence gives rise to an orbit of period n.

Choose a sequence $s_0 = 1, s_i = -1$, for $i = 1, \ldots, n-1$, according to (4), the periodic orbit generated by it consists of the following n points:

$$x^{(i)} = (I - A^n)^{-1} \sum_{j=0}^{n-1} A^{n-j-1} bs_{j+i}.$$

We show that these n points are different, therefore this orbit is n-periodic.

To this end, we will prove that $x^{(0)}, x^{(1)}, \ldots, x^{(n-1)}$ are linearly independent. As a matter of fact, if we denote the following matrix:

$$\Gamma = \begin{bmatrix} 1 & -1 & \cdots & -1 \\ -1 & 1 & \cdots & -1 \\ & & \ddots & \\ -1 & -1 & \cdots & 1 \end{bmatrix}$$

then it is routine to check that

$$\left(x^{(0)}, x^{(n-1)}, x^{(n-2)}, \ldots, x^{(1)}\right) = (I - A^n)^{-1}(A^{n-1}b \cdots Ab\, b)\Gamma.$$

For $n \geq 3, \Gamma$ is invertible and one can derive

$$\Gamma^{-1} = \begin{bmatrix} \frac{n-3}{2n-4} & -\frac{1}{2n-4} & \cdots & -\frac{1}{2n-4} \\ -\frac{1}{2n-4} & \frac{n-3}{2n-4} & \cdots & -\frac{1}{2n-4} \\ & & \ddots & \\ -\frac{1}{2n-4} & -\frac{1}{2n-4} & \cdots & \frac{n-3}{2n-4} \end{bmatrix}.$$

Therefore, $x^{(0)}, x^{(1)}, \ldots, x^{(n-1)}$ are linearly independent, and the proof is complete.

Of course, a Δ-modulated feedback system can have periodic orbits with periods higher than the dimension of the system. We use the following two-dimensional (2-D) example to show that there can be very "large" periods.

Consider a 2-D system

$$\begin{aligned} x_1^+ &= \lambda_1 x_1 + \mathrm{sgn}(c^T x) \\ x_2^+ &= \lambda_2 x_2 + \mathrm{sgn}(c^T x) \end{aligned}$$

where $|\lambda_1| < 1, |\lambda_2| < 1$. Given any n, it can be verified that the following construction of c gives a $2n$-periodic orbit:

$$\begin{aligned} c^T = \Big(\left(1 - \lambda_2^{n-1}\right) / \left((1 + \lambda_2^n)(1 - \lambda_2)\right), \\ - \left(1 - \lambda_1^{n-1}\right) / \left((1 + \lambda_1^n)(1 - \lambda_1)\right) \Big) \end{aligned}$$

and the $2n$-periodic orbit starts from

$$\left((1 - \lambda_1^n) / \left((1 + \lambda_1^n)(1 - \lambda_1)\right), (1 - \lambda_2^n) / \left((1 + \lambda_2^n)(1 - \lambda_2)\right) \right).$$

Fig. 1 shows three orbits generated in this way for $\lambda_1 = 0.5, \lambda_2 = 0.8196$.

Criterion (3) is useful in deriving concrete results about the existence of periodic orbits of a certain order.

Proposition 1:

i) System (1) under the Δ-modulation of (2) has a fixed (1-periodic) point if and only if

$$c^T (I - A)^{-1} b \geq 0.$$

ii) System (1) under the Δ-modulation of (2) has a 2-periodic orbit if and only if

$$c^T (I + A)^{-1} b < 0.$$

iii) System (1) under the Δ-modulation of (2) has a 3-periodic orbit if and only if

$$\begin{aligned} 2 \max\{c^T (I - A^3)^{-1} b, c^T (I - A^3)^{-1} Ab\} \\ < c^T (I - A)^{-1} b \leq 2c^T (I - A^3)^{-1} A^2 b. \end{aligned}$$

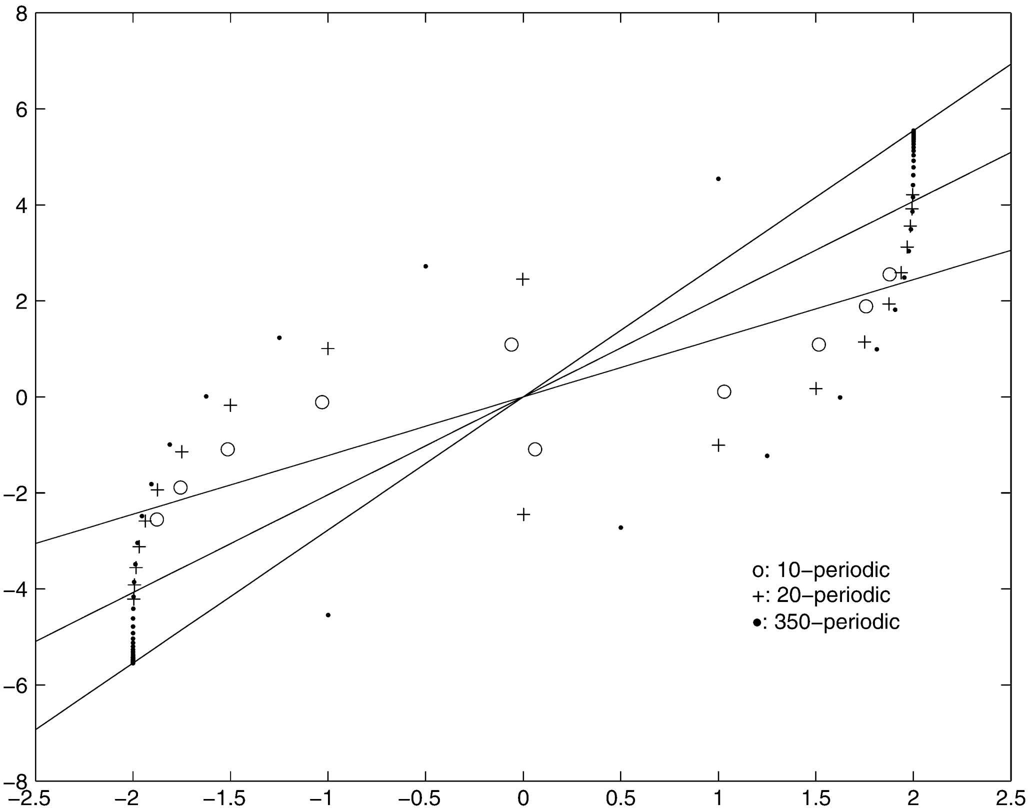

Fig. 1. 10-, 20-, and 350-periodic orbits.

iv) System (1) under the Δ-modulation of (2) has a 4-periodic orbit if and only if

$$2\max_{0\le i\le 2} c^T(I-A^4)^{-1}A^i b < c^T(I-A)^{-1}b \le 2c^T(I-A^4)^{-1}A^3 b$$

or

$$c^T(I-A)^{-1}b > 2\max\{c^T(I-A^4)^{-1}(A+I)b \\ c^T(I-A^4)^{-1}(A^2+A)b\}.$$

Proof: Item i) is a special case of (3) when $L=1$. To prove ii), note that the only 2-periodic binary sequence is $\{1,-1\}$ ($\{-1,1\}$ is regarded as equivalent to $\{1,-1\}$). By invoking (3), we have

$$c^T(I-A^2)^{-1}(A-I)b > 0$$

which is equivalent to

$$c^T(I+A)^{-1}b < 0.$$

Similarly, for the case iii) the only 3-periodic binary sequence is $\{1,-1,-1\}$. By invoking (3), we have

$$c^T(I-A^3)^{-1}(A^2-A-I)b \ge 0$$
$$c^T(I-A^3)^{-1}(A^2+A-I)b > 0$$
$$c^T(I-A^3)^{-1}(A^2-A+I)b > 0.$$

These inequalities can be rewritten as stated in iii).

In item iv), there are only two 4-periodic binary sequences $\{1,-1,-1,-1\}$ and $\{1,-1,-1,1\}$. Along similar lines, one can get the inequalities in iv). □

III. Polyhedra and Periodicity

In this section, we study interesting properties of some polyhedra and their relationship to the *maximal length* of a periodic modulated orbit. By the maximal length of a binary sequence, we mean the maximal number of consecutive ones or minus ones. For example, the 4-periodic sequence $\{1,-1,-1,-1\}$ has a maximal length of 3, while $\{1,-1,-1,1\}$ has a maximal length of 2.

Define

$$S_0^+ = \{x \mid c^T x \ge 0\}$$
$$\vdots$$
$$S_k^+ = S_{k-1}^+ \cap \{x \mid c^T A^k x + c^T(A^{k-1}+A^{k-2}+\cdots+A+I)b \ge 0\}$$

and, similarly

$$S_0^- = \{x \mid c^T x < 0\}$$
$$\vdots$$
$$S_k^- = S_{k-1}^- \cap \{x \mid c^T A^k x + c^T(A^{k-1}+A^{k-2}+\cdots+A+I)b < 0\}.$$

Lemma 1:

i) For $k = 1, 2, \ldots$

$$S_k^+ = \{c^T A x + c^T b \ge 0 \mid x \in S_{k-1}^+\}$$

and

$$S_k^- = \{c^T A x - c^T b < 0 \mid x \in S_{k-1}^-\}.$$

ii) If for some $k \ge 0, S_{k+1}^+ = S_k^+ (S_{k+1}^- = S_k^-)$, then $S_j^+ = S_k^+ (S_j^- = S_k^-)$, for $j \ge k$.

Proof: Item i) is implied by definition of S_k^+ and S_k^-.

Note that S_k^+'s are polyhedra. We make use of Farkas' Lemma [2] to prove ii).

If $S_{k+1}^+ = S_k^+$, then by [2, Th. 4.7], there is a *nonnegative* vector $p = (p_0, p_1, \ldots, p_{k-1})^T, p_i \geq 0$, for $i = 0, 1, \ldots, k-1$, such that

$$c^T A^k = p^T \begin{pmatrix} c^T \\ c^T A \\ \vdots \\ c^T A^{k-1} \end{pmatrix}$$

$$p^T \begin{pmatrix} 0 \\ c^T b \\ \vdots \\ c^T(A^{k-3} + \cdots + A + I)b \\ c^T(A^{k-2} + \cdots + A + I)b \end{pmatrix} \leq c^T(A^{k-1} + \cdots + A + I)b.$$

It can be easily verified that

$$c^T A^{k+1} = p^T \begin{pmatrix} c^T A \\ c^T A^2 \\ \vdots \\ c^T A^{k-1} \\ c^T A^k \end{pmatrix} = \tilde{p}^T \begin{pmatrix} c^T \\ c^T A \\ \vdots \\ c^T A^{k-1} \end{pmatrix}$$

where

$$\tilde{p}^T = \left(p_{k-1}p_0, p_0 + p_{k-1}p_1, \ldots, p_{k-3} + p_{k-1}p_{k-2}, p_{k-2} + p_{k-1}^2\right).$$

Clearly, $\tilde{p}$ is a nonnegative vector.

To proceed, note that

$$\begin{aligned} & c^T(A^k + \cdots + A + I)b \\ &= c^T A^k b + c^T(A^{k-1} + \cdots + A + I)b \\ &\geq p_0 c^T b + p_1 c^T Ab + \cdots + p_{k-1} c^T A^{k-1} b \\ &\quad + p_1 c^T b + \cdots + p_{k-1}(c^T(A^{k-2} + \cdots + A + I)b) \\ &= p_0 c^T b + p_1 c^T(A + I)b + \cdots \\ &\quad + p_{k-2}(c^T(A^{k-2} + \cdots + A + I)b) \\ &\quad + p_{k-1}(c^T(A^{k-1} + \cdots + A + I)b) \\ &\geq p_0 c^T b + p_1 c^T(A + I)b + \cdots \\ &\quad + p_{k-2}(c^T(A^{k-2} + \cdots + A + I)b) \\ &\quad + p_{k-1}p_1 c^T b + p_{k-1}p_2 c^T(A + I)b + \cdots \\ &\quad + p_{k-1}^2(c^T(A^{k-2} + \cdots + A + I)b) \\ &= \tilde{p}^T \begin{pmatrix} 0 \\ c^T b \\ \vdots \\ c^T(A^{k-3} + \cdots + A + I)b \\ c^T(A^{k-2} + \cdots + A + I)b \end{pmatrix}. \end{aligned}$$

This, again by [2, Th. 4.7], proves that $S_{k+2}^+ = S_k^+$. A mathematical induction argument shows that $S_j^+ = S_k^+$, for all $j \geq k$.

Following completely similar lines, we can prove the case for S_k^-.

Then, we can define

$$S_\infty^+ = \bigcap_i S_i^+$$

and

$$S_\infty^- = \bigcap_i S_i^-.$$

If P is a polyhedron, denote $AP + b$ as

$$AP + b = \{Ax + b \mid x \in P\}.$$

By the definition of a polyhedron and [2, Cor. 2.5], $AP + b$ is a polyhedron.

Firstly, we have the following simple result.

Proposition 2: If the Δ-modulated system (1) and (2) has a fixed point, then

$$(AS_0^+ + b) \cap S_\infty^+ \neq \emptyset.$$

Proof: When there is a fixed point, there is always a fixed point x satisfying $c^T x \geq 0$. In fact, when y satisfying $c^T y < 0$ is a fixed point, then $y = Ay - b$ and, therefore, $-y = A(-y) + b$. So, $-y$ is a fixed point in S_0^+.

It is then easy to see that

$$x \in \left(AS_0^+ + b\right) \cap S_\infty^+.$$

Proposition 3:

a) If there is a finite integer k such that $S_k^+ = \emptyset$, then the maximal length of any periodic modulated orbit is smaller than k.
b) If there is a finite integer k such that $(AS_0^+ + b) \cap S_k^- = \emptyset$, then the maximal length of any periodic modulated orbit is smaller than k.

Proof:

i) By definition of S_k^+, any orbit starting from S_0^+ can only stay in S_0^+ for at most k times. Also, by symmetry, $S_k^- = \emptyset$ when $S_k^+ = \emptyset$. Therefore, any periodic modulated orbit cannot have the same sign for more than k times.
ii) A periodic modulated orbit with a maximal length 1 corresponds to a fixed point (1-periodic). For any periodic orbit with a periodic greater than 1, there are points in both S_0^+ and S_0^-. Suppose $\bar{x}$ is a periodic point in S_0^+ followed by a point $y = Ax + b$ in S_0^-. By assumption, $y \notin S_k^-$, y cannot be followed by more than $k-1$ points in S_0^-. This proves that the maximal length of consecutive minus ones is less than k. By symmetry, the maximal length of consecutive ones is also less than k.

IV. CONCLUDING REMARKS

We have derived necessary and sufficient conditions for the existence of periodic points of a finite-order arising from Δ-modulated control along an arbitrary direction of a single-input, discrete-time, linear stable system. Some definite results about the existence of a certain order of periodic points are also obtained. The relationship between certain polyhedra and the periodicity of the Δ-modulated orbit is also explored.

Some of the results can be extended to the multiple-input case and this is currently under investigation.

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A Gramian-Based Controller for Linear Periodic Systems

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***Abstract*—This note proposes a new design method for the control of linear time-periodic systems. The method is based on the reachability Gramian and a specific form for the feedback gain matrix to build a novel control law for the closed-loop system. The new controller allows assignment of all the invariants of the system. Calculating the feedback requires solving a matrix integral equation for the periodic Floquet factor of the state-transition matrix of the closed-loop system.**

***Index Terms*—Floquet, Gramian, invariant factors, periodic feedback, periodic systems.**

I. Introduction

Linear time-invariant (LTI) systems are the most common way of analyzing engineering processes. Consequently, they have been extensively studied, and many different strategies have been developed over the years for their control. Yet, modeling real-world processes often leads to a linear time-periodic (LTP) system; see, e.g., [1] and the references therein.

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Unfortunately, results established for LTI systems do not usually hold for time-varying systems. LTP systems are an exception in that they all exhibit similar behavior, thus forming a unified class. Moreover, several aspects of Floquet–Lyapunov theory for LTP systems have connections with LTI systems, raising the prospect of being able to take advantage of this well-established body of knowledge.

A. Notation and Definitions

Let $\mathbb{R}(\mathbb{R}^n)[\mathbb{R}^{m\times/n}]$ denote the real field (space of real n-vectors) [set of real matrices with m rows and n columns], $\mathbb{C}(\mathbb{C}^n)[\mathbb{C}^{m\times n}]$ denote the complex field (space of complex n-vectors) [set of complex matrices with m rows and n columns], $\mathbb{N}$ denote the set $\{1, 2, \ldots\}$, $\mathbf{I}$ denote the identity matrix of order n, and superscript $T(-1)$ denote matrix transpose (inverse). Consider the continuous-time system described by the differential equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \tag{1}$$

and its corresponding uncontrolled form

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) \tag{2}$$

where $\mathbf{A}(\,\cdot\,) \in \mathbb{R}^{n\times n}, \mathbf{B}(\,\cdot\,) \in \mathbb{R}^{n\times r}$ are piecewise continuous, T-periodic matrix functions. Denote by $\mathbf{\Phi}(\cdot, 0)$ the state-transition matrix (STM) of (2). The matrix $\mathbf{\Phi}(T, 0)$ is called the *monodromy matrix*.

B. Floquet Theory

We give the main results and refer to [2] and [3] for a complete treatment. For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ define the following set of matrix functions:

$$\begin{aligned} \mathcal{L}_T^{\mathbb{K}} = \{&\mathbf{L}(\,\cdot\,) : \mathbb{R} \to \mathbb{K}^{n\times n} : \\ &\mathbf{L}(0) = \mathbf{I}, \mathbf{L}(t+T) = \mathbf{L}(t), \det\mathbf{L}(t) \neq 0 \qquad \forall t \\ &\mathbf{L}(\,\cdot\,)\text{absolutely continuous with} \\ &\text{a piecewise-continuous derivative}\}. \end{aligned}$$

Theorem 1: (Representation Theorem) The STM $\mathbf{\Phi}(\cdot, 0)$ of system (2) can be factored as

$$\mathbf{\Phi}(t, 0) = \mathbf{L}(t)\exp(t\mathbf{F}), \text{where } \mathbf{L}(\,\cdot\,) \in \mathcal{L}_T^{\mathbb{C}}, \mathbf{F} \in \mathbb{C}^{n\times n}. \tag{3}$$

Theorem 2: (Reducibility) The Lyapunov transformation $\mathbf{x}(t) = \mathbf{L}(t)\mathbf{z}(t)$ transforms the original LTP system into an LTI system $\dot{\mathbf{z}}(t) = \mathbf{F}\mathbf{z}(t)$, where $\mathbf{L}(\,\cdot\,)$ and $\mathbf{F}$ are the same as those in (3).

One disadvantage of Theorems 1 and 2 is that the *Floquet factors* $\mathbf{L}(t)$ and $\mathbf{F}$ may be complex even if $\mathbf{\Phi}(T, 0)$ is real. It is well known (see e.g., [4]) that it is always possible to obtain real Floquet factors by treating (2) as having $2T$-periodic coefficients. However, in this case calculations must be made over two periods. Recently, [5] and [6] demonstrated how to obtain a real representation from computations performed on one period by generalizing a result from [3]. The two main results are reproduced as follows.

Theorem 3: Consider (2) and let $\mathbf{\Phi}(\cdot, 0)$ be its (real) state-transition matrix. Let $\mathbf{Y} \in \mathbb{R}^{n\times n}$ such that (i) $\mathbf{Y\Phi}(T, 0)$ has a real logarithm; (ii) $\mathbf{\Phi}^k(T, 0) = [\mathbf{Y\Phi}(T, 0)]^k$ for some positive integer k. Then, for any $\mathbf{F}_Y \in \mathbb{R}^{n\times n}$ satisfying $\exp(T\mathbf{F}_Y) = \mathbf{Y\Phi}(T, 0)$, the real