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Parameter Identifiability of Nonlinear Systems With Time-Delay

J. Zhang, X. Xia, and C. H. Moog

***Abstract*—In this note, various parameter identifiability concepts for nonlinear systems with time-delay are defined, complete characterizations of these concepts as well as easily checkable criteria are provided. It is proved that geometric identifiability is equivalent to identifiability with known initial conditions, algebraic identifiability implies geometric identifiability. As for identifiability with partially known initial conditions, an easy characterization is also provided.**

***Index Terms*—Identifiability, linear algebraic approach, nonlinear systems, time-delay.**

I. Introduction

In this note, we consider the parameter identifiability problem of a nonlinear system with time-delay

$$\Sigma_\theta : \begin{cases} \dot{x} &= f(x(t-i), \theta, u(t-j) : i, j \in S_-) \\ y &= h(x(t-i), \theta, u(t-j) : i, j \in S_-) \\ x(t) &= x_0(t) \qquad \forall t \in [-s, 0] \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$, and $\theta \in R^q$ is the parameter. This problem grows out of the same problem for systems without time-delay (see [9]–[11] for historical account and some recent results), and it has major applications for time-delay systems, for example, the identification of the death rate in the SIS epidemic model with maturation delay (see [2]) as displayed in Section IV.

The study of the identifiability of control systems with time-delay has been scarce, and limited to linear systems with time-delay. In [8], [7], [1], aspects of identifiability of linear time-delay system such as system parameters, transfer function coefficients as well as time-delays are brought forward. Our results in this note are only concerned about parameter identifiability of nonlinear systems with time-delay. As discussions in [7] and [1] reveal, the parameter identifiability of a linear time-delay system is itself an intricate problem. For nonlinear time-delay systems, the problem remains largely open and difficult. On the other hand, a generic version of the problem lends itself to complete characterizations in light of the algebraic framework developed in [12] and the rigorous approach taken in [11] for nonlinear systems without time-delay.

In this note, various parameter identifiability concepts for nonlinear systems with time-delay are defined, complete characterizations of these concepts and easily checkable criteria for all of them are obtained.

Specifically, the results of [11] are generalized to nonlinear systems with time-delay. Similar results as in [11] are established. That is, we prove that geometric identifiability is equivalent to identifiability with known initial conditions, algebraic identifiability implies geometric identifiability, and some easy criteria for the two kinds of identifiability. As for identifiability with partially known conditions, an easy characterization is also provided.

It is worthy to note that there are fundamental differences between the systems with and without time-delay. For example, Theorems 2 and 3 are not direct generalizations of the corresponding results in [11], since extra operations depending the delay operator have to be done.

The note is organized as follows. In Section II, we give some definitions. The main results are established in Section III. Section IV is devoted to examples. The last section offers some concluding remarks.

II. Definitions

To make things more precise, assume that in the system (1), the functions f and h are meromorphic functions which are defined as the quotients of convergent power series with real coefficients. The integer s is nonnegative, and the set $S_- := \{0, 1, \ldots, s\}$ is a finite set of constant time delays, and

$$\begin{aligned} &f(x(t-i), \theta, u(t-j), i, j \in S_-) \\ &:= f(x(t), x(t-1), \ldots, x(t-s), \theta, u(t), u(t-1), \ldots, u(t-s)). \end{aligned}$$

The function x_0 denotes a continuous function of initial condition. Assume that $\operatorname{rank}(\partial h/\partial x) = p$, that is, for any fixed θ, and u in some open sets, the p components of h are independent functions of x and its shifts. The variable θ is the parameter to be identified and it is assumed to belong to $\mathcal{P}$ which is an open subset of $\mathbb{R}^q$. Moreover, without loss of generality, x_0 is assumed to be independent of θ and u. Denote by $\mathcal{M} := C[-s, 0]$ the set of initial functions on $[-s, 0]$.

For any open subset $U \subseteq \mathbb{R}^m$, an admissible input function $u(t) : [-s, T] \to U$ is defined to be an input on $[-s, T]$ such that the differential equation in (1) admits a unique (local) solution. For any initial function x_0 and an admissible input $u(t)$ on $[-s, T]$, there exists a parameterized solution $x(t, \theta, x_0, u)$ on some interval $[-s, \bar{T}]$, $\bar{T} \leq T$. Denote the corresponding output by $y(t, \theta, x_0, u)$. The following definitions are generalizations of the corresponding ones in [11].

Definition 1: The system Σ_θ is said to be x_0-identifiable at θ through an admissible input u (on $[-s, T]$) if there exists an open set $\mathcal{P}^0 \subset \mathcal{P}$ containing θ such that for any two distinct $\theta_1, \theta_2 \in \mathcal{P}^0$, the solutions $x(t, \theta_1, x_0, u)$ and $x(t, \theta_2, x_0, u)$ exist on $[-s, \epsilon]$, $0 < \epsilon \leq T$, and their corresponding outputs satisfy $y(t, \theta_1, x_0, u) \neq y(t, \theta_2, x_0, u)$ on $t \in [-s, \epsilon]$.

Now, consider the generic property of identifiability. The same topology for the input function spaces as in [11] is used in this note,

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that is, for any $T > 0$ and an integer $N > 0$, the function space $C^N[-s,T]$ and its topology, the set $C^N_{\mathcal{U}}[-s,T]$ of all admissible inputs on $[-s,T]$, the topology of $C^N_{\mathcal{U}}[-s,T] \times C^N_{\mathcal{U}}[-s,T]$, and the topology of M-fold product $(C^N_{\mathcal{U}}[-s,T])^M$, are all defined exactly as [11]. Define $W_k := \mathcal{P} \times \mathcal{M} \times C^k_{\mathcal{U}}[-s,T]$ and a natural map Ψ from W_k to $\mathcal{P} \times (C[-s,T])^{(k+1)^2 p} \times (C[-s,T])^{(k+1)^2 m}$ such that

$$(\theta, x_0, u) \mapsto (\theta, y^{(i)}(t-j), u^{(i)}(t-j), \qquad i,j = 0,1,\ldots,k).$$

Then Ψ is a continuous map under the topology defined previously.

Definition 2: The system Σ_θ is said to be geometrically identifiable if there exist a $T > 0$, an integer $k \geq 0$, an open subset S_1 of W_k, a function ϕ which is meromorphic in its arguments, such that $\theta = \phi(y^{(i)}(t-j), u^{(i)}(t-j), x(t-j) : i,j = 0,1,\ldots,k)$ holds for all $(\theta, x_0, u) \in S_1$.

The following one is slightly different from that of [11] by removing the dense condition.

Definition 3: The system Σ_θ is said to be algebraically identifiable if there exist an integer $k \geq 0$, a $T > 0$, an open subset S_1 of W_k, a meromorphic function ϕ, such that

$$\theta = \phi(y^{(i)}(t-j), u^{(i)}(t-j), \qquad i,j = 0,1,\ldots,k) \tag{2}$$

holds for all $(\theta, x_0, u) \in S_1$.

An example in Section IV will show that the geometric identifiability is a weaker concept than the algebraic identifiability.

Definition 4: The system Σ_θ is said to be identifiable with known initial conditions if there exist a $T > 0$, an integer $k \geq 0$, an open subset S_2 of W_k, and a meromorphic function ϕ such that

$$\theta = \phi(x_0(-i_1^+), u^{(j)}(-i_2^+), y^{(j)}(-i_2^+), \qquad i_1 \in S_-, i_2, j = 0,1,\ldots,k)$$

holds for all $(\theta, x_0, u) \in S_2$, where the notation i_1^+ means the right limit of i_1.

III. Main Results

A. Notations

The notations and linear algebraic tools in [12] are used to build up the main results. Let $\mathcal{K}$ be the field of meromorphic functions of a finite number of variables in the set $\mathcal{C} = \{x(t-i), \theta, u^{(j)}(t-i) : i,j \in \mathbb{Z}_{\geq 0}\}$, where $\mathbb{Z}_{\geq 0}$ denotes the set of nonnegative integers. For simplicity, $u^{(j)}(t-i)$ is also denoted by $D^j\delta^i u(t)$, where $D = \mathrm{d}/\mathrm{d}t$. For any matrix $A = (a_{ij}(t))$, define $\delta(A)$ to be the matrix with (i,j)-element $\delta(a_{ij}(t)) = a_{ij}(t-1)$. Let $\mathcal{K}(\delta]$ be the noncommutative ring which is defined as the set of polynomials in δ with coefficients in $\mathcal{K}$ (see [12]). Define $E = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\mathcal{K}\}$, which is the set of linear combinations of a finite number of one-forms from $\mathrm{d}x(t-i), \mathrm{d}\theta, \mathrm{d}u^{(j)}(t-i)$, $\mathrm{d}y^{(j)}(t-i)$ with row vector coefficients whose elements are in $\mathcal{K}$. The elements in E are called one-forms.

The differentiation of a function $\phi(x(t-i_1), \theta, u^{(j)}(t-i_2) : i_1, i_2 \in S_-; j = 0,1,\ldots,k)$ and a one-form $\omega = \sum_i \kappa_x^i \mathrm{d}x(t-i) + \kappa_\theta \mathrm{d}\theta + \sum_{ij} \eta_i \mathrm{d}u^{(j)}(t-i) \in E$ along the dynamics of the system (1) are defined as

$$\begin{aligned}
\dot{\phi} =& \sum_{i_1=0}^{s} \frac{\partial \phi}{\partial x(t-i_1)} \delta^{i_1} f \\
&+ \sum_{i_2=0}^{s} \sum_{j=0}^{k} \frac{\partial \phi}{\partial u^{(j)}(t-i_2)} u^{(j+1)}(t-i_2) \\
\dot{\omega} =& \sum_i \dot{\kappa}_x^i \mathrm{d}x(t-i) + \dot{\kappa}_\theta \mathrm{d}\theta + \sum_{ij} \dot{\eta}_i \mathrm{d}u^{(j)}(t-i) \\
&+ \sum_i \kappa_x^i \mathrm{d}\delta^i f + \sum_{ij} \eta_i \mathrm{d}u^{(j+1)}(t-i)
\end{aligned}$$

where

$$\begin{aligned}\delta^{i_1} f =& f(x(t-i_3), \theta, u(t-i_2) \\ & : i_2, i_3 = i_1, i_1+1, \ldots, i_1+s).\end{aligned}$$

It is important to note that $\delta(\theta) = \theta$, $g(\delta)(\theta) = g(1)(\theta) = g(1)\theta$ for any $g(\delta) \in \mathcal{K}(\delta]$.

B. Algebraic Identifiability

Denote $\mathcal{Y} = \mathrm{span}_{\mathcal{K}(\delta]}\{\mathrm{d}y^{(j)} : j \in \mathbb{Z}_{\geq 0}\}$, $\mathcal{X} = \mathrm{span}_{\mathcal{K}(\delta]}\{\mathrm{d}x\}$, $\mathcal{U} = \mathrm{span}_{\mathcal{K}(\delta]}\{\mathrm{d}u^{(j)} : j \in \mathbb{Z}_{\geq 0}\}$, $\Theta = \mathrm{span}_{\mathcal{K}(\delta]}\{\mathrm{d}\theta\}$, where $\mathcal{Y}$, $\mathcal{X}$, $\mathcal{U}$ and Θ are the linear combinations of the generators with row vector coefficients whose elements are in $\mathcal{K}$. The notation $\mathcal{X} + \Theta$ means the span of $\{\mathrm{d}x, \mathrm{d}\theta\}$ with suitable row vector coefficients.

Theorem 1: The system is algebraically identifiable if and only if $\mathrm{d}\theta \in \mathcal{Y} + \mathcal{U}$ holds for all $(\theta, x_0, u) \in S_1$, where S_1 is an open subset of some W_k and W_k is defined as in Section II.

Proof: One only needs to consider the sufficiency. By $\mathrm{d}\theta \in \mathcal{Y} + \mathcal{U}$, one has

$$\mathrm{d}\theta = \sum_{i,j=0}^{k} (a_{ij} D^i \delta^j \mathrm{d}y + b_{ij} D^i \delta^j \mathrm{d}u) \tag{3}$$

where a_{ij}, b_{ij} are functions in $\mathcal{T} := \{D^i\delta^j y, D^i\delta^j u : i,j = 0,1,\ldots,k\}$. Without loss of generality suppose that the functions in $\{\theta, D^i\delta^j y, D^i\delta^j u : i,j = 0,1,\ldots,k\}$ are independent and the equality (3) holds on a subset of $\mathbb{R}^{n_1}$, where $n_1 = (k+1)^2(p+m) + q$. Then it must hold on an open subset $U \subseteq \mathbb{R}^{n_2}$ by the implicit function theorem, where $q < n_2 \leq n_1$ and U is holomorphic to $\mathbb{R}^{n_2}$. The set U corresponds to a set of functions $\{\theta, D^{i_1}\delta^{i_2} y, D^{i_3}\delta^{i_4} u : i_j \in I_j, j = 1,2,3,4\}$ where I_j is an index set with cardinality $|I_j|$ and $|I_1| + |I_2| + |I_3| + |I_4| = n_2 - q$. Now define a similar map Ψ' as in Section II

$$\begin{aligned}\Psi' : (\theta, x_0, u) \in W_k \mapsto& (\theta, D^{i_1}\delta^{i_2} y, D^{i_3}\delta^{i_4} u \\ & : i_j \in I_j, \qquad j = 1,2,3,4).\end{aligned}$$

Then, Ψ' is continuous with respect to the underlying topology. Thus, $(\Psi')^{-1}(U)$ is open in W_k. By the Poincaré Lemma, there exists a meromorphic function ϕ such that $\theta = \phi(D^{i_1}\delta^{i_2} y, D^{i_3}\delta^{i_4} u : i_j \in I_j, j = 1,2,3,4)$ holds for all $(\theta, x_0, u) \in (\Psi')^{-1}(U)$. QED

In the following, an easily checkable criterion for algebraic identifiability is provided after some preliminaries.

By the condition $\mathrm{rank}\,(\partial h/\partial x) = p$, the function h in (1) satisfies $\mathrm{d}h = H_1(\delta)\mathrm{d}x + H_2\mathrm{d}\theta + H_3(\delta)\mathrm{d}u$, where the elements of $H_1(\delta)$ and $H_3(\delta)$ are in $\mathcal{K}(\delta]$, and $H_1(\delta)$ is of full-row rank in the sense that the rows of it are independent over $\mathcal{K}(\delta]$. As done in [11], define the so-called observability indexes for system (1). That is, let $\mathcal{F}_k := \mathcal{X} \cap \left(\mathrm{span}_{\mathcal{K}(\delta]}\{\mathrm{d}y, \mathrm{d}\dot{y}, \ldots, \mathrm{d}y^{(k-1)}\} + \mathcal{U} + \Theta\right)$ for $k = 1,\ldots,n$. It is important to note that $\mathcal{F}_n = \mathcal{X} \cap (\mathcal{Y} + \mathcal{U} + \Theta)$ and $\mathrm{rank}_{\mathcal{K}(\delta]}\mathcal{F}_n \leq n$, where the definition of the rank of a module over a noncommutative ring can be found in any textbook in noncommutative ring (see, for example, [5]). Consider the filtration of $\mathcal{K}(\delta]$-modules $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_n$. Define $d_1 := \mathrm{rank}\,\mathcal{F}_1$, $d_k := \mathrm{rank}\,\mathcal{F}_k - \mathrm{rank}\,\mathcal{F}_{k-1}$, $k = 2,\ldots,n$, $k_i := \max\{k \mid d_k \geq i\}$ (see [3]). Then, $\{k_1, k_2, \ldots, k_p\}$ are observability indexes and d_k is the number of observability indexes which are greater than or equal to k, for $k = 1,\ldots,n$. It is obvious that $k_1 \geq k_2 \geq \cdots \geq k_p$. By the condition that $H_1(\delta)$ is of full row rank, the p observability indexes are well-defined, i.e., each $k_i \geq 1$, $i = 1,\ldots,p$.

Now, compute

$$\mathrm{d}y_i^{(j-1)} = \xi_{ij}(\delta)\mathrm{d}x + \gamma_{ij}\mathrm{d}\theta (\bmod\, \mathcal{U}) \tag{4}$$

for $i = 1, \dots, p$ and $j = 1, \dots, k_i$. Define

$$\Gamma_0 = (\gamma_{11}^T, \dots, \gamma_{1,k_1}^T, \gamma_{21}^T, \dots, \gamma_{2,k_2}^T, \dots, \gamma_{p,k_p}^T)^T$$
$$\Xi = (\xi_{11}^T, \dots, \xi_{1,k_1}^T, \xi_{21}^T, \dots, \xi_{2,k_2}^T, \dots, \xi_{p,k_p}^T)^T$$
$$\bar{y} = (y_i^{(j)} : j = 0, \dots, k_i - 1; i = 1, \dots, p)^T$$

then $\mathrm{d}\bar{y} \equiv \Xi \mathrm{d}x + \Gamma_0 \mathrm{d}\theta (\mathrm{mod} \quad \mathcal{U})$. Let $N_0 := k_1 + k_2 + \dots + k_p$, then $N_0 \leq n$, Ξ is a matrix of the size $N_0 \times n$, and one has the following result whose lengthy proof has been omitted due to limited space (see [13] for details).

Lemma 1: Let $\mathcal{F}_0 = \mathcal{X} \cap (\Theta + \mathcal{U})$ then $\mathcal{F}_0 = 0$. Furthermore, the matrix Ξ defined previously is of full-row rank in the sense that all the rows of Ξ are linearly independent over $\mathcal{K}(\delta]$.

For any general index of integers $\alpha = (\alpha_1, \dots, \alpha_p)$, suppose $\mathrm{d}y_i^{(j-1)} \equiv \xi_{ij}(\delta)\mathrm{d}x + \gamma_{ij}\mathrm{d}\theta(\mathrm{mod} \quad \mathcal{U})$, where $j = 1, \dots, \alpha_i$, $i = 1, \dots, p$. If $\alpha_i \geq k_i$, $i = 1, \dots, p$, then denote by

$$y^\alpha = (\bar{y}^T, y_1^{(k_1)}, y_1^{(k_1+1)}, \dots, y_1^{(\alpha_1-1)}, y_2^{(k_2)}, y_2^{(k_2+1)} \\ \dots, y_2^{(\alpha_2-1)}, \dots, y_p^{(k_p)}, y_p^{(k_p+1)}, \dots, y_p^{(\alpha_p-1)})^T$$

otherwise

$$y^\alpha = (y_1, y_1^{(1)}, \dots, y_1^{(\alpha_1-1)}, y_2, y_2^{(1)} \\ \dots, y_2^{(\alpha_2-1)}, \dots, y_p, y_p^{(1)}, \dots, y_p^{(\alpha_p-1)})^T.$$

Define Γ^α and Ξ^α to be the matrices such that the previous relation can be written as

$$\mathrm{d}y^\alpha \equiv \Xi^\alpha \mathrm{d}x + \Gamma^\alpha \mathrm{d}\theta (\mathrm{mod}\, \mathcal{U}). \tag{5}$$

Given any matrix $A(\delta)$ with elements in $\mathcal{K}(\delta]$, denote by L_i its ith row. Define the following elementary row operations:

Use $h_1(\delta)L_i + h_2(\delta)L_j$ to substitute L_j, where $h_1(\delta)$, $h_2(\delta) \in \mathcal{K}(\delta]$, $h_2(\delta)$ has no nontrivial factor in $\mathcal{K}(\delta]$, and $h_2(\delta)$ is not a factor of $h_1(\delta)$ in $\mathcal{K}(\delta]$.

When the above elementary operation acts on the identity matrix, one obtains the so-called *elementary matrix*. Suppose $\alpha_i \geq k_i$, $i = 1, \dots, p$, then Ξ is just the submatrix formed by the first N_0 rows of Ξ^α. Now use the above elementary matrix, or equivalently, elementary operation, on the relation $\mathrm{d}y^\alpha \equiv \Xi^\alpha \mathrm{d}x + \Gamma^\alpha \mathrm{d}\theta(\mathrm{mod}\, \mathcal{U})$. By Lemma 1, the N_0 rows of Ξ are independent, then one can use the N_0 rows of Ξ to eliminate the other rows of Ξ^α (see [12, Lemma 1]). At last Ξ^α can be transformed into $(\Xi^T, 0)^T$, while Γ^α is transformed into $((\Gamma_0^\alpha)^T, (\Gamma_a^\alpha)^T)^T$, where Γ_0^α is still the first N_0 rows of Γ^α. Thus, there exists a matrix $B(\delta)$ such that

$$B(\delta)\mathrm{d}y^\alpha \equiv \Gamma_a^\alpha \mathrm{d}\theta(\mathrm{mod}\, \mathcal{U}). \tag{6}$$

Now, one has an easy characterization for algebraic identifiability.

Theorem 2: System (1) is algebraically identifiable if and only if there exist integers $J \geq 0$, $k \geq 0$, $\alpha_i \geq k_i$, $i = 1, \dots, p$, a $T > 0$, an open subset S_1 of W_k, such that $\mathrm{rank}_{\mathcal{K}} \left((\Gamma_a^\alpha)^T, (\delta(\Gamma_a^\alpha))^T, \dots, (\delta^J(\Gamma_a^\alpha))^T\right)^T = q$ holds for all $(\theta, x_0, u) \in S_1$, where $\alpha = (\alpha_1, \dots, \alpha_p)$. Furthermore, k can be determined in the following way: Suppose Γ^{-1} is the left inverse of $((\Gamma_a^\alpha)^T, \delta(\Gamma_a^\alpha)^T, \dots, \delta^J(\Gamma_a^\alpha)^T)^T$, then by multiplying Γ^{-1} in both sides of (6) and its shifts by $\delta, \dots, \delta^J$ one has $\mathrm{d}\theta = \mu\mathrm{d}y + \beta\mathrm{d}u$ for suitable matrices μ and β. The elements in μ and β are functions of $\{\delta^i y^{(j)}, \delta^i u^{(j)}, j = 0, \dots, k; i \geq 0\}$.

Proof: By Theorem 1 one only needs to prove necessity. Suppose $\mathrm{d}\theta \equiv \sum A_{j,r}^i \mathrm{d}\delta^i y_r^{(j)}$ $(\mathrm{mod}\, \mathcal{U})$. Arrange the set $\mathcal{S} := \{\delta^i y_r^{(j)}, i, j \geq 0; r = 1, \dots, p\}$ by the following ordering:

$\delta^i y_r^{(j)} > \delta^{i'} y_{r'}^{(j')}$ if and only if

$$[(j = j', r = r', i < i') \text{ or } (j = j', r > r') \text{ or } (j > j')].$$

For any subset $\mathcal{S}_1$ of $\mathcal{S}$, define $LT(\mathcal{S}_1)$ to be the greatest element in $\mathcal{S}_1$ by the aforementioned ordering. For any two subsets $\mathcal{S}_1, \mathcal{S}_2$, $\mathcal{S}_1$ is said to be greater than $\mathcal{S}_2$ if $LT(\mathcal{S}_1 \backslash \mathcal{S}_2) > LT(\mathcal{S}_2 \backslash \mathcal{S}_1)$. Without loss of generality suppose that the functions in set $\mathcal{S}_0 := \{\delta^i y_r^{(j)} : A_{j,r}^i \neq 0\}$ are locally independent modulo $\mathcal{U}$ and $\mathcal{S}_0$ is the smallest possible subset by the previous ordering. Let v be the vector $(\delta^{i_1} y_{r_1}^{(j_1)}, \dots, \delta^{i_q} y_{r_q}^{(j_q)})^T$, where $\mathcal{S}_1 := \{\delta^{i_1} y_{r_1}^{(j_1)}, \dots, \delta^{i_q} y_{r_q}^{(j_q)}\}$ is the largest possible subset of $\mathcal{S}_0$ with cardinality q. Then there exists an analytic function ψ such that $v = \psi(\theta, \tilde{u}, \tilde{y})$, where $\tilde{y}$ denotes functions in $\mathcal{S}_0 \backslash \mathcal{S}_1$, $\tilde{u}$ denotes the derivatives and shifts of the variables $\{u_1, \dots, u_m\}$ such that the functions in $\{\theta, \tilde{u}, \tilde{y}\}$ are independent locally. It is obvious that $\mathrm{rank}_{\mathcal{K}}(\partial\psi/\partial\theta) = q$.

By the definition of Γ_a^α there exist $\alpha = (\alpha_1, \dots, \alpha_p)$ and integer $r \geq 0$ such that $B(\delta)\mathrm{d}y^\alpha \equiv \Gamma_a^\alpha \mathrm{d}\theta(\mathrm{mod} \quad \mathcal{U})$, and by applying δ suitably times for the rows of this equation one can solve $\delta^r v = \phi(\theta, \tilde{u}', \tilde{y}')$, where $\tilde{y}'$ denotes the set of derivatives and shifts of y which is less than the set $\{\delta^{r+i_1} y_{r_1}^{(j_1)}, \dots, \delta^{r+i_q} y_{r_q}^{(j_q)}\}$, and $\tilde{u}'$ denotes the derivatives and shifts of the variables $\{u_1, \dots, u_m\}$ such that the functions in $\{\theta, \tilde{u}', \tilde{y}'\}$ are independent locally. Thus $\delta^r \psi(\theta, \tilde{u}, \tilde{y}) \equiv \phi(\theta, \tilde{u}', \tilde{y}')$. By the construction of $\tilde{u}$, $\tilde{y}$, $\tilde{u}'$ and $\tilde{y}'$ one has $\partial\phi/\partial\theta = \delta^r \partial\psi/\partial\theta$, and $\mathrm{rank}_{\mathcal{K}}(\partial\phi/\partial\theta) = \mathrm{rank}_{\mathcal{K}}(\delta^r \partial\psi/\partial\theta) = q$. However, the matrix $\partial\phi/\partial\theta$ is a submatrix of $((\Gamma_a^\alpha)^T, \delta(\Gamma_a^\alpha)^T, \dots, \delta^J(\Gamma_a^\alpha)^T)^T$ for J large enough, thus $\mathrm{rank}_{\mathcal{K}}((\Gamma_a^\alpha)^T, \delta(\Gamma_a^\alpha)^T, \dots, \delta^J(\Gamma_a^\alpha)^T)^T = q$. QED

Remark 1: It is worthy noting that the elements in the fraction field of $\mathcal{K}(\delta]$ are not well-defined operators on $\mathrm{d}\theta$. For example, suppose they are well-defined, then $(1 - \delta)\mathrm{d}\theta = \mathrm{d}\theta - \delta\mathrm{d}\theta = 0$, $\mathrm{d}\theta = (1 - \delta)^{-1}(1 - \delta)\mathrm{d}\theta = (1 - \delta)^{-1}0 = 0$. The equality $\mathrm{d}\theta = 0$ contradicts with the fact that θ is an arbitrary parameter. Thus one can not define the actions of the elements in the fraction field of $\mathcal{K}(\delta]$ on $\mathrm{d}\theta$. Due to this reason we can not eliminate $\mathrm{d}x$ directly to define Γ_a as what was done in [11], and therefore work only in the ring $\mathcal{K}(\delta]$. Another important thing to note is that the above Theorem 2 requires also the shifts of Γ_a^α in the rank condition, while the corresponding result in [11] does not. It is because that Definition 3 permits one to represent θ by the shifts of y, u and their derivatives too, while in [11] shifts are not permitted. In this sense, Definition 3 is weaker than the definition of algebraic identifiability in [11] (see Example 6 of [13]). If delays are not permitted in $\mathcal{K}$, the two definitions coincide for systems without time-delay.

C. Geometric Identifiability

Theorem 2 gives an easy criterion to tell if a system is algebraically identifiable. The similar result holds also for geometric identifiability.

Theorem 3: The following statements are equivalent.

i) The system Σ_θ is geometrically identifiable.

ii) There exist integers $r \geq 0$, $k \geq 0$, α_i, $i = 1, \dots, p$, a $T > 0$, an open subset S_1 of W_k, such that $\mathrm{rank}_{\mathcal{K}} \left((\Gamma_g)^T, (\delta(\Gamma_g))^T, \dots, (\delta^r(\Gamma_g))^T\right)^T = q$ holds for all $(\theta, x_0, u) \in S_1$, where $\alpha = (\alpha_1, \dots, \alpha_p)$, $\Gamma_g = \Gamma^\alpha$. Furthermore k can be determined in the following way: Suppose Γ^{-1} is the left inverse of $(\Gamma_g^T, \delta(\Gamma_g)^T, \dots, \delta^r(\Gamma_g)^T)^T$, then by multiplying Γ^{-1} in both sides of (5) and its shifts by $\delta, \dots, \delta^r$ one has $\mathrm{d}\theta = \mu\mathrm{d}y + \beta\mathrm{d}u + \gamma\mathrm{d}x$ for suitable matrices μ, β and γ. The elements in μ, β and γ are functions of $\{\delta^i y^{(j)}, \delta^i u^{(j)}, \delta^j x : j = 0, \dots, k; i \geq 0\}$.

iii) The relation $\mathrm{d}\theta \in (\mathcal{X} + \mathcal{Y} + \mathcal{U})$ holds for all $(\theta, x_0, u) \in S$, where S is an open subset of some W_k.

iv) The system Σ_θ is identifiable with known initial conditions.

Proof: The proof of i) $\Rightarrow$ iii) is obvious, while iii) $\Rightarrow$ i) and ii) $\Rightarrow$ i) can be proved similarly as Theorem 1.

i) $\Rightarrow$ ii)*:* Since i) implies iii) one has $\mathrm{d}\theta \equiv C(\delta)\mathrm{d}y^{\alpha} + \widetilde{W}(\delta)\mathrm{d}x(\mathrm{mod}\ \mathcal{U})$ for some α, $C(\delta)$ and $\widetilde{W}(\delta)$, where $C(\delta) = \sum_{i=1}^{r} C_i\delta^i$. Suppose $\mathrm{d}y^{\alpha} \equiv \Xi^{\alpha}(\delta)\mathrm{d}x + \Gamma_g\mathrm{d}\theta(\mathrm{mod}\ \mathcal{U})$, then $(C(\delta)\Xi^{\alpha}(\delta) + \widetilde{W}(\delta))\mathrm{d}x \equiv (I - C(\delta)\Gamma_g)\mathrm{d}\theta(\mathrm{mod}\ \mathcal{U})$. By Lemma 1 one has $0 = (I - C(\delta)\Gamma_g)\mathrm{d}\theta = (I - C(\delta)(\Gamma_g))\mathrm{d}\theta$, $I = C(\delta)(\Gamma_g) = (C_0, C_1, \ldots, C_r)(\Gamma_g^T, \delta(\Gamma_g)^T, \ldots, \delta^r(\Gamma_g)^T)^T$, thus $(\Gamma_g^T, \delta(\Gamma_g)^T, \ldots, \delta^r(\Gamma_g)^T)^T$ is of full-column rank.

iii) $\Rightarrow$ iv)*:* Since $\mathrm{d}\theta = \beta\mathrm{d}x + \sum_{i=0}^{k}(\xi_i\mathrm{d}y^{(i)} + \eta_i\mathrm{d}u^{(i)})$, for suitable matrices β, ξ_i, η_i and all $t \in [0, T], T < 1$, one obtains by the proof of Theorem 1 that there exists a meromorphic function ϕ such that $\theta = \phi(\delta^{i_1}x, D^j\delta^{i_2}y, D^j\delta^{i_2}u : i_1, i_2, j = 0, \ldots, k)$ holds for some open subset S_2 of W_k. Since $T < 1$ and $x(t)$ exists on $[0, T]$, one has $i_1 \leq s$, i.e., $i_1 \in S_-$. Let $t = 0$ in ϕ then

$$\theta = \phi(x_0(-i_1^+), u^{(j)}(-i_2^+), y^{(j)}(-i_2^+) : i_1 \in S_-, i_2, j = 0, 1, \ldots, k)$$

for all $(\theta, x_0, u) \in S_2$.

iv) $\Rightarrow$ ii)*:* By

$$\theta = \phi(x_0(-i_1^+), u^{(j)}(-i_2^+), y^{(j)}(-i_2^+) : i_1 \in S_-, i_2, \qquad j = 0, 1, \ldots, k)$$

the map from θ to y must be one-to-one. Suppose $\mathrm{rank}\,\Gamma = r < q$ for all $\alpha = (\alpha_1, \ldots, \alpha_p)$ and (θ, x_0, u) in any open set $V_0 \times \bar{W}$, where $\Gamma = (\Gamma_g^T, \delta(\Gamma_g)^T, \ldots, \delta^r(\Gamma_g)^T)^T$ and V_0 is any open set in $\mathbb{R}^q$. Fix some $(\theta_0, x_0, u_0) \in V_0 \times \bar{W}$. Let $\bar{\lambda} \in Ker\Gamma$, $\|\bar{\lambda}\| = c$. Note that Γ, $\bar{\lambda}$ can be viewed as functions of t since (θ_0, x_0, u_0) is fixed. Define the curve segment L with parameter $v = t/a$, $a > 0$, in the open ball $U_\epsilon \subseteq \mathbb{R}^q$, which has radius ϵ and center θ_0, such that $L = \{\theta(v) : \mathrm{d}\theta = \bar{\lambda}\mathrm{d}v,\ \theta(v)|_{v=0} = \theta_0,\ v \in [0, 1]\}$. When ϵ, c and a are small enough, one can assume that $L \subset U_\epsilon \subset V_0$. Let $\mathrm{d}u = 0$ and θ varies on L, denote $y' = ((y^{\alpha})^T, \delta(y^{\alpha})^T, \ldots, \delta^r(y^{\alpha})^T)^T$, then there exists a matrix G such that $\mathrm{d}y' = \Gamma\mathrm{d}\theta + G\mathrm{d}x = \Gamma\bar{\lambda}\mathrm{d}v + G\mathrm{d}x = G\mathrm{d}x$. Now, y' is a function of x and its shifts locally. When θ varies on L, the function y does not change since x and its shifts are independent of θ by Lemma 1. Thus the initial values of y and its shifts and derivatives do not change. This contradicts with Definition 4. QED

The previous results shows that one only needs to check the ranks of some matrices to know if a system is algebraically or geometrically identifiable, and geometric identifiability is equivalent to identifiability with known initial conditions.

The following proposition and two subsections are trivial generalizations of [11] and the above results, therefore we simply list the definitions and results and refer the proofs to [11].

Proposition 1:

i) If a system is algebraically identifiable, then it is geometrically identifiable.

ii) If $\mathcal{X} \cap (\mathcal{Y} + \Theta + \mathcal{U}) = \mathcal{X} \cap (\mathcal{Y} + \mathcal{U})$, then the system is algebraically identifiable if and only if it is geometrically identifiable.

iii) If the system is algebraically identifiable, then $\mathcal{X} \cap (\mathcal{Y} + \Theta + \mathcal{U}) = \mathcal{X} \cap (\mathcal{Y} + \mathcal{U})$.

D. Identifiability With Partially Known Initial Conditions

Denote the i_0th component of the vector function x_0 by $(x_0)_{i_0}$. Assume that initial conditions are partially known for $(x_0)_{i_0}(-i_1^+)$, $i_0 \in \{v_1, \ldots, v_l\} \subseteq \{1, \ldots, n\}$, $i_1 \in S_-$, and the identifiability problem in this case is to find whether the parameter θ can be expressed as a meromorphic function of $(x_0)_{i_0}(-i_1^+)$, $i_0 = v_1, \ldots, v_l \in \{1, \ldots, n\}$, $i_1 \in S_-$, and u, y and their derivatives or delays. Define $\mathcal{X}_p = \mathrm{span}_{\mathcal{K}(\delta]}\{\mathrm{d}x_i : i = v_1, v_2, \ldots, v_l\}$.

Definition 5: The system Σ_θ is said to be identifiable with partially known initial conditions $(x_0)_{i_0}(-i_1^+)$, $i_0 \in \{v_1, \ldots, v_l\} \subseteq \{1, \ldots, n\}$, $i_1 \in S_-$, if there exists a meromorphic function ϕ such that

$$\theta = \phi((x_0)_{i_0}(-i_1^+), u^{(j)}(-i_2^+), y^{(j)}(-i_2^+) : i_0 = v_1, \ldots, v_l; i_1 \in S_-; i_2, \qquad j = 0, 1, \ldots, k)$$

holds for all $(\theta, x_0, u) \in S_3$, where S_3 is an open subset of some W_k.

Theorem 4: The system is identifiable with known $(x_0)_{i_0}(-i_1^+)$, $i_0 \in \{v_1, \ldots, v_l\} \subseteq \{1, \ldots, n\}$, $i_1 \in S_-$ if and only if $\mathrm{d}\theta \in (\mathcal{X}_p + \mathcal{Y} + \mathcal{U})$ holds for all $(\theta, x_0, u) \in S_3$, where S_3 is an open subset of some W_k.

Corollary 1: For any two sets of known initial conditions $\mathcal{X}_p^1$, $\mathcal{X}_p^2$, and $\mathcal{X}_p^1 \supset \mathcal{X}_p^2$, if the system is identifiable with $\mathcal{X}_p^2$, then it is identifiable with $\mathcal{X}_p^1$.

E. Persistent Excitation and Singularities

Definition 6: A pair (x_0, u) is algebraically (respectively, geometrically) persistently exciting for θ_0 if there exist J, α, such that $((\Gamma_a^{\alpha})^T, \delta(\Gamma_a^{\alpha})^T, \ldots, \delta^J(\Gamma_a^{\alpha})^T)^T$ (respectively, $(\Gamma_g^T, \delta(\Gamma_g)^T, \ldots, \delta^r(\Gamma_g)^T)^T)$ is of rank q for when evaluated at x_0, θ, and u.

Note that the matrices $\delta^i(\Gamma_a^{\alpha})$ and $\delta^i(\Gamma_g)$ are functions of x_0, θ, u and derivatives of u. To make the matrices to be of full column rank, x_0, θ, u and derivatives of u need to provide "sufficiently rich" signals. In this sense, the previous definition conforms with the linear concept.

It is clear that when (x_0, u) is algebraically (geometrically) persistently exciting, the parameters can be determined at least locally around (x_0, u) by u, y, x_0 and their derivatives and delays. When (x_0, u) is not algebraically (geometrically) persistently exciting for θ_0, then it is called to have an algebraical (a geometrical) singular point for the identifiability of θ_0.

IV. EXAMPLES

Example 1 shows the essential difference between the method in [11] for systems without time-delay and the method in this note for systems with time-delay. In fact, [11] uses the inverse of some coefficient submatrix of $\mathrm{d}x$ to represent $\mathrm{d}x$ as a linear combination of $\mathrm{d}\theta$ and the derivatives of $\mathrm{d}y$. In the case of time-delay systems, the corresponding submatrix may not be invertible in $\mathcal{K}(\delta]$, and $\mathcal{K}(\delta]$ is itself a noncommutative ring where division is not defined, one has to use row operations defined in this note instead. The difference shows again that the method of the present note is not simply a trivial generalization of [11].

Example 1: Consider the following example:

$$\dot{x}_1 = \theta_1 x_2 \quad \dot{x}_2 = \theta_2 x_1^2,\ y = x_1.$$

Following the calculation steps in [11], one has that

$$\begin{pmatrix} \mathrm{d}y \\ \mathrm{d}\dot{y} \\ \mathrm{d}\ddot{y} \\ \mathrm{d}y^{(3)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \theta_1 \\ 2\theta_1\theta_2 x_1 & 0 \\ 2\theta_1^2\theta_2 x_2 & 2\theta_1^2\theta_2 x_1 \end{pmatrix} \begin{pmatrix} \mathrm{d}x_1 \\ \mathrm{d}x_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ x_2 & 0 \\ \theta_2 x_1^2 & \theta_1 x_1^2 \\ 4\theta_1\theta_2 x_1 x_2 & 2\theta_1^2 x_1 x_2 \end{pmatrix} \begin{pmatrix} \mathrm{d}\theta_1 \\ \mathrm{d}\theta_2 \end{pmatrix}.$$

It is obvious that the system is geometrically identifiable generically since the coefficient matrix of $(\mathrm{d}\theta_1, \mathrm{d}\theta_2)^T$ has full column rank generically. Since the submatrix $\mathrm{diag}(1, \theta_1)$ is invertible, one can represent $(\mathrm{d}x_1, \mathrm{d}x_2)^T$ as a function of $\mathrm{d}\theta_1$, $\mathrm{d}\theta_2$, $\mathrm{d}y$, $\mathrm{d}\dot{y}$ by multiplying the inverse of $\mathrm{diag}(1, \theta_1)$ to the equations of $\mathrm{d}y$ and $\mathrm{d}\dot{y}$. Substituting this into the equations for $\mathrm{d}\ddot{y}$ and $\mathrm{d}y^{(3)}$ one has that the coefficient matrix for

$(\mathrm{d}\theta_1, \mathrm{d}\theta_2)^T$ is not of full rank, that is, the system is not algebraically identifiable.

Now, consider a time-delay system which has the same state equations as above but with a delay in the output $y(t) = x_1(t-1) = \delta(x(t))$. Similarly, one has

$$\begin{pmatrix} \mathrm{d}y \\ \mathrm{d}\dot{y} \\ \mathrm{d}\ddot{y} \\ \mathrm{d}y^{(3)} \end{pmatrix} = \begin{pmatrix} \delta & 0 \\ 0 & \theta_1\delta \\ 2\theta_1\theta_2\delta(x_1)\delta & 0 \\ 2\theta_1^2\theta_2\delta(x_2)\delta & 2\theta_1^2\theta_2\delta(x_1)\delta \end{pmatrix} \begin{pmatrix} \mathrm{d}x_1 \\ \mathrm{d}x_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \delta(x_2) & 0 \\ \theta_2\delta(x_1^2) & \theta_1\delta(x_1^2) \\ 4\theta_1\theta_2\delta(x_1x_2) & 2\theta_1^2\delta(x_1x_2) \end{pmatrix} \begin{pmatrix} \mathrm{d}\theta_1 \\ \mathrm{d}\theta_2 \end{pmatrix}.$$

By the convention of this note, the previous two coefficient matrices are denoted by Ξ^3 and Γ^3 respectively. Note that the matrix Ξ^1, which consists of the first two rows of Ξ^3, is not invertible in $\mathcal{K}(\delta]$, and one can not solve $\mathrm{d}x_1, \mathrm{d}x_2$ as what was done in the case without time-delay. However, the third and fourth rows of Ξ^3 can be eliminated by the first two rows through the elementary row operations defined in this note. At last, one obtains

$$\begin{pmatrix} \mathrm{d}\ddot{y} - 2\theta_1\theta_2\delta(x_1)\mathrm{d}y \\ \mathrm{d}y^{(3)} - 2\theta_1^2\theta_2\delta(x_2)\mathrm{d}y - 2\theta_1\theta_2\delta(x_1)\mathrm{d}\dot{y} \end{pmatrix} = \begin{pmatrix} \theta_2\delta(x_1^2) & \theta_1\delta(x_1^2) \\ 2\theta_1\theta_2\delta(x_1x_2) & 2\theta_1^2\delta(x_1x_2) \end{pmatrix} \begin{pmatrix} \mathrm{d}\theta_1 \\ \mathrm{d}\theta_2 \end{pmatrix}.$$

The aforementioned coefficient matrix is just Γ_a^3. Note that in the case of systems without time-delay, one only needs to check if this coefficient matrix of $\mathrm{d}\theta$ has full-rank over $\mathcal{K}$. In the case of systems with time-delay, one needs to check the rank of Γ_a^3 over $\mathcal{K}(\delta]$ instead of $\mathcal{K}$. That is, one needs to consider if there exists a j such that the matrix $\left((\Gamma_a^3)^T, (\delta(\Gamma_a^3))^T, \ldots, (\delta^j(\Gamma_a^3))^T\right)$ has rank 2 over $\mathcal{K}$. However the rank is always 1. Thus the time-delay system is geometrically identifiable generically but not algebraically identifiable.

Models of population dynamics [4] or biomedical dynamical systems [6] provide practical examples of time delay systems due to reproduction or to out-spreading of some disease. One elementary model is taken in Example 2, from [2]. The time-delay is the developmental or maturation time and is "naturally" known for some population, whereas the death rate parameter needs to be identified, depending on the population conditions.

Example 2: Consider a population N whose size changes according to the population growth equation [2]

$$\dot{N} = B(N(t-\tau))N(t-\tau) - \theta N \tag{7}$$

where $\theta > 0$ is the constant death rate, in the absence of disease. $B(N(t-\tau))N(t-\tau)$ is a general nonlinear birth rate and τ is the average developmental or maturation time. When the population is subject to an epidemic disease causing death, then the death rate increases and should be identified. Obviously

$$\theta = \frac{B(N(t-\tau)N(t-\tau) - \dot{N}(t)}{N(t)}$$

and system (7) is algebraically identifiable from the measured output $N(t)$ in the sense of Definition 3, whenever $N \neq 0$, which is not restrictive in practise.

V. Conclusion

In this note, different notions of parameter identifiability for nonlinear systems with time-delay are presented. The relations between them are characterized by linear algebra based on differential forms. The easily checkable criteria for algebraic and geometric identifiability have been given. The whole approach was developed for systems with commensurate delays. If several noncommensurate delays are present in the system, then it is possible to employ "polynomials" of several variables. This adds complexity to the notations and is not explicitly developed here in this note. We also believe that the results and methods in this note should be able to apply to the nonlinear discrete time system with time-delay. We leave this to interested readers.

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