Complex Dynamics of Systems Under Delta-Modulated Feedback

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***Abstract*—In this paper, we cast the design of Δ-modulated control of a high-order system into the study of control Lyapunov functions. We classify the complex dynamics of the closed-loop system in three cases. In the first case, we show how Δ-modulated feedback introduces a finite set of globally attracting periodic points. We find the numbers and periods of all possible such periodic orbits. In addition, we characterize the attracting region for each of such periodic points. In the second case, we show that there is a maximal "stabilizable" region, and inside this region, there is a local attractor. In the last case, we show that all the states stabilizable by the Δ-modulated feedback constitute a Cantor set. This Cantor set is a repeller, and the closed-loop system is chaotic on the Cantor set.**

***Index Terms*—Attracting region, attractor, Cantor set, chaos, control Lyapunov function, delta-modulation, repeller.**

I. Introduction

We CONSIDER a discrete-time control system of order n

$$x^{n+k} = ax^k + u \tag{1}$$

where , a, x and u are real, $a \neq 0$, and the scalar control u takes a Δ-modulated feedback form

$$u = -\Delta \mathrm{sgn}(ax^k) \tag{2}$$

in which $\Delta > 0$, and $\mathrm{sgn}(x) = 1$, when $x \geq 0$, and $\mathrm{sgn}(x) = -1$, when $x < 0$.

A practical example of this kind of control is the transmitting power control of a mobile unit in the direct sequence code division multiple access (DS-CDMA) cellular network. A simplest model is given by (1) and (2) with $a = 1$ and $n = 1$, and the "state" x is the error of the mobile unit's power level received at the base station with respect to the desired value (both in decibel, dB). The control action stems from a simple and intuitive idea: When the level of the received power is higher than the desired level, it is decreased by Δ dB, and when the level of the received power is lower than the desired level, it is increased by the same amount [1]. There is only one design parameter, Δ, and the power increment is either Δ or $-\Delta$. This scheme is called a delta modulation (DM) transmitting power control. An advantage of such a control is that Δ can be stored at the base station or the active mobile unit, and the base station only needs to send 1 or -1 to command the increase or decrease of the power level. In other words, only one bit of datum is necessary for the implementation of the DM control. The requirement of one bit for transmitting power control is the standard of IS-95 [21]. Other control schemes with multiple power levels have been proposed for the third generation of CDMA technology, but some of them offer only marginal improvement in the practical multifading environment [26]. Δ-modulation is a method of converting analog signals to digital signals. In electronic circuits, such a method of analog-to-digital conversion is also called $\Sigma\Delta$ modulation, which was introduced much earlier in [18], [5] and studied in depth for the "leaky" case (when $a \neq 1$ and $n = 1$) in a number of later publications [19], [15], [23]. Δ-modulation has since been widely used in digital electronics and telecommunication. One recent study [6] is on digital bang–bang phase-locked loops, where periodic orbits and the control performance are evaluated. The interests of Δ-modulation in digital electronics are demodulation schemes, statistical properties of the digital outputs as well as the complex dynamics involved.

In this paper, one of our objectives is to justify the design in (2) in a control system framework. This is achieved through an investigation of control Lyapunov functions (CLFs) for a general class of systems

$$x^+ = Ax + bu \tag{3}$$

where $x \in R^n$ is the state, x^+ denotes the system state at the next discrete-time, A is an $n \times n$ matrix of real numbers, and b is a column vector of n real numbers.

The results we obtain in this paper are natural generalizations of those in [28], where we considered the first order system (1) (when $n = 1$).

Our first results concern with "stable" systems of (1) when $|a| \leq 1$. In [28], we showed that Δ-modulated feedback can generate one periodic solution with period 2 or two periodic solutions with period 1 (that is, two equilibria). In particular, there is one periodic-2 orbit, but no periodic-1 orbit, on the closed invariant interval $[-\Delta, \Delta]$. This is a major departure

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from what the famous Sarkovskii theorem [25] claims about periodic points of continuous mappings on a closed-interval. Note that Δ-modulation introduces discontinuity, and the theory of continuous dynamical systems does not apply to it. We will show that a Δ-modulated feedback also introduces periodic orbits of different periods in the higher order case. We will furthermore find all possible such periods, and for each of such periods, we will also find the number of periodic orbits. In addition, we will show that the set of these periodic points has 2^n points and the set is a globally invariant attractor for the closed-loop system. Finally, we will characterize the attracting region for each of such periodic points. We will note that high order results are no simple generalizations of the scalars ones. For example, a sixth-order system (1) with $a = -0.5$ does not only have 6-periodic points, but also 1-, 2-, and 3-periodic points. A sixth-order system with $a = 0.5$ has only 4-periodic and 12-periodic points.

The determination of self-excited oscillations or limit cycles, originated in the work of Poincaré and Lyapunov, is an old and difficult problem in the classic qualitative theory of dynamical systems [16]. It is known that the mechanism of generating periodic orbits by Δ-modulated feedback is very complex [30], [31]. Apart from their theoretical interests, these results are of practical importance. For instance, the digital output signals of $\Sigma\Delta$ modulation are nonwhite [15], the spectrum information of these signals are crucial in helping developing preventive measures if they are necessary.

Δ-modulated control is bounded, bang–bang, and also a special kind of *quantized control*, which are topics of longstanding interests in the control community [17], [7], [8], [4], [22], [11], [2], [12], [13], [20]. Δ-modulated control is a two-level quantized control, and a quantized control is a cascade of Δ-modulated controls. The study of Δ-modulated control will eventually be helpful in the implementation of a quantized control. On the other hand, the recent interest in quantized control has been focused on the design of quantization levels for the purpose of stabilization, the quantification of information flow and the convergence time. Particular interests are given to (controlled) invariance arising from quantized control.

In this regard, our second and third results to be briefly described below are very much relevant.

Our second results are devoted to the case when $1 < |a| \leq 2$. We will show that there is a maximal "stabilizable" region, and inside this region, there is a local attractor whose size is independent of the value of a. The control-theoretical significance of this result is obvious: Δ-modulated feedback offers some, though limited, stabilizability capabilities for unstable systems. This result also captures a convergence property of the proposed Δ-modulated feedback control. For scalar systems under quantized feedback control, this kind of convergence property is characterized by the so-called (I, J)-stability in [12] where the interest is to see whether two "intervals" exist such that all points in the "stabilizable region" I are steered to the "controlled-invariant attractor" J.

Our last result shows that when $|a| > 2$, all the states stabilizable by the designed Δ-modulated feedback constitute a Cantor set. This Cantor set is a repeller, that is, any state outside this set is steered away from this set (actually to infinity). On the Cantor set itself, the system is chaotic. Chaos have long been found in association with quantization in digital filters and digital control systems [7], [8]. The construction of the Cantor set offers a detailed study of chaotic behaviors. In particular, this shows that the Cantor set is the closure of all the periodic points of the Δ-modulated orbits of the system, therefore, it is a minimal controlled-invariant set containing all the periodic points. This result is also insightful: Though most of the research in controlled-invariance introduced by (quantized) feedback control is focused on the maximal controlled-invariant set (see [3], [24] and the references therein), a minimal controlled-invariant set can sometimes be properly defined and constructed. Note also our discussions on Cantor set here is constructive, compared with some existence proofs of the scalar case in [28].

The layout of the paper is as follows. In Section II, we study some properties of control Lyapunov functions (CLFs) and describe the design procedure of Δ-modulated feedback based on a CLF. Sections III, IV and V are devoted to detailed studies of the above three cases, respectively. Some concluding remarks are given in VI.

II. CLF AND Δ-MODULATED FEEDBACK

Assume that (3) is stabilizable. It is, therefore, quadratically stabilizable in the sense that there is a control input u, which is a function of x, that makes a quadratic function of the state a valid Lyapunov function for the closed-loop system. Such Lyapunov functions are called CLFs.

Given a quadratic CLF, $V(x) = x^T P x$ with $P > 0$, where P is always assumed to be symmetric in this paper, we look for a control input u such that $V(x)$ is decreasing along the trajectories of system (3), i.e., for $x \neq 0$

$$\begin{aligned}\mathcal{D}V(x) &\stackrel{\text{def}}{=} V(x^+) - V(x)\\ &= x^T(A^T PA - P)x + 2b^T PAxu + b^T Pbu^2 > 0. \quad (4)\end{aligned}$$

Given x, it is easily verified that the following input:

$$u \stackrel{\text{def}}{=} -\frac{b^{\mathrm{T}}\mathrm{PA}}{b^{\mathrm{T}}\mathrm{PB}}x \stackrel{\text{def}}{=} k_{\mathrm{GD}}^T x \quad (5)$$

defines the gradient descent direction making $V(x)$ decrease the most along the trajectories. Under feedback (5), we have, for the closed-loop system

$$\mathcal{D}V(x) = x^T\left(A^{\mathrm{T}}\mathrm{PA} - P - \frac{A^{\mathrm{T}}\mathrm{Pbb^{T}PA}}{\mathrm{b^{T}Pb}}\right)x.$$

For convenience, denote

$$Q = P - A^{\mathrm{TPA}} + \frac{A^{\mathrm{T}}\mathrm{Pbb^{T}PA}}{\mathrm{b^{T}Pb}}. \quad (6)$$

By the assumption that $V(x)$ is a CLF, $Q > 0$.

Given x, two inputs making $\mathcal{D}V(x)$ [as defined in (4)] zero can be found as

$$u^{(1),(2)} = k_{\mathrm{GD}}^T x \mp \sqrt{\frac{x^{\mathrm{T}}\mathrm{Qx}}{b^{\mathrm{T}}\mathrm{Pb}}}. \quad (7)$$

Lemma 1: [11] Let $V(x) = x^T P x, P > 0$, be a CLF of system (3). For any $x \neq 0$, define the following set: $U(x) = \{u \in R \,|\, \mathcal{D}V(x) \leq 0\}$. Then

i) $U(x) = \{u \in R \,|\, u^{(1)}(x) \leq u \leq u^{(2)}(x)\}$;
ii) $U(\alpha x) = \alpha U(x)$ for $\alpha > 0$;
iii) $u^{(1)}(x) = -u^{(2)}(-x)$ for any $x \in R^n$. ■

Further properties of $u^{(1)}(x)$ and $u^{(2)}(x)$, and of the CLF $V(x)$, are described by the following lemmas.

Lemma 2: The following statements are equivalent.

i) System (3) is stable.
ii) There is a *Lyapunov function*, $V(x) = x^T P x, P > 0$, such that $A^T P A - P \leq 0$.
iii) There is a CLF $V(x) = x^T P x, P > 0$, such that the corresponding $u^{(1)}(x)$ and $u^{(2)}(x)$ in (7) satisfy $u^{(1)}(x) u^{(2)}(x) \leq 0$.

Proof: The conclusion follows from the following calculations:

$$\begin{aligned} u^{(1)}(x)u^{(2)}(x) &= \frac{x^T A^T P b b^T P A x}{(b^T P b)^2} - \frac{x^T Q x}{b^T P b} \\ &= \frac{1}{b^T P b} x^T \left(\frac{A^T P b b^T P A}{b^T P b} - Q \right) x \\ &\overset{6}{=} \frac{1}{b^T P b} x^T \left(A^T P A - P \right) x. \end{aligned}$$

■

The interpretation of this lemma is clear: there are input values with different signs at any state to decrease the CLF if and only if the open-loop system is stable.

Proposition 1: Let $V(x) = x^T P x$ be a CLF for system (3), and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the generalized eigenvalues of the matrix pair $(A^T P A, P)$ in a decreasing order. If $\rho(A)$, the spectrum radius of A, is greater than 1, then $\lambda_1 > 1 > \lambda_2$.

Two different proofs of this result can be found in [10] and [29], respectively. The later contains also results for continuous-time systems.

Define a set, $\Gamma = \{x \,|\, u^{(1)}(x) \geq -\Delta, u^{(2)}(x) \leq \Delta\}$. By Lemma 1, if $x \in \Gamma$, then there is control input bounded by Δ such that $V(x)$ is decreasing at x along the system trajectories.

Lemma 3: Γ is a bounded and closed set containing the origin.

Proof: By the definition of Γ, we have

$$\begin{aligned} k_{\mathrm{GD}} x - \sqrt{\frac{x^T Q x}{b^T P b}} &\geq -\Delta \\ k_{\mathrm{GD}} x + \sqrt{\frac{x^T Q x}{b^T P b}} &\leq \Delta. \end{aligned}$$

These imply $x^T Q x \leq b^T P b \Delta^2$. ■

From Lemma 1, we see that there is a Δ-modulated control to decrease the CLF $V(x)$ if $\Delta \in U(x)$ or $-\Delta \in U(x)$. The next lemma gives a natural choice of such a Δ-modulated feedback.

Lemma 4: If $\Delta \in U(x)$ or $-\Delta \in U(x)$, then $\Delta \,\mathrm{sgn}(k_{\mathrm{GD}}^T x) \in U(x)$.

Proof: We show that $\Delta \in U(x)$ implies $\Delta \mathrm{sgn}(k_{\mathrm{GD}}^T x) \in U(x)$.

This implication is obvious when $k_{\mathrm{GD}}^T x$ is nonnegative.

When $k_{\mathrm{GD}}^T x$ is negative, first of all, $\Delta \mathrm{sgn}(k_{\mathrm{GD}}^T x) = -\Delta < 0 < u^{(2)}(x)$, in which the last inequality follows from the assumption that $\Delta \leq u^{(2)}(x)$.

On the other hand, since

$$\Delta \leq k_{\mathrm{GD}}^T x + \sqrt{\frac{x^T Q x}{b^T P b}}$$

we have

$$-k_{\mathrm{GD}}^T x - \sqrt{\frac{x^T Q x}{b^T P b}} \leq -\Delta$$

and $k_{\mathrm{GD}}^T x$ is negative, so we further have

$$k_{\mathrm{GD}}^T x - \sqrt{\frac{x^T Q x}{b^T P b}} \leq -k_{\mathrm{GD}}^T x - \sqrt{\frac{x^T Q x}{b^T P b}} \leq -\Delta$$

or $u^{(1)}(x) \leq -\Delta$. That is, $\Delta \mathrm{sgn}(k_{\mathrm{GD}}^T x) \in U(x)$.

Similarly, we can prove that $\Delta \mathrm{sgn}(k_{\mathrm{GD}}^T x) \in U(x)$ if $-\Delta \in U(x)$. ■

Due to this result, we assume that the Δ-modulated feedback is designed according to

$$u = \Delta \mathrm{sgn}(k_{\mathrm{GD}}^T x). \tag{8}$$

Lemma 5: Let $V(x) = x^T P x$ be a CLF for (3). For the closed-loop system under control of the Δ-modulated feedback (8), define the following set:

$$\Omega_i = \left\{ x \,|\, \mathcal{D}V(x) = V(x^+) - V(x) \geq 0 \right\}.$$

i) When $\rho(A) < 1$

$$\begin{aligned} \Omega_i &= \left\{ x \big| |u^{(1)}(x)| \leq \Delta, |u^{(2)}(x)| \leq \Delta \right\} \\ &= \left\{ x \,\Big|\, |k_{\mathrm{GD}}^T x| + \sqrt{\frac{x^T Q x}{b^T P b}} \leq \Delta \right\} \end{aligned}$$

and it is the intersection of two ellipsoids in R^n with centers at $x = \Delta (P - A^T P A)^{-1} A^T P b$ and $x = -\Delta (P - A^T P A)^{-1} A^T P b$, respectively.

ii) When $\rho(A) > 1, \Omega_i$ consists of three parts: The intersection of the intersecting sheets of two hyperboloids of two sheets in R^n, and the insides of the two nonintersecting sheets of the two hyperboloids.

Proof: First of all, note that when $\rho(A) < 1$, system (3) is stable; therefore, from Lemma 2, $u^{(1)}(x) \leq 0$ and $u^{(2)}(x) \geq 0$. Then, it is easy to see (from Lemma 1) that

$$\begin{aligned} &\{x \big| |u^{(1)}(x)| \leq \Delta, |u^{(2)}(x)| \leq \Delta\} \\ &= \left\{ x \,\Big|\, |k_{\mathrm{GD}}^T x| + \sqrt{\frac{x^T Q x}{b^T P b}} \leq \Delta \right\} \subset \Omega_i. \end{aligned}$$

On the other hand, if $x \notin \Omega_i$, then $u^{(1)}(x) < \pm\Delta < u^{(2)}(x)$. From here, it is easy to see that $x \notin \{x \,|\, |u^{(1)}(x)| \leq \Delta, |u^{(2)}(x)| \leq \Delta\}$.

Note moreover that in both cases of $\rho(A) < 1$ and $\rho(A) > 1, P - A^T P A$ is nonsingular: in the first case, $P - A^T P A$ is positive definite; and in the second case, according to Proposition 1, $\lambda_1 > 1$ and $\lambda_i < 1$, for $i = 2, 3, \ldots, n$, so $P - A^T P A$ is also nonsingular.

Having this, in order to prove the rest of parts i) and ii), we first do the following calculations (denote $\bar{\Delta} = \Delta(P - A^TPA)^{-1}A^TPb$):

$$\begin{aligned} &\mathcal{D}V(x) = (Ax - \Theta\Delta b)^T P(Ax - \Theta\Delta b) - x^TPx \geq 0 \\ &x^T(P - A^TPA)x + 2\Theta\Delta b^TPAx \leq b^TPb\Delta^2 \\ &(x + \Theta\bar{\Delta})^T(P - A^TPA)(x + \Theta\bar{\Delta}) \\ &\quad \leq \Delta^2 b^T(P + PA(P - A^TPA)^{-1}A^TP)b \end{aligned} \tag{9}$$

where $\Theta = \mathrm{sgn}(k_{\mathrm{GD}}^T x)$.

When $\Theta > 0$, the previous inequality becomes

$$\begin{aligned} &(x + \bar{\Delta})^T(P - A^TPA)(x + \bar{\Delta}) \\ &\quad \leq \Delta^2 b^T(P + PA(P - A^TPA)^{-1}A^TP)b. \end{aligned}$$

When $\Theta < 0$, it becomes

$$\begin{aligned} &(x - \bar{\Delta})^T(P - A^TPA)(x - \bar{\Delta}) \\ &\quad \leq \Delta^2 b^T(P + PA(P - A^TPA)^{-1}A^TP)b. \end{aligned}$$

Now, it is easy to see that when $\rho(A) < 1$, the last two inequalities define two ellipsoids, and when $\rho(A) > 1$, they are two hyperboloids of two sheets.

The two situations are depicted in Fig. 1. ■

Now, we return our attention to (1) in the rest of this paper. A controllable canonical realization of the system (1) takes the following form ($a_n = -a$):

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -a_n & 0 & \cdots & 0 & \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{10}$$

For such class of systems, the following result characterizes when a diagonal matrix gives a CLF.

Proposition 2: For (10), $V(x) = x^TPx, P = \mathrm{diag}(p_1, \ldots, p_n)$, is a CLF if and only if $p_n > p_{n-1} > \cdots > p_1 > 0$.

Proof: The conclusion is implied by the following calculations:

$$\begin{aligned} A^TPA &= \begin{bmatrix} 0 & -a_n \\ E_{n-1} & 0 \end{bmatrix} P \begin{bmatrix} 0 & E_{n-1} \\ -a_n & 0 \end{bmatrix} \\ &= \mathrm{diag}\left(a_n^2 p_n, p_1, \ldots, p_{n-1}\right) \\ b^TPA &= (0, \ldots, 0, p_n) \begin{bmatrix} 0 & E_{n-1} \\ -a_n & 0 \end{bmatrix} \\ &= (-a_n p_n, 0, \ldots, 0) \\ Q &= P - A^TPA + \frac{A^TPbb^TPA}{b^TPb} \\ &= \mathrm{diag}(p_1, p_2 - p_1, \ldots, p_n - p_{n-1}). \end{aligned}$$

Due to this result, the Δ-modulated feedback (8) for this class of systems becomes

$$u = \Delta \mathrm{sgn}(a_n x_1) \tag{11}$$

where x_1 is the first component of x. For notational convenience, denote the closed-loop mapping of system (10) under the feedback (11) as

$$f(x) = Ax + b\Delta \mathrm{sgn}(a_n x_1). \tag{12}$$

1) Definition 2: When $a_n < 0$, system (10) is classified to be of type-I, and when $a_n > 0$, type-II.

III. Globally Attracting Periodic Solutions: Case $|a_n| < 1$

From now on, we denote $a = |a_n|$, and define the following set of 2^n points:

$$\begin{aligned} \Omega_{ca} = \{&(\Delta/(1+a))\theta, \theta = (\theta_1, \theta_2, \ldots, \theta_n)^T \\ &\theta_i \in \{-1, 1\}, i = 1, 2, \ldots, n\}. \end{aligned}$$

The following property of Ω_{ca} is immediate; its proof is therefore omitted.

Lemma 6: For any CLF $V(x) = x^TPx$ of system (10), $P = \mathrm{diag}(p_1, \ldots, p_n)$, and for the closed-loop system under control of the Δ-modulated feedback (11), define

$$\Omega_b = \{x \,|\, \mathcal{D}V(x) = V(x^+) - V(x) = 0\}.$$

Then, $\Omega_{ca} \subset \Omega_b \cap S_r$, where S_r is a sphere centered at the origin with radius $r = (\sqrt{n}\Delta)/(1+a)$. ■

A. Periodicity

The set Ω_{ca} contains exactly all the periodic points. The following results also characterizes all the periods.

Theorem 1: Any $x \in \Omega_{ca}$ is a periodic point of the closed-loop system under control of the Δ-modulated feedback (11). For type-I systems, a positive integer l is a period for some $x \in \Omega_{ca}$ if and only if l is not a divisor of n, but a divisor of $2n$. For type-II systems, a positive integer l is a period for some $x \in \Omega_{ca}$ if and only if l is a divisor of n.

Proof: For any $x = \theta\Delta/(1+a) \in \Omega_{ca}$, if we denote $x^{(1)} = f(x)$, then

$$\begin{aligned} x^{(1)} &= Ax + b\Delta \mathrm{sgn}(a_n x_1) \\ &= \left(x_2, \ldots, x_n, -a_n\theta_1 \frac{\Delta}{1+a} + \Delta \mathrm{sgn}(a_n\theta_1)\right)^T \\ &= \left(x_2, \ldots, x_n, \Delta \mathrm{sgn}(a_n\theta_1)\left(1 - \frac{|a_n\theta_1|}{1+|a_n|}\right)\right)^T \\ &= \left(x_2, \ldots, x_n, \Delta \mathrm{sgn}(a_n\theta_1)\frac{1}{1+a}\right)^T \\ &= (x_2, \ldots, x_n, \mathrm{sgn}(a_n)x_1)^T. \end{aligned}$$

Similarly, if we denote $x^{(k)} = f^k(x)$, then for $k = 1, 2, \ldots, n$

$$x^{(k)} = (x_{k+1}, \ldots, x_n, \mathrm{sgn}(a_n)x_1, \ldots, \mathrm{sgn}(a_n)x_k)^T. \tag{13}$$

For type-I systems, $a_n < 0$, (13) leads to $x^{(n)} = -x$, which implies that n is not the period for any point in Ω_{ca} and, there-

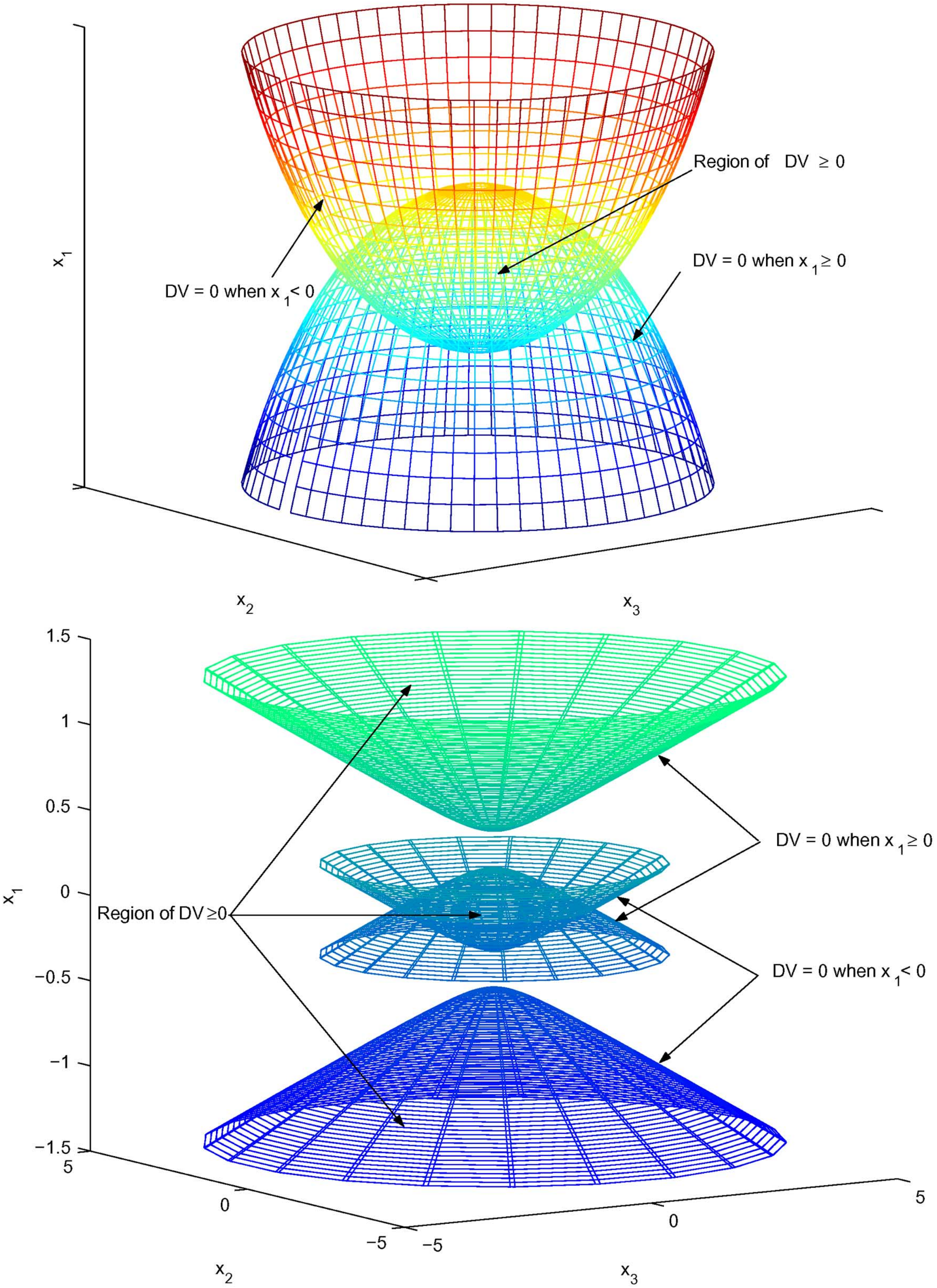

Fig. 1. Illustration of Lemma 5, with $\mathcal{D}V(x) \geq 0$.

fore, a positive integer l; that is, the period of a point in Ω_{ca} does not divide n.

With the same kind of reasoning, we have $x^{(2n)} = x$. Therefore, every point in Ω_{ca} is a periodic point of f, and a positive integer l is the period of a point of Ω_{ca} only if it is a divisor of $2n$ but a divisor of n.

On the other hand, if a positive integer l is a divisor of $2n$, but not a divisor of n, then, according to the integer division algorithm, there exists a non-negative integer α and a positive integer β, strictly less than l, such that

$$n = \alpha l + \beta. \tag{14}$$

From the aforementioned expression of n, it follows that l divides $2n$ implies l divides 2β, that is, $2\beta = k\,l$ for some positive integer k. Because $0 < \beta < l$, it follows necessarily that $k = 1$, and therefore $l = 2\beta$. From (14), we have $n = (2\alpha+1)l/2 = (2\alpha+1)\beta$.

Construct the following vector of n numbers:

$$\Bigg(\overbrace{\underbrace{1,\ldots,1}_{\beta},\underbrace{-1,\ldots,-1}_{\beta},\ldots,\underbrace{1,\ldots,1}_{\beta},\underbrace{-1,\ldots,-1}_{\beta},\underbrace{1,\ldots,1}_{\beta}}^{2\alpha+1}\Bigg)^T.$$

It is easily seen that the vector obtained via multiplying this vector by $\Delta/(1+a)$ belongs to Ω_{ca} and has period l.

For type-II systems, $a_n > 0$, and (13) leads to $x^{(n)} = x$. This proves that every point in Ω_{ca} is a periodic point of f, and an integer l is the period of a point in Ω_{ca} only if l divides n.

On the other hand, if l divides n, we can construct the following vector of n numbers:

$$c = \frac{\Delta}{1+a}\Bigg(\underbrace{1,-1,\ldots,-1}_{l},\ldots,\underbrace{1,-1,\ldots,-1}_{l}\Bigg)^T.$$

This vector belongs to Ω_{ca} and has period l. ■

In the following, we will prove that when $0 < a < 1$, there are no other periodic points outside Ω_{ca}. To this end, we first prove the following lemmas.

Lemma 7: For any $x \in R^n$, denote the i-th components of x and $f^n(x), 1 \le i \le n$, by x_i and $f_i^n(x)$, respectively. Then

$$\lim_{l\to\infty}\left|f_i^{ln}(x)\right| = \frac{\Delta}{1+a}. \tag{15}$$

Proof: By the definition of the Δ-modulated feedback (11), for any $x \in R^n$

$$f(x) \quad = \quad (x_2, x_3, \ldots, -a_n x_1 + \Delta\mathrm{sgn}(a_n x_1))^T.$$

Generally, for each $1 \le k \le n-1$, we have

$$\begin{aligned} f^k(x) = (x_{k+1},\ldots,x_n,-a_n x_1 + \Delta\mathrm{sgn}(a_n x_1)\\ \ldots, -a_n x_k + \Delta\mathrm{sgn}(a_n x_k))^T. \end{aligned} \tag{16}$$

In particular

$$f^n(x) = -a_n x + \Delta\mathrm{sgn}(a_n x). \tag{17}$$

where $\mathrm{sgn}(a_n x) = (\mathrm{sgn}(a_n x_1), \ldots, \mathrm{sgn}(a_n x_n))^T$.

From (17), for any two points $x, y \in R^n$

i) when $\mathrm{sgn}(x_i) = \mathrm{sgn}(y_i), f_i^n(y) - f_i^n(x) = -a_n(y_i - x_i)$;

ii) when $\mathrm{sgn}(x_i) \ne \mathrm{sgn}(y_i), f_i^n(y) + f_i^n(x) = -a_n(y_i + x_i)$.

These show that $\||f_i^n(y)| - |f_i^n(x)\|| \le a\||y_i| - |x_i\||$.

Generally, for any natural number l, we have $\||f_i^{ln}(y)| - |f_i^{ln}(x)\|| \le a^l\||y_i| - |x_i\||$. Therefore, $\lim_{l\to\infty}\||f_i^{ln}(y)| - |f_i^{ln}(x)\|| \le \lim_{l\to\infty} a^l\||y_i| - |x_i\|| = 0$.

In particular, take $y \in \Omega_{ca}$, we know from the definition of Ω_{ca} and Theorem 1, that $|f_i^{ln}(y)| \equiv \Delta/(1+a)$ for all $l \ge 0$. Hence, we arrive at conclusion (15). ■

Lemma 8: Suppose $0 < a < 1$ and let $x \in R^n$ be an initial state. Then, for the orbit $\{x^{(k)}, k \ge 0\}$ starting from x, the following conclusions hold true.

i) For $a_n > 0$, the sub-sequence $x^{(mn)}, m = 0, 1, \ldots$, has a limit when $m \to \infty$, and $\lim_{m\to\infty} x^{(mn)} \in \Omega_{ca}$.

ii) For $a_n < 0$, the sub-sequence $x^{(2mn)}, m = 0, 1, \ldots$, has a limit when $m \to \infty$, and $\lim_{m\to\infty} x^{(2mn)} \in \Omega_{ca}$.

Proof: For any given point $x \in R^n$, denote $\{|x_1|, |x_2|, \ldots, |x_n|\}$ by $|x|$. According to Lemma 7, the sequence $\{|x^{(0)}|, |x^{(n)}|, \ldots, |x^{(mn)}|, \ldots\}$ must be convergent and converges to the point $x^\star = \theta\Delta/(1+a) \in \Omega_{ca}$, where $\theta = \{1, 1, \ldots, 1\}$. Remember the norm $\|x\|_\infty = \max_{i=1}^n |x_i|$ for $x \in R^n$. It is calculated that $\|x^\star\|_\infty = \Delta/(1+a)$, there exists a natural number M such that $\|x^{(mn)}\|_\infty < \Delta/a$ for all $m \ge M$. Therefore, the following hold.

i) When $a_n > 0$, it is easily to see from formula (17) that $\mathrm{sgn}(x)$ and $\mathrm{sgn}(f^n(x))$ are the same when $\|x\|_\infty < \Delta/a$. Hence, one can conclude that the sequence $\{x^{(0)}, x^{(n)}, \ldots, x^{(mn)}, \ldots\}$ is convergent and converges to some point in Ω_{ca}.

ii) When $a_n < 0$, one can verify, based on formula (17) again, that $\mathrm{sgn}(f^n(x)) = -\mathrm{sgn}(x)$ when $\|x\|_\infty < \Delta/a$. Hence, the sequence $\{x^{(0)}, x^{(2n)}, \ldots, x^{(2mn)}, \ldots\}$ is convergent and converges to some point in Ω_{ca}.

■

Theorem 2: Ω_{ca} is a global attractor of system (12).

Proof: We first prove the invariance of Ω_{ca}.

For any $x = (\theta_1, \ldots, \theta_n)^T\Delta/(1+a) \in \Omega_{ca}$, in which $\theta_i \in \{-1, 1\}, i = 1, 2, \ldots, n$, we have proven in (13) that

$$f(x) = \frac{\Delta}{1+a}(\theta_2, \theta_3, \ldots, \mathrm{sgn}(a_n)\theta_1)^T \in \Omega_{ca}$$

implying that $f(\Omega_{ca}) \subset \Omega_{ca}$. On the other hand, for any $x \in \Omega_{ca}$, suppose l is the period of x. Then, $f^{l-1}(x) \in \Omega_{ca}, x = f(f^{l-1}(x)) \in \Omega_{ca}$, hence, $\Omega_{ca} \in f(\Omega_{ca})$.

We next prove that Ω_{ca} is globally attracting. For any given point $x \in R^n$, let $x^{(k)} = f^k(x), k = 0, 1, \ldots, 2n-1$. Then, from Lemma 8, for $a_n > 0$, these sub-sequences $x^{(mn+k)} = f^{mn}(x^k), k = 0, 1, \ldots, n-1$, and for $a_n < 0$, $x^{(2mn+k)} = f^{2mn}(x^k), k = 0, 1, \ldots, 2n-1$, are all convergent (or *forward asymptotic* [9]) to Ω_{ca}. This implies that Ω_{ca} is globally attracting. ■

Theorem 3: When $n > 1$, denote $n = \prod_{i=1}^{q} p_i^{n_i}$, where $n_i > 0$, and $p_i, i = 1, 2, \ldots, q$, are different prime factors of n in increasing order. Then, the following conclusions hold true.

i) When $n = 1, \Omega_{ca}$ consists of only one forward orbit of period 2 for systems of type-I, and two equilibria for systems of type-II.

ii) When $n > 1$, for systems of type-II, a positive integer l is the period of a point $x \in \Omega_{ca}$ if and only if it takes the form

$$l = \prod_{i=1}^{q} p_i^{d_i} \quad 0 \le d_i \le n_i, \qquad i = 1, 2, \ldots, q. \tag{18}$$

There are two periodic-1 (fixed) points in Ω_{ca}.

For any $l \neq 1$ of the form (18), let $p_m, 1 \leq m \leq q$, be the smallest prime factor of l, i.e., $d_1 = \cdots = d_{m-1} = 0, d_m \neq 0$. Denote $\bar{l} = (l)/(p_m)$. Then, the number of distinct periodic orbits in Ω_{ca} with period l equals $2^l - 2^{\bar{l}}$.

For systems of type-I, factorize n into the form

$$n = 2^{n_0} \prod_{i=1}^{q} p_i^{n_i}, \quad p_i \neq 2,\ n_0 \geq 0,\ n_i > 0,\ i = 1, \ldots, q. \tag{19}$$

Then a positive integer l is the period of a point $x \in \Omega_{ca}$ if and only if it takes the form

$$l = 2^{n_0+1} \prod_{i=1}^{q} p_i^{d_i}, \qquad 0 \leq d_i \leq n_i, \quad i = 1, 2, \ldots, q. \tag{20}$$

There is one periodic-2 orbit in Ω_{ca}.

For any $l \neq 2$ of the form (20), let $p_m, 1 \leq m \leq q$, be the second smallest prime factor of l, i.e., $d_1 = \cdots = d_{m-1} = 0, d_m \neq 0$. Denote $\bar{l} = (l)/(p_m)$. Then, the number of distinct periodic orbits in Ω_{ca} with period l equals $2^l - 2^{\bar{l}}$.

Proof:

i) When $n = 1, \Omega_{ca}$ contains only two points. According to Theorem 1, it is clear that both points are equilibria for systems of type-II, and they form a periodic orbit with period 2 for systems of type-I.

ii) According to Theorem 1, a positive integer is the period of some periodic point if and only if it is a divisor of n. The expression (18) is a parameterization of all the factors of n. Since 2^l is the number of periodic orbits in Ω_{ca} with periods less than or equal to l, it follows the conclusion about the total number of periodic orbits with period l.

iii) Again, according to Theorem 1, a positive integer is the period of some point if and only if it is a divisor of $2n$ but n. It is straightforward to verify that the expression in (19) is a parameterization of all such integers. ■

B. Attracting Regions

Since all the periodic points in Ω_{ca} are attracting, it will be useful to know the attracting region for each of the periodic points.

First, we introduce a concept. For any real number x, the characteristic index κ is defined as the following non-negative integer:

$$\kappa = \left\lfloor \log_a \left(\frac{\Delta}{\Delta + (1-a)|x|} \right) \right\rfloor$$

where $\lfloor \cdot \rfloor$ denotes the integer part of a real number. For any $x \in R^n$, the characteristic index is an n-vector $(\kappa_1, \ldots, \kappa_n)^T$, in which κ_i is the characteristic index of x_i, for $i = 1, 2, \ldots, n$.

Lemma 9:

i) For any $x \in R^n$, and for each $i = 1, 2, \ldots, n$, the characteristic index κ_i is the smallest non-negative integer m such that $|x_i^{(mn)}| < \Delta/a$.

ii) For systems of type-I, κ_i is the smallest non-negative integer m such that $x_i^{(mn)}$ and $x_i^{((m+1)n)}$ have opposite signs.

iii) For systems of type-II, κ_i is the smallest non-negative integer m such that $x_i^{(mn)}$ and $x_i^{((m+1)n)}$ have the same sign.

Proof: We prove the result only for systems of type-I. Proof for systems of type-II can be worked out in similar lines, and is therefore omitted.

For systems of type-I, according to Lemma 8 and the procedure of the proof for the lemma, it is easy to understand that $|x_i^{(mn)}| < \Delta/a$ if and only if $x_i^{(mn)}$ and $x_i^{((m+1)n)}$ have different signs.

For $m \leq \kappa_i$, by (17), we have

— if $x_i \geq 0$, then

$$\begin{aligned} x_i^{(mn)} &= a^m x_i - a^{m-1}\Delta - \cdots - a\Delta - \Delta \\ &= a^m |x_i| - \frac{(1-a^m)}{(1-a)}\Delta; \end{aligned}$$

— if $x_i < 0$, then

$$\begin{aligned} x_i^{(mn)} &= a^m x_i + a^{m-1}\Delta + \cdots + a\Delta + \Delta \\ &= -a^m |x_i| + \frac{(1-a^m)}{(1-a)}\Delta. \end{aligned}$$

It is straightforward to verify that the real number $s = \log_a \Delta/(\Delta + (1-a)|x_i|)$ satisfies

$$a^s |x_i| - \frac{(1-a^s)}{(1-a)}\Delta = 0.$$

Therefore, it is easy to see that $\kappa_i = \lfloor s \rfloor$ is the smallest integer such that $x_i^{((m+1)n}$ changes sign. The conclusions of the lemma follow. ■

The analysis in the previous proof can be useful in finding the limiting periodic points in Ω_{ca}. We do this separately for the two types of systems.

Note that for systems of type-II, we have

$$x^{((m+1)n)} = \begin{bmatrix} -ax_1^{(mn)} + \operatorname{sgn}\left(x_1^{(\kappa_1 n)}\right)\Delta \\ -ax_2^{(mn)} + \operatorname{sgn}\left(x_2^{(\kappa_2 n)}\right)\Delta \\ \vdots \\ -ax_n^{(mn)} + \operatorname{sgn}\left(x_n^{(\kappa_n n)}\right)\Delta \end{bmatrix}.$$

By ii.2) of Lemma 9, $x_i^{(mn)}$ has the same sign as $x_i^{(\kappa_i n)}$, for $m \geq \kappa_i, i = 1, 2, \ldots, n$. Therefore, for $m \geq \max(\kappa_1, \kappa_2, \ldots, \kappa_n)$, we have

$$x^{((m+1)n)} = \begin{bmatrix} -ax_1^{(mn)} + \operatorname{sgn}\left(x_1^{(\kappa_1 n)}\right)\Delta \\ -ax_2^{(mn)} + \operatorname{sgn}\left(x_2^{(\kappa_2 n)}\right)\Delta \\ \vdots \\ -ax_n^{(mn)} + \operatorname{sgn}\left(x_n^{(\kappa_n n)}\right)\Delta \end{bmatrix}.$$

Denote the limit of $x^{(mn)}$ by x^*. Then, for each $i = 1, 2, \dots, n$, we can solve x_i^* from

$$x_i^* = -ax_i^* + \mathrm{sgn}\left(x_i^{(\kappa_i n)}\right)\Delta$$

to obtain

$$x_i^* = \frac{\mathrm{sgn}\left(x_i^{(\kappa_i n)}\right)\Delta}{1+a}.$$

For systems of type-I, first let κ_i^e be the next even integer or zero after κ_i (that is, $\kappa_i^e = \kappa_i$ if κ_i is even (or zero), and $\kappa_i^e = \kappa_i + 1$ if κ_i is odd). Then, from ii.1) of Lemma 9, $x_i^{(2mn)}$ has the same sign as $x_i^{(\kappa_i^e n)}$, for $m \geq (\kappa_i^e)/(2), i = 1, 2, \dots, n$. Therefore, for $2m \geq \max(\kappa_1^e, \kappa_2^e, \dots, \kappa_n^e)$, we have

$$x^{(2(m+1)n)} = \begin{bmatrix} a^2 x_1^{(mn)} + (1-a)\mathrm{sgn}\left(x_1^{(\kappa_1^e n)}\right)\Delta \\ a^2 x_2^{(mn)} + (1-a)\mathrm{sgn}\left(x_2^{(\kappa_1^e n)}\right)\Delta \\ \vdots \\ a^2 x_n^{(mn)} + (1-a)\mathrm{sgn}\left(x_n^{(\kappa_1^e n)}\right)\Delta \end{bmatrix}.$$

Denote the limit of $x^{(2mn)}$ by x^*. Then, for each $i = 1, 2, \dots, n$, we can solve x_i^* from

$$x_i^* = a^2 x_i^* + (1-a)\mathrm{sgn}(x_i^{(\kappa_i^e n)})\Delta$$

to obtain

$$x_i^* = \frac{\mathrm{sgn}\left(x_i^{(\kappa_i^e n)}\right)\Delta}{1+a}.$$

Summarizing the above development, we have the following characterization of the attracting region for a periodic point in Ω_{ca}.

Theorem 4: For any $x \in R^n$, denote its characteristic index as $(\kappa_1, \kappa_2, \dots, \kappa_n)^T$. A generic periodic point in Ω_{ca} is denoted as $\Theta = (\Delta)/(1+a)(\theta_1, \theta_2, \dots, \theta_n)^T, \theta_i \in \{-1, 1\}$, for $i = 1, 2, \dots, n$.

i) For systems of type-I, x belongs to the attracting region of Θ if and only if for $i = 1, 2, \dots, n$,

$$\mathrm{sgn}\left(x_i^{(\kappa_i n)}\right) = \begin{cases} \theta_i, & \text{for even } \kappa_i \\ -\theta_i, & \text{for odd } \kappa_i. \end{cases}$$

ii) For systems of type-II, x belongs to the attracting region of Θ if and only if for $i = 1, 2, \dots, n, \mathrm{sgn}(x_i^{(\kappa_i n)}) = \theta_i$.

The attracting region for a two-dimensional case $(0 < a_n < 1)$ is visualized in Fig. 2. In this case, there are four periodic points: the two in the first and third quadrants are fixed (1-periodic) points, and the two in the second and the fourth quadrants are 2-periodic points. The plane is divided into four (disconnected) parts marked by different strips. All points in each of these four parts are attracted to the corresponding periodic point in the same part.

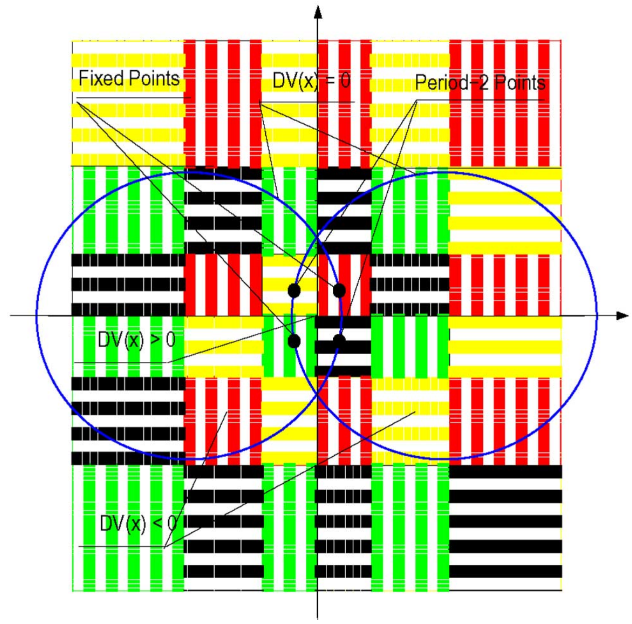

Fig. 2. Attracting region: The two-dimensional case with $a_n > 0$.

IV. Locally Attractor With Bounded Attracting Region: Case $1 \leq |a_n| \leq 2$

For convenience, denote the product of k times of a region $D, \underbrace{D \times D \times \cdots \times D}_{k}$, by D^k. We also use the following notations:

$$\Omega^k = [-\Delta, \Delta)^k, \overline{\Omega}^k = [-\Delta, \Delta]^k$$
$$\Gamma_0^k = \left(-\frac{\Delta}{a-1}, \frac{\Delta}{a-1}\right)^k, \overline{\Gamma}_0^k = \left[-\frac{\Delta}{a-1}, \frac{\Delta}{a-1}\right]^k.$$

Theorem 5:

i) When $1 \leq |a_n| \leq 2, \Omega^n$ is an invariant set of the closed-loop system (12), i.e., $f(\Omega^n) = \Omega^n$.

ii) When $1 < |a_n| < 2, \Omega^n$ is an attractor with the attracting region Γ_0^n.

Proof:

i) We only prove the results for $a_n < 0$. Note

$$f(x) = (x_2, x_3, \dots, x_n, ax_1 - \mathrm{sgn}(x_1)\Delta)^T \tag{21}$$

it is obvious that, when $x \in \Omega^n, x_i \in \Omega$ for all $1 \leq i \leq n$. $x_1 \in \Omega$, in turn, implies that $ax_1 - \mathrm{sgn}(x_1)\Delta \in \Omega$. That is, $f(x) \in \Omega^n$, or $f(\Omega^n) \subset \Omega^n$. On the other hand, a point $y \in \Omega^n$ is equivalent to $y_i \in \Omega$ for all $1 \leq i \leq n$. Hence, we have $0 \leq x_1 = (y_n + \Delta)/a < (\Delta)/(a) \leq \Delta$, when $-\Delta \leq y_n < 0$, and $-\Delta \leq -\Delta/a \leq x_1 = (y_n - \Delta)/a < 0$, when $0 \leq y_n < 0\Delta$.

One verifies that for $x = (x_1, y_1, \dots, y_{n-1})^T, y = f(x)$. Hence, we have proved $\Omega^n \subset f(\Omega^n)$.

i) At first, we show that

$$\|f(x)\|_\infty \geq \frac{\Delta}{a-1}$$

when $\|x\|_\infty \geq \Delta/(a-1)$. As a matter of fact, for any $x \in R^n \setminus \Gamma_0^n$, there exists an $i \in \{1, 2, \ldots, n\}$, such that $x_i \in R \setminus \Gamma_0$. We have

$$\begin{cases} \|f(x)\|_\infty \geq \Delta/(a-1), & \text{if } i \neq 1 \\ ax_1 - \mathrm{sgn}(x_1)\Delta \geq \Delta/(a-1), & \text{if } i = 1 \text{ and } x_1 \geq \Delta/(a-1) \\ ax_1 - \mathrm{sgn}(x_1)\Delta \leq -\Delta/(a-1), & \text{if } i = 1 \text{ and } x_1 \leq -\Delta/(a-1). \end{cases}$$

Therefore, $f(x) \in R^n \setminus \Gamma_0^n$.

Furthermore, when $x \in \Gamma_0^n \setminus \Omega^n$, it follows that there exists an $i \in \{1, 2, \ldots, n\}$, such that $x_i \in \Gamma_0 \setminus \Omega$, and

$$\begin{cases} 0 \leq ax_i - \Delta < x_i, & \text{when } \Delta \leq x_i < 0\Delta/(a-1) \\ x_i < ax_i + \Delta < 0, & \text{when } -\Delta/(a-1) \leq x_1 < -\Delta. \end{cases}$$

That is, $\|f^n(x)\|_\infty < \|x\|_\infty$, any point in $\Gamma_0^n \setminus \Omega^n$ tends to the invariant set Ω^n. ■

The "boundary" case when $a = 1$ deserves special attention.

Theorem 6: When $a = 1$, the following conclusions hold true.

i) The region Ω^n is a global attractor.

ii) A point $x \in R^n$ is a periodic point if and only if $x \in \Omega^n$. In particular, every point $x \in \Omega^n$ is a periodic point with a period $2n$.

iii) For Type-I systems: iii.1) There is no $(2k+1)n$ periodic point for any $k \geq 0$.

iv) A positive integer l is a prime period of some point if and only if l is even, and a divisor of $2n$ but n. In this case, if denote $l = 2\beta$ and $n = (2\alpha + 1)\beta$, then a point $x \in \Omega^n$ is an l-periodic point if and only if

$$\begin{aligned} x &= \left(x^1, x^2, \ldots, x^{2\alpha+1}\right)^T \\ x^1 &= x^3 = \cdots = x^{2\alpha+1} = (\xi, \xi, \ldots \xi)^T \in R^\beta \\ x^2 &= x^4 = \cdots = x^{2\alpha} = x^1 - \mathrm{sgn}(x^1)\Delta \end{aligned}$$

for some $0 \neq \xi \in [-(\Delta/a), (\Delta/a))$.

v) For Type-II systems:

vi) a point x is of period n if and only if

$$x = \frac{\Delta}{2}(\theta_1, \theta_2, \ldots, \theta_n)^T$$

where $|\theta_i| = 1, i = 1, 2, \ldots, n$;

vii) a positive integer l is the prime period of some point $x \in \Omega^n$ if and only if l is a divisor of $2n$.

Proof: First of all, for a given positive integer l and $z = (z_1, z_2, \ldots, z_l)^T \in R^l$, denote

$$(\mathrm{sgn}(z_1), \mathrm{sgn}(z_2), \ldots, \mathrm{sgn}(z_l))^T$$

by $\mathrm{sgn}(z)$.

i) Here, we only prove the conclusion for Type-I systems. The proof of the conclusion for Type-II systems can be similarly carried out.

From (21) and the inequalities

$$\begin{cases} -\Delta \leq x_1 - \Delta \leq 0 & \forall x_1 \in [0, \Delta] \\ 0 \leq x_1 + \Delta < \Delta & \forall x_1 \in [-\Delta, 0) \end{cases}$$

we have

$$f(x) \in \Omega^n \qquad \forall x \in \overline{\Omega}^n.$$

Besides, when $\|x\|_\infty > \Delta$, since for any $1 \leq k \leq n$

$$f^k(x) = (x_{k+1}, x_{k+2}, \ldots, x_n, x_1 - \mathrm{sgn}(x_1)\Delta \\ \ldots, x_k - \mathrm{sgn}(x_k)\Delta)^T \tag{22}$$

and from the expression $x_i - \mathrm{sgn}(x_i)\Delta$, for each $1 \leq i \leq k$, we have

$$\begin{cases} 0 < x_i - \Delta < x_i & \forall x_i < \Delta \\ 0 > x_i + \Delta > x_i & \forall x_i < -\Delta. \end{cases}$$

Therefore, for every $1 \leq i \leq n-1$, it follows that

$$\|f^{i+1}(x)\|_\infty \leq \|f^i(x)\|_\infty.$$

In particular

$$\|f^n(x)\|_\infty < \|x\|_\infty.$$

Furthermore, it is not hard to verify that the inequality

$$\|f^{i+kn}(x)\|_\infty < \|f^{i+(k-1)n}(x)\|_\infty$$

holds true for all $k \geq 1$ and $0 \leq i < n$. Clearly, this inequality implies that conclusion i) holds true for Type-I systems.

i) From conclusion i), it is easily seen that there is no periodic point in the region $R^n \setminus \Omega^n$. For Type-I systems, when $x = (x_1, x_2, \ldots, x_n)^T \in \Omega^n$, by (22), one gets

$$f^n(x) = x - \mathrm{sgn}(x)\Delta.$$

The equality implies obviously $\mathrm{sgn}(f^n(x)) = -\mathrm{sgn}(x)$, for all $x \in \Omega^n$. Moreover, it follows that $f^{2n}(x) = x$. Similarly, one can verify the conclusion for Type-II systems.

ii) At first, from the proof of the conclusion ii), it is obvious that $f^n(x) \neq x$ for all $x \in \Omega^n$. Generally, for any nonnegative integer k, denote $x^{(l)} = f^{ln}(x)$ for each $0 \leq l \leq 2k+1$. Then, it follows that

$$f^{(2k+1)n}(x) = x - \Delta \sum_{l=0}^{2k} \mathrm{sgn}(x^{(l)}).$$

It is obvious that the second term at the right side of the previous equality does not equal zero, so $f^{(2k+1)n}(x) \neq x$.

iii) The proof of the conclusion can be similarly done as the proof of Theorem 1.

iv) The two conclusions are two special cases of Theorem 1 for Type-II systems.

V. CHAOTIC REPELLER OF CANTOR SET: CASE $|a_n| > 2$

We will show in this case that the "stabilizable" set is a Cantor set. We will give a construction of this set, and show that it is a repeller of the closed-loop system, and the closed-loop system is chaotic on it.

A. Preliminaries

In this paper, we use the definition of chaos as follows.

Definition 2 [14]: For an attractor or repeller F of the dynamical system (14), the motion of (12) is called chaotic on F if

i) there is an $x \in F$, such that the orbit $\{f^k(x)\}$ is dense in F;

ii) the set of periodic points of f in F is dense in F;

iii) f is sensitive to initial conditions, that is, for any x, there is a $\delta > 0$, a y in arbitrary close vicinity of x, and an integer k such that $|f^k(x) - f^k(y)| \geq \delta$.

First of all, we have the following result about periodic points.

Lemma 10: Suppose $|a_n| \geq 2$. Then, for each $m \geq 1$, a point $z^{(0)}$ is a periodic point with period m of Type-I systems if and only if it has the following form:

$$z^{(0)} = \frac{\Delta}{a^m - 1}\sum_{i=0}^{m-1} a^{m-i-1}\theta_i. \tag{23}$$

It is a periodic point with period m of Type-II systems if and only if it has the following form:

$$z^{(0)} = \frac{\Delta}{a^m + (-1)^{m+1}}\sum_{i=0}^{m-1} (-1)^i a^{m-i-1}\theta_i \tag{24}$$

where $\theta_i = (\theta_{1i}, \theta_{2i}, \ldots, \theta_{ni})^T, \theta_{ji} \in \{-1, 1\}$ for all $0 \leq i \leq m-1$ and $1 \leq j \leq n$.

The same result was obtained in [30] for the scalar case $n = 1$. The proof of Lemma 10 can be worked out in similar lines, thus omitted here.

B. Construction of Cantor Set

The construction of the Cantor set is given in the following lemmas.

Lemma 11:

i) Suppose $|a_n| > 2$. For $i = 1, 2, \ldots$, there exists a closed set $\overline{\Gamma}_i \subset \overline{\Gamma}_{i-1}$ such that

$$\overline{\Gamma}_i^n = \left\{x \,|\, \|f^n(x)\|_\infty \leq \frac{\Delta}{a-1}, x \in \overline{\Gamma}_{i-1}^n\right\}.$$

ii) For $l \geq 1, f^n(\overline{\Gamma}_{l-1}^n) \bigcap \overline{\Gamma}_{l-1}^n = f^n(\overline{\Gamma}_l^n)$, and $f^n(\overline{\Gamma}_l^n) \bigcap \overline{\Gamma}_l^n = \overline{\Gamma}_l^n$.

iii) Define for $l = 1, 2, \ldots$

$$\overline{\Gamma}_{l,+}^n = \left\{x = (x_1, x_2, \ldots, x_n)^T \middle| x \in \overline{\Gamma}_l^n, x_j > 0, 1 \leq j \leq n\right\} \tag{25}$$

then

$$f^n(\overline{\Gamma}_l^n) = \overline{\Gamma}_{l-1}^n \tag{26}$$
$$f^n(\overline{\Gamma}_{l,+}^n) = \overline{\Gamma}_{l-1}^n. \tag{27}$$

iv) $m(\overline{\Gamma}_l^n)/m(\overline{\Gamma}_{l-1}^n) = 2^n/a^n$, where $m(\cdot)$ is the Euclidean measure of a region in the n-dimensional space.

Proof:

i) Due to (17), it is readily constructed that

$$\overline{\Gamma}_1 = \overline{\Gamma}_0 \setminus \left(-\frac{\Delta(a-2)}{a(a-1)}, \frac{\Delta(a-2)}{a(a-1)}\right). \tag{28}$$

Similar constructions exist for $\overline{\Gamma}_i$, for $i = 2, 3, \ldots$.

ii) According to the definition of the system of the regions $\overline{\Gamma}_l^n$, one sees that $x \in \overline{\Gamma}_l^n \setminus \overline{\Gamma}_{l+1}^n$ if and only if $f^{ln}(x) \in \overline{\Gamma}_0^n \setminus \overline{\Gamma}_1^n$.

Furthermore, since for all $l \geq 1, \overline{\Gamma}_{l-1}^n = \overline{\Gamma}_l^n \bigcup (\overline{\Gamma}_{l-1}^n \setminus \overline{\Gamma}_i^n)$ and $f^n(\overline{\Gamma}_{l-1}^n \setminus \overline{\Gamma}_l^n) \bigcap \overline{\Gamma}_{l-1}^n = \emptyset$, the first equality in ii) holds true.

It is easily verified that the second equality in ii) holds true for $i = 0$. Assume it holds true for some $i > 0$. Then, we have

$$f^n(\overline{\Gamma}_{i+1}^n) \bigcap \overline{\Gamma}_{i+1}^n \overset{(30)}{=} f^n(\overline{\Gamma}_i^n) \bigcap \overline{\Gamma}_i^n \bigcap \overline{\Gamma}_{i+1}^n$$
$$= \overline{\Gamma}_i^n \bigcap \overline{\Gamma}_{i+1}^n = \overline{\Gamma}_{i+1}^n.$$

i) Formula (26) follows directly from conclusion ii). To prove (27), we first discuss the case $i = 1$. Let x be a point in $\in \overline{\Gamma}_{1,+}^n$. Then, for Type-I systems, by the expression $f^n(x) = (ax_1 - \Delta, ax_2 - \Delta, \ldots, ax_n - \Delta)^T$, one can verify that the inequalities $-\Delta/(a-1) \leq ax_i - \Delta \leq \Delta/(a-1)$ hold true for all $1 \leq i \leq n$, therefore, $f^n(\overline{\Gamma}_{1,+}^n) = \overline{\Gamma}_0^n$.

For Type-II systems, by the expression $f^n(x) = -(ax_1 - \Delta, ax_2 - \Delta, \ldots, ax_n - \Delta)^T$, one can similarly get the result $f^n(\overline{\Gamma}_{1,+}^n) = \overline{\Gamma}_0^n$.

Generally, with respect to the aforementioned result, for every $y \in \overline{\Gamma}_{i-1}^n \subset \overline{\Gamma}_0^n$, there exists a point $x \in \overline{\Gamma}_{1,+}^n$ such that $y = f^n(x)$. Moreover, $f^{(i-1)n}(y) = f^{in}(x)$, therefore, according to the definition of the region $\overline{\Gamma}_{i-1}^n$, we have $\|f^{in}(x)\|_\infty \leq \Delta/(a-1)$. This means, of course, that $x \in \overline{\Gamma}_{i,+}^n \subset \overline{\Gamma}_i^n$.

ii) By mathematical induction, when $i = 1$, for both Type-I and Type-II systems, it is routine to verify the following results:

$$\overline{\Gamma}_1^n = \left\{x \middle| \frac{\Delta(a-2)}{a(a-1)} \leq |x_i| \leq \frac{\Delta}{a-1} \quad \forall 1 \leq i \leq n\right\}$$
$$\overline{\Gamma}_{1,+}^n = \left\{x \middle| \frac{\Delta(a-2)}{a(a-1)} \leq x_i \leq \frac{\Delta}{a-1} \quad \forall 1 \leq i \leq n\right\}.$$

Hence, we get that

$$m(\overline{\Gamma}_1^n) = 2^n m(\overline{\Gamma}_{1,+}^n) = \Delta^n 2^{2n}/(a^n(a-1)^n)$$

and thus $m(\overline{\Gamma}_1^n)/m(\overline{\Gamma}_0^n) = 2^n/a^n$.

Assume iv) holds true for some $l > 1$. For $l+1$, by (26) and the assumption of the induction, one gets

$$\frac{m(f^n(\overline{\Gamma}_{l+1}^n))}{m(f^n(\overline{\Gamma}_l^n))} = \frac{m(\overline{\Gamma}_l^n))}{m(\overline{\Gamma}_{l-1}^n))} = \frac{2^n}{a^n}.$$

On the other hand, by (27) and the monotonicity of the mapping $f^n(\cdot)$ along the direction of each coordinate axis on the region $\overline{\Gamma}_{1,+}^n$, it is easily verified that, for every $i \geq 1, m(f^n(\overline{\Gamma}_i^n)) = m(f^n(\overline{\Gamma}_{i,+}^n)) = a^n m(\overline{\Gamma}_{i,+}^n)$. Hence, we conclude that

$$\frac{m(\overline{\Gamma}_{l+1}^n)}{m(\overline{\Gamma}_l^n)} = \frac{m(\overline{\Gamma}_{l+1,+}^n)}{m(\overline{\Gamma}_{l,+}^n)} = \frac{m(f^n(\overline{\Gamma}_{l+1,+}^n))}{m(f^n(\overline{\Gamma}_{l,+}^n))} = \frac{m(\overline{\Gamma}_l^n)}{m(\overline{\Gamma}_{l-1}^n)} = \frac{2^n}{a^n}.$$

Lemma 12:

i) For $l \geq 1$ and $1 \leq k \leq n$

$$\overline{\Gamma}_l^k \times \overline{\Gamma}_{l-1}^{n-k} = \left\{ x \,\middle|\, \left\| f^{(l-1)n+k}(x) \right\|_\infty \leq \frac{\Delta}{a-1}, x \in \overline{\Gamma}_l^{k-1} \times \overline{\Gamma}_{l-1}^{n-k+1} \right\}. \quad (29)$$

ii) $\overline{\Gamma}_l^k \times \overline{\Gamma}_{l-1}^{n-k}$ is symmetric about the origin of the n-dimension space.

Proof:

i) This can be routinely verified by the definition of $\overline{\Gamma}_l$ and the expressions of $f^{(l-1)n+k}(x)$ [refer to (16)].

ii) From (28), it is clear that the two sets $\overline{\Gamma}_1^k$ and $\overline{\Gamma}_0^{n-k}$ are symmetric. Therefore, the symmetry of these two regions means the symmetry of the set $\overline{\Gamma}_1^k \times \overline{\Gamma}_0^{n-k}$.

Assume that assertion i) holds true for some $l > 1$ and $0 \leq k \leq n-1$. For $k+1$, by the definition of the region $\overline{\Gamma}_l^{k+1} \times \overline{\Gamma}_{l-1}^{n-k-1}$, it is clear that $x \in \overline{\Gamma}_l^k \times \overline{\Gamma}_{l-1}^{n-k} \setminus \overline{\Gamma}_l^{k+1} \times \overline{\Gamma}_{l-1}^{n-k-1}$ if and only if $f^{(l-1)n+k}(x) \in \overline{\Gamma}_1^k \times \overline{\Gamma}_0^{n-k} \setminus \overline{\Gamma}_1^{k+1} \times \overline{\Gamma}_0^{n-k-1}$. Furthermore, by the symmetry of the region $\overline{\Gamma}_l^k \times \overline{\Gamma}_{l-1}^{n-k}$, a point $x = (x_1, x_2, \ldots, x_j, \ldots, x_n)^T \in \overline{\Gamma}_l^k \times \overline{\Gamma}_{l-1}^{n-k}$ if and only if $(x_1, x_2, \ldots, -x_j, \ldots, x_n)^T \in \overline{\Gamma}_l^k \times \overline{\Gamma}_{l-1}^{n-k}$ for each $1 \leq k \leq n$.

Denote $f^{(l-1)n+k}(x)$ by $z = (z_1, z_2, \ldots, z_n)^T$. If $f^{(l-1)n+k+1}(x) > \Delta/(a-1)$, then for Type-I systems, $|az_1 - \mathrm{sgn}(z_1)\Delta| > \Delta/(a-1)$. Therefore, when $z_1 \geq 0$, $|az_1 - \Delta| > \Delta/(a-1)$, which is equivalent to the inequality $az_1 - \Delta < -\Delta/(a-1)$. Furthermore, it is not hard to understand that the inequality holds true if and only if $0 \leq z_1 < \Delta(a-2)/(a(a-1))$, i.e., $-az_1 + \Delta > \Delta/(a-1)$. When $z_1 < 0$, one can similarly get that $|az_1 + \Delta| > \Delta/(a-1)$, which is equivalent to the inequality $az_1 + \Delta > \Delta/(a-1)$. By the same argument, the inequality holds true if and only if $0 > z_1 > -\Delta(a-2)/(a(a-1))$, i.e., $-az_1 - \Delta < -\Delta/(a-1)$. Thus, we see that the region $\overline{\Gamma}_l^{k+1} \times \overline{\Gamma}_{l-1}^{n-k-1}$ is also symmetric.

It is easily seen that the above procedure in the proof of conclusion for Type-I systems shows that the conclusion also is true for Type-II systems. ■

Lemma 13:

i) Under the Δ-modulated feedback (11), the set of all stabilizable states of system (12), i.e.,

$$\overline{\Gamma}^n \stackrel{\text{def}}{=} \bigcap_{l=1}^{\infty} \bigcap_{k=1}^{n} \overline{\Gamma}_l^k \times \overline{\Gamma}_{l-1}^{n-k} \quad (30)$$

is a Cantor set in R^n.

ii) The *box dimension* [14] of the set $\overline{\Gamma}^n$ is given by $\dim_B(\overline{\Gamma}^n) = n \ln 2/\ln a$.

Proof: In the following, we prove the result only for the case of Type-I systems. The proof for the case of Type-II systems can be similarly carried out.

i) It is not hard to understand that a state $x \in R^n$ is stabilizable under the Δ-modulated feedback (11) if and only if $x \in \overline{\Gamma}^n$.

In order to show that the set $\overline{\Gamma}^n$ is a Cantor set, we need only to prove that the set is compact, perfect, and totally disconnected according to the definition of a Cantor set.

i) By definition, every region $\overline{\Gamma}_l^k \times \overline{\Gamma}_{l-1}^{n-k}$ is compact and all these regions form a nesting sequence of compact sets. Therefore, $\overline{\Gamma}^n$ is also compact. Furthermore, it is easily verified, for each $l \geq 1$ and $1 \leq k \leq n$, that there are some points on the boundary of $\overline{\Gamma}_l^k \times \overline{\Gamma}_{l-1}^{n-k}$ which belong to $\overline{\Gamma}^n$, for example, the point $(\Delta, \Delta, \ldots, \Delta)^T/(a-1)$. So, $\overline{\Gamma}^n$ is nonempty.

ii) Now, we prove that the set $\overline{\Gamma}^n$ is perfect. Firstly, a point $x \in \overline{\Gamma}^n$ must lie in $\overline{\Gamma}_l^n$ for every $l \geq 1$. By Lemma 11.iv), we see that $\lim_{l \to \infty} m(\overline{\Gamma}_l^n) = 0$. This shows that those points on the boundary of $\overline{\Gamma}^n$ converge to x, i.e., every point in $\overline{\Gamma}^n$ is an accumulation point.

iii) The following fact that

$$0 \leq m(\overline{\Gamma}^n) = m\left(\bigcap_{l=1}^{\infty} \bigcap_{k=1}^{n} \overline{\Gamma}_l^k \times \overline{\Gamma}_{l-1}^{n-k}\right) \leq m\left(\bigcap_{l=1}^{\infty} \overline{\Gamma}_l^n\right) = 0 \quad (31)$$

implies that there is no connected region in $\overline{\Gamma}^n$, i.e., $\overline{\Gamma}^n$ is totally disconnected.

iv) By the definitions of $\overline{\Gamma}_1$ and $\overline{\Gamma}_1^n$, we know that the region $\overline{\Gamma}_0^n \setminus \overline{\Gamma}_1 \times \overline{\Gamma}_0^{n-1}$ separates $\overline{\Gamma}_0^n$ into two n-dimensional cuboids with the length of the short sides given by

$$L_1 = \frac{2\Delta}{a(a-1)}.$$

Similarly, the region $\overline{\Gamma}_0^n \setminus \overline{\Gamma}_1^2 \times \overline{\Gamma}_0^{n-2}$ separates $\overline{\Gamma}_0^n$ into four n-dimensional cuboids with the length of the short sides being the same as L_1. Therefore, we can conclude that the region $\overline{\Gamma}_0^n \setminus \overline{\Gamma}_1^n$ separates $\overline{\Gamma}_0^n$ into 2^n n-dimension cubes with the length of sides equal to L_1. i.e., the region $\overline{\Gamma}_1^n$ is the collection of 2^n n-dimensional cubes with the side length L_1. Generally, by the symmetry of $\overline{\Gamma}_l^n$ and the proportion formula in Lemma 11.iv) of volumes about $\overline{\Gamma}_l^n, l \geq 0$, it is easily understood that the region $\overline{\Gamma}_l^n$ is a collection of 2^{ln} cubes with the side length

$$L_l = \frac{2\Delta}{a-1}\frac{1}{a^l}. \tag{32}$$

Therefore, it is obvious that, for every $l \geq 1$, the collection $\overline{\Gamma}_l^n$ of 2^{ln} cubes is a covering of the region $\overline{\Gamma}^n$. Hence, by the calculation formula of *the box dimension* [14], the box dimension of the region $\overline{\Gamma}^n$ is obtained as

$$\begin{aligned}\dim_B(\overline{\Gamma}^n) &= \lim_{l\to\infty}\left(-\frac{\ln 2^{ln}}{\ln \frac{2\sqrt{n}\Delta}{a-1}\frac{1}{a^l}}\right)\\ &= \lim_{l\to\infty}\left(-\frac{ln\ln 2}{\ln\frac{2\sqrt{n}\Delta}{a-1} - l\ln a}\right) = \frac{n\ln 2}{\ln a}.\end{aligned}$$

C. Chaotic Repeller

Theorem 7:

i) $\overline{\Gamma}^n$ is invariant under the Δ-modulated feedback control (11) and is a repeller of the closed-loop system (12).

ii) A point $x^{(0)} \in \overline{\Gamma}^n$ if and only if the point has the form

$$x^{(0)} = \Delta\sum_{i=0}^{\infty}\theta_i a^{-(i+1)} \tag{33}$$

where the definition of θ_i is the same as those vector parameters in Lemma 10.

iii) The closed-loop system (12) is chaotic in $\overline{\Gamma}^n$.

Proof: Again, we prove the result only for the case of Type-I systems.

i) First of all, we prove that $\overline{\Gamma}^n$ is invariant under the Δ-modulated feedback (11). As a matter of fact, by the definition of the region $\overline{\Gamma}^n$, we know that a point $x \in \overline{\Gamma}^n$ is equivalent to $\|f^k(x)\|_\infty \leq \Delta/(a-1)$ for all $k \geq 0$. Therefore, we must have that $f(x) \in \overline{\Gamma}^n$, i.e., $f(\overline{\Gamma}^n) \subset \overline{\Gamma}^n$.

On the other hand, the assertion $\overline{\Gamma}^n \subset f(\overline{\Gamma}^n)$ is directly implied by (26).

i) It is clear that $\|x\|_\infty > \Delta/(a-1)$ if and only if there exists a subscript $1 \leq i \leq n$, such that $\|x_i\|_\infty > \Delta/(a-1)$. In the following, we prove that $\|f(x)\|_\infty > \Delta/(a-1)$ if $\|x\|_\infty > \Delta/(a-1)$. In fact, the assertion follows directly from (21) and the inequalities

$$\begin{cases} ax_1 - \Delta > \Delta/(a-1) & \forall x_1 > \Delta/(a-1)\\ ax_1 + \Delta < 0 - \Delta/(a-1) & \forall x_1 < -\Delta/(a-1).\end{cases}$$

The previous conclusion shows that an orbit of the closed-loop Type-I systems will be outside of the region $\overline{\Gamma}_0^n \supset \overline{\Gamma}^n$ if its initial state is outside of this region.

Moreover, according to Lemma 11, one gets that

$$\overline{\Gamma}_0^n \setminus \overline{\Gamma}^n = \bigcup_{i=0}^{\infty}\overline{\Gamma}_i^n \setminus \overline{\Gamma}_{i+1}^n. \tag{34}$$

Therefore, a point $x \in \overline{\Gamma}_0^n \setminus \overline{\Gamma}^n$ only if $x \in \overline{\Gamma}_0^n \setminus \overline{\Gamma}_l^n$ for some $l \geq 1$, hence, by the definition of $\overline{\Gamma}_0^n \setminus \overline{\Gamma}_l^n$, we know that $\|f^{(l+1)n}(x)\|_\infty > \Delta/(a-1)$.

From the aforementioned discussion, we see that the set $\overline{\Gamma}_0^n \setminus \overline{\Gamma}^n$ is a set of all unstable states of the system in $\overline{\Gamma}_0^n$.

i) A point $x^{(0)} \in \overline{\Gamma}^n$ if and only if $\|f^{mn}(x^{(0)})\|_\infty \leq \Delta/(a-1)$ for all $m \geq 0$. When $x^{(0)} \in \overline{\Gamma}^n$, denote $x^{(m)} = f^{mn}(x^{(0)})$. Then, for all $m \geq 0$, by the equality

$$x^{(m)} = a^m x^{(0)} - \Delta\sum_{i=0}^{m-1}a^{m-i-1}\mathrm{sgn}(x^{(i)})$$

we get that

$$\frac{x^{(m)} - x^{(0)}}{a^m - 1} = x^{(0)} - \frac{\Delta}{a^m-1}\sum_{i=0}^{m-1}a^{m-i-1}\mathrm{sgn}(x^{(i)}).$$

Letting m tend to infinity, and combining the equality

$$\begin{aligned}\lim_{m\to\infty}\frac{x^{(m)} - x^{(0)}}{a^m-1} &= x^{(0)}\\ &- \lim_{m\to\infty}\frac{\Delta}{a^m-1}\sum_{i=0}^{m-1}a^{m-i-1}\mathrm{sgn}(x^{(i)})\end{aligned}$$

with the two facts that

$$\lim_{m\to\infty}(x^{(m)} - x^{(0)})/(a^m-1) = 0$$

and $\lim_{m\to\infty} a^m/(a^m-1) = 1$, we have

$$x^{(0)} = \Delta\sum_{i=0}^{\infty}a^{-(i+1)}\mathrm{sgn}(x^{(i)})$$

i.e., $x^{(0)}$ has the form of (33).

Assume a point $x^{(0)}$ has the form of (33). Then, it means that

$$\begin{aligned}x^{(0)} &= \lim_{m\to\infty}\Delta\sum_{i=0}^{m-1}\theta_i a^{-(i+1)}\\ &= \lim_{m\to\infty}\frac{\Delta}{a^m-1}\sum_{i=0}^{m-1}\theta_i a^{m-i-1}.\end{aligned}$$

Since, for every $m \geq 1$ and $1 \leq j \leq n$, the following inequalities hold:

$$\begin{aligned}\frac{a^m - 2a^{m-1}+1}{a-1} &= a^{m-1} - \sum_{i=0}^{m-2}a^i\\ &\leq \sum_{i=0}^{m-1}\theta_{j0}\theta_{ji}a^{m-i-1} \leq \frac{a^m-1}{a-1}\end{aligned}$$

it follows that, when $\theta_{j0} = 1$

$$\frac{a-2}{a(a-1)}\Delta \leq \frac{\Delta}{a^m-1}\sum_{i=0}^{m-1}\theta_{ji}a^{m-i-1} \leq \frac{\Delta}{a-1}$$

and when $\theta_{j0} = -1$,

$$-\frac{\Delta}{a-1} \leq \frac{\Delta}{a^m - 1}\sum_{i=0}^{m-1}\theta_{ji}a^{m-i-1} \leq -\frac{a-2}{a(a-1)}\Delta.$$

The previous results show that $\|x^{(0)}\|_\infty \leq \Delta/(a-1)$ and $x^{(0)}$ is a continuous point of both $f(\cdot)$ and $f^n(\cdot)$ when $x^{(0)}$ has the form of (33). Therefore, the following calculation procedure is feasible:

$$\begin{aligned} x^{(1)} = f^n(x^{(0)}) &= \lim_{l\to\infty} f^n\left(\frac{\Delta}{a^l-1}\sum_{i=0}^{l-1}\theta_i a^{l-i-1}\right) \\ &= \lim_{l\to\infty}\frac{\Delta}{a^l-1}\left(\sum_{i=0}^{l-2}\theta_{i+1}a^{l-i-1} + \theta_0\right) \\ &= \Delta\sum_{i=0}^{\infty}\theta_{i+1}a^{-(i+1)}. \end{aligned}$$

By the same kind of argument, we can prove that the point $x^{(1)}$ is also a continuous point of the maps $f(\cdot)$ and $f^n(\cdot)$. Generally, one can verify, for all $l \geq 1$, the following:

$$x^{(m)} = f^{mn}(x^{(0)}) = \Delta\sum_{i=0}^{\infty}\theta_{i+m}a^{-(i+1)}.$$

Hence, we can conclude that the point $x^{(0)} \in \overline{\Gamma}^n$.

ii) We first prove that the set of periodic points of the closed-loop system (12) is dense in $\overline{\Gamma}^n$. For a given point $x^{(0)} \in \overline{\Gamma}^n$, we have that

$$x^{(m)} = a^m x^{(0)} - \Delta\sum_{i=0}^{m-1}a^{m-i-1}\mathrm{sgn}(x^{(i)}).$$

By Lemma 10, we know that the point

$$z^{(0)} = \frac{\Delta}{a^m-1}\sum_{i=0}^{m-1}a^{m-i-1}\mathrm{sgn}(x^{(i)})$$

is a periodic point with period m; therefore

$$\mathrm{sgn}(f^{in}(z^{(0)})) = \mathrm{sgn}(x^{(i)})$$

and

$$z^{(0)} = a^m z^{(0)} - \Delta\sum_{i=0}^{m-1}a^{m-i-1}\mathrm{sgn}(x^{(i)}).$$

Hence, noticing that $\|x^{(m)}\|_\infty \leq \Delta/(a-1)$ for all $m \geq 0$, we conclude that $x^{(m)} - z^{(0)} = a^m\left(x^{(0)} - z^{(0)}\right)$. Since $\lim_{m\to\infty}\left(x^{(m)} - z^{(0)}\right)/a^m = 0$ and, therefore

$$x^{(0)} = \lim_{m\to\infty} z^{(0)} = \lim_{m\to\infty}\frac{\Delta}{a^m-1}\sum_{i=0}^{m-1}a^{m-i-1}\mathrm{sgn}(x^{(i)})$$

we know that the set of periodic points of the closed-loop system (12) is dense in $\overline{\Gamma}^n$.

iii) In the following, we prove that the closed-loop system (12) has sensitive dependence on the initial conditions on the set $\overline{\Gamma}^n$. To this end, we first reveal a characteristic of orbits of $f^n(\cdot)$, that is, for any two given orbits $\{x^{(0)}, x^{(1)}, \ldots\}$ and $\{z^{(0)}, z^{(1)}, \ldots\}$ of $f^n(\cdot)$ in $\overline{\Gamma}^n$, there exists an $m \geq 0$ such that

$$\|z^{(m)} - x^{(m)}\|_\infty \geq \frac{2(a-2)\Delta}{a(a-1)}$$

if $\mathrm{sgn}\left(z_i^{(m)}\right) \neq \mathrm{sgn}\left(x_i^{(m)}\right)$ for some subscript i. The assertion follows directly from the fact that a point $x \in \overline{\Gamma}^n$ only if

$$|x_i| \geq \frac{(a-2)}{a(a-1)}\Delta$$

for all subscripts $1 \leq i \leq n$.

Let $\delta = 2(a-2)\Delta/(a(a-1))$. If there exists a point $x^{(0)} \in \overline{\Gamma}^n$ and a positive number ε such that there is a point $y^{(0)} \in \overline{\Gamma}^n$ which satisfies $\|y^{(0)} - x^{(0)}\|_\infty < \varepsilon$, and

$$\|y^{(m)} - x^{(m)}\|_\infty < \frac{2(a-2)\Delta}{a(a-1)}$$

for all $m \geq 0$, then, based on the previous discussion, we know that the equality $\mathrm{sgn}(y^{(m)}) = \mathrm{sgn}(x^{(m)})$ holds true for all $m \geq 0$. Hence, every point in the straight line $z^{(0)} = \alpha x^{(0)} + (1-\alpha)y^{(0)}, 0 \leq \alpha \leq 1$, has the property: $\mathrm{sgn}(z^{(0)}) = \mathrm{sgn}(x^{(0)})$. Besides, the following equalities:

$$z^{(1)} = f^n(z^{(0)}) = az^{(0)} - \mathrm{sgn}\left(z^{(0)}\right)\Delta = \alpha x^{(1)} + (1-\alpha)y^{(1)}$$

imply that $\mathrm{sgn}\left(z^{(1)}\right) = \mathrm{sgn}\left(x^{(1)}\right)$. Generally, one can verify by mathematical induction that the two equalities $z^{(m)} = \alpha x^{(m)} + (1-\alpha)y^{(m)}$ and $\mathrm{sgn}\left(z^{(m)}\right) = \mathrm{sgn}\left(x^{(m)}\right)$ hold true for all $m \geq 0$. Hence, one gets

$$\begin{aligned} \|z^{(m)}\|_\infty &= \|\alpha x^{(m)} + (1-\alpha)y^{(m)}\|_\infty \\ &\leq \alpha\|x^{(m)}\|_\infty + (1-\alpha)\|y^{(m)}\|_\infty = \frac{\Delta}{a-1}. \end{aligned}$$

By the definition of $\overline{\Gamma}^n, z^{(0)} = \alpha x^{(0)} + (1-\alpha)y^{(0)} \in \overline{\Gamma}^n$ for all $0 \leq \alpha \leq 1$. This means that the straight-line $z^{(0)} = \alpha x^{(0)} + (1-\alpha)y^{(0)}$ must belong to some cube in $\overline{\Gamma}_l^n$ for every $l \geq 1$. It is obviously impossible since the length of side of every cube in $\overline{\Gamma}_l^n$ tends to zero when l tends to infinity.

i) Finally, we prove that the system has an orbit which is dense in $\overline{\Gamma}^n$. To this end, we need the following preparative knowledge: For any given $\varepsilon > 0$, let M be the smallest integer which satisfies the inequality $M > (\ln 2 - \ln\varepsilon - \ln(a-1))/\ln a$, then, for all $\theta_i = (\theta_{1i}, \theta_{2i}, \ldots, \theta_{ni}), \theta_{ji} \in \{-1, 1\}, i \geq M$ and $1 \leq j \leq n$, the following inequality holds true:

$$\left\|\sum_{i=M}^{\infty}2\theta_i a^{-(i+1)}\right\|_\infty \leq 2\sum_{i=M}^{\infty}a^{-(i+1)} < \varepsilon.$$

Moreover, as we have shown that $\overline{\Gamma}^n$ is a compact set, according to the well-known finite covering theorem, there are a finite number of open sets which cover $\overline{\Gamma}^n$. In particular, for any given $\varepsilon > 0$, from (32), there must exist $l \geq 1$ such that the length of the diagonal of each cube in $\overline{\Gamma}_l^n$ is less than $\varepsilon/2$. Hence, when we denote the center of the kth cube of $\overline{\Gamma}_l^n$ by $z^{(k)}, k = 1, 2, \ldots, 2^{ln}$, the set $\overline{\Gamma}_l^n$ is covered by these 2^{ln} open balls $D(z^{(k)}, \varepsilon/2)$ and so is $\overline{\Gamma}^n$. When $D(z^{(k)}, \varepsilon/2) \cap \overline{\Gamma}^n \neq \emptyset$, denote by $y_{(k)} = \Delta \sum_{i=0}^{\infty} \theta_i^{(k)} a^{-(i+1)}$, an arbitrarily chosen point in $D(z^{(k)}, \varepsilon/2) \cap \overline{\Gamma}^n, k = 1, 2, \ldots, 2^{ln}$. Let us now consider the point

$$x^{(0)} = \Delta \left(\sum_{i=0}^{M-1} \theta_i^{(1)} a^{-(i+1)} + \sum_{i=0}^{M-1} \theta_i^{(2)} a^{-(i+1+M)} + \cdots + \sum_{i=0}^{M-1} \theta_i^{(2^{ln})} a^{-(i+1+(2^{ln}-1)M)} + \sum_{i=2^{ln}}^{\infty} \theta_i a^{-(i+1)} \right) \quad (35)$$

where the meanings of $\theta_i^{(k)}$ and θ_i are the same as those in (33). Based on the previous discussions, it is not hard to verify that, for every $1 \leq k \leq 2^{ln} - 1$,

$$x^{(kM)} = \Delta \sum_{i=0}^{M-1} \theta_i^{(k+1)} a^{-(i+1)} + \cdots + \Delta \sum_{i=0}^{M-1} \theta_i^{(2^{ln})} a^{-(i+1+(2^{ln}-k-1)M)} + \cdots. \quad (36)$$

By (35) and (36), for every $1 \leq k \leq 2^{ln}, \|y_{(k)} - x^{((k-1)M)}\|_\infty < \varepsilon/2$.

This shows that, for any $\varepsilon > 0$, there is an orbit in $\overline{\Gamma}^n$ with the property that for any point $y \in \overline{\Gamma}^n$, there exists at least a point x in the orbit such that $\|y - x\|_\infty < \varepsilon$.

VI. Conclusion

In this paper, we have presented a design of a Δ-modulated feedback control of a high-order system based on the technique of control Lyapunov functions. We have classified the complex dynamics of the closed-loop system in three cases. In the first case, we have found all possible periodic orbits for their numbers and periods, together with characterizations of their attracting regions. In the second case, we have shown that there is a maximal "stabilizable" region, and that inside this region, there is a local attractor whose size is independent of the value of the parameter a. In the last case, we have shown that all the states stabilizable by the Δ-modulated feedback constitute a Cantor set; this Cantor set is a repeller; and the closed-loop system is chaotic on the Cantor set. This study, along with [26]–[28] and [30]–[33], has now provided a relatively complete picture about the dynamics of the Δ-modulated feedback control mechanism, useful for control engineering design and applications.

References

[1] S. Ariyavisitakul and L. Chang, "Simulation of a CDMA system performance with feed-back power control," *Electron. Lett.*, vol. 27, pp. 2127–2128, 1991.

[2] A. Bicchi, A. Marigo, and B. Piccoli, "On the reachability of quantized control systems," *IEEE Trans. Autom. Control*, vol. 47, no. 4, pp. 546–563, Apr. 2002.

[3] F. Blanchini, "Set invariance in control," *Automatica*, vol. 35, pp. 1747–1767, 1999.

[4] R. W. Brockett and D. Liberzon, "Quantized feedback stabilization of linear systems," *IEEE Trans. Autom. Control*, vol. 45, no. 7, pp. 1279–1289, Jul. 2000.

[5] J. C. Candy, "A use of limit cycle oscillations to obtain robust analog-to-digital converters," *IEEE Trans. Commun.*, vol. COM-22, no. 3, pp. 298–305, Mar. 1974.

[6] N. Da Dalt, "A design-oriented study of the nonlinear dynamics of digital bang-bang PLLs," *IEEE Trans. Circuits Syst. I*, vol. 52, no. 1, pp. 21–31, Jan. 2005.

[7] D. F. Delchamps, "Some chaotic consequences of quantization in digital filters and digital control systems," in *Proc. 27th IEEE Conf. Dec. Control*, Austin, TX, 1988, pp. 405–410.

[8] D. F. Delchamps, "Stablilizing a linear system with quantized state feedback," *IEEE Trans. Autom. Control*, vol. 35, no. 8, pp. 916–924, Aug. 1990.

[9] R. L. Devaney, *An Introduction to Chaotic Dyanmical Systems*. New York: Addison-Wesley, 1987.

[10] N. Elia and E. Frazzoli, C. J. Tomlin and M. R. Greenstreet, Eds., "Quantized stabilization of two-input linear systems: A lower bound on the minimal quantization density," in *Proc. 5th Int. Workshop Hybrid Syst.: Comput. Control*, Mar. 2002, vol. 2289, pp. 179–193, Lecture Notes in Computer Science.

[11] N. Elia and S. K. Mitter, "Stabilization of linear systems with limited information," *IEEE Trans. Autom. Control*, vol. 46, no. 9, pp. 1384–1400, Sep. 2001.

[12] F. Fagnani and S. Zampieri, "Stability analysis and synthesis for scalar linear system with a quantized feedback," *IEEE Trans. Autom. Control*, vol. 48, no. 9, pp. 1569–1584, Sep. 2003.

[13] F. Fagnani and S. Zampieri, "Performance evaluations of quantized stabilizers," in *Proc. 42nd IEEE Conf. Dec. Control*, Maui, HI, 2003, pp. 1897–1901.

[14] K. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*. London, U.K.: Wiley, 1989.

[15] O. Feely and L. O. Chua, "Nonlinear dynamics of a class of analog-to-digital converters," *Int. J. Bifurcation Chaos*, vol. 2, pp. 325–340, 1992.

[16] P. Hartman, *Ordinary Differential Equations*, 2nd ed. New York: Wiley, 1973.

[17] T. Hu, Z. Lin, and L. Qiu, "Stabilization of exponentially unstable linear systems with saturating actuators," *IEEE Trans. Autom. Control*, vol. 46, no. 6, pp. 973–979, Jun. 2001.

[18] H. Inose and Y. Yasuda, "A unity bit coding method by negative feedback," *Proc. IEEE*, vol. 51, pp. 1524–1535, 1963.

[19] N. N. Leonov, "Map of the line into itself," *Radiofisica*, vol. 2, pp. 942–956, 1959.

[20] K. Li and J. Baillieul, "Robust quantization for digital finite communication bandwidth (DFCB) control," in *Proc. 42nd IEEE Conf. Dec. Control*, Maui, HI, 2003, pp. 3130–3135.

[21] J. C. Liberti, Jr. and T. S. Rappaport, *Smart Antennas for Wireless Communications: IS-95 & Third Generation CDMA Applications*. Upper Saddle River, NJ: Prentice-Hall, 1999.

[22] G. Nair and R. Evans, "Stabilization with data-rate-limited feedback: Tightest attainable bounds," *Syst. Control Lett.*, vol. 41, pp. 49–56, 2000.

[23] S. J. Park and R. M. Gray, "Sigma-delta modulation with leaky integration and constant input," *IEEE Trans. Inform. Theory*, vol. 38, no. 9, pp. 1512–1533, Sep. 1992.

[24] B. Picasso, F. Gouaisbaut, and A. Bicchi, "Construction of invariant and attracting sets for quantized-input linear systems," in *Proc. IEEE Conf. Dec. Control*, Las Vegas, NV, 2002, pp. 824–829.

[25] A. N. Sarkovskii, "Coexistence of cycle of a continuous map of a line into itself," *Ukr. Mat. Z.*, vol. 16, pp. 61–71, 1964.

[26] T.-C. Song, X. Xia, and L. P. Linde, "A new W-CDMA transmit power control technique," in *Proc. 3rd Int. Conf. Control Theory Appl.*, Pretoria, South Africa, Dec. 2001, pp. 2–14.

[27] X. Xia, "Periodic orbits arising from two-level quantized feedback control," *Chaos, Solitons & Fractals*, doi: 10.1016/j.chaos.2006.01.093.

[28] X. Xia and G. Chen, "On delta-modulated control: A simple system with complex dynamics," *Chaos, Solitons & Fractals*, doi: 10.1016/j.chaos.2006.01.092.

[29] X. Xia, G. Chen, and R. Gai, "On Control Lyapunov Modes of Linear Control Systems," in *Proc. 1st African Control Conf.*, Cape Town, South Africa, Dec. 2003, p. 5.

[30] X. Xia, R. Gai, and G. Chen, "Periodic orbits arising from Delta-modulated feedback control," *Chaos, Solitons Fractals*, vol. 19, pp. 581–595, 2004.

[31] X. Xia and A. S. Zinober, "Periodic orbits from Δ-modulation of stable linear systems," *IEEE Trans. Autom. Control*, vol. 49, no. 8, pp. 1376–1380, Aug. 2004.
[32] X. Xia and A. S. Zinober, "Δ-modulated feedback in discretization of sliding mode control," *Automatica*, vol. 42, pp. 771–776, 2006.
[33] R. Gai, G. Chen, and X. Xia, "Global dynamics of unbalanced Delta-modulated feedback-controlled discrete-time systems," *Dyna. Continuous, Discrete Impul. Syst.*, ser. B, vol. 13, pp. 635–679, Mar. 2006.

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