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Periodicity in Delta-modulated feedback control

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Abstract: The Delta-modulated feedback control of a linear system introduces nonlinearity into the system through switchings between two input values. It has been found that Delta-modulation gives rise to periodic orbits. The existence of periodic points of all orders of Sigma-Delta modulation with "leaky" integration is completely characterized by some interesting groups of polynomials with "sign" coefficients. The results are naturally generalized to Sigma-Delta modulations with multiple delays, Delta-modulations in the "downlink", unbalanced Delta-modulations and systems with two-level quantized feedback. Further extensions relate to the existence of periodic points arising from Delta-modulated feedback control of a stable linear system in an arbitrary direction, for which some necessary and sufficient conditions are given.

Keywords: Switching; Periodic orbit; Delta-modulation; Sign polynomial; Nonlinear control

1 Introduction

The determination of self-excited oscillations or limit cycles first studied by Poincaré and Lyapunov is an old but difficult problem in the classic qualitative theory of dynamical systems [1]. For discrete-time systems, the problem has been tackled from different points of view, ranging from counting the number of types of periodic orbits [2], to the arithmetic of the number of periodic points [3], to the existence [4] and calculation [5] of the periodic points. Hybrid systems resulting from the switching of controllers constitute a special class of nonlinear dynamical systems [6]. Although stability properties around a specific limit cycle (periodic orbits) have been discussed [7], there are very few results on the existence and characterization of periodic points induced by switchings. Worth mentioning is [8], where the existence of a globally attractive periodic behaviour is proved for some switched flow networks.

This paper reports our recent studies on the periodic orbits arising from a typical switching system — a Δ-modulated control system:

$$x^{+} = Ax + bu, \tag{1}$$

$$u = \mathrm{sgn}\ (c^{\mathrm{T}}x), \tag{2}$$

where $x \in \mathbb{R}^n$ is the state, x^{+} denotes the system state at the next discrete time step, $u \in \mathbb{R}$ is the scalar input, A is an $n \times n$ matrix of real numbers, b is a column vector of n real numbers, and $c \in \mathbb{R}^n \backslash \{0\}$ is called the modulation direction. As usual, the function $\mathrm{sgn}\ (x)$ is defined as

$$\mathrm{sgn}\ (x) = \begin{cases} 1, & \text{when } x \geqslant 0, \\ -1, & \text{when } x < 0. \end{cases}$$

The term Sigma-Delta ($\Sigma\Delta$) modulation comes from analog-digital electronics [9, 10]. Δ-modulated feedback has been applied to, for example, transmitting power regulation of a mobile unit in the Direct Sequence Code Division Multiple Access (DS-CDMA) cellular network [11]. An advantage of such a control method is that only one bit of datum is necessary for implementing the controller. This is the standard in IS-95 [12] for transmitting power control. Δ-modulated control is bounded, bang-bang, and also a special kind of quantized control, which are topics of longstanding interest in the control community [13~16]. Δ-modulated feedback is a switching between two values, typically $+1$ and -1. The resulting switching system is a special kind of piecewise-linear system [6, 17, 18]. Discretizing the equivalent-control-based sliding-mode controllers also results in Δ-modulated type of feedback [19~21].

Notably, periodic points have been found in all the aforementioned situations (see also [22~24]). Results are partial and methods are not systematic, however.

In our recent works [25~31], we have made efforts to find complete solutions in some of the simple yet long-standing cases, as well as unified methodologies in some general cases; in particular, a complete characterization of the scalar

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setting with $n = 1$ in (1), which is the Sigma-Delta modulation. Some necessary and sufficient conditions have been obtained for stable system (1) in an arbitrary modulation dimension.

We now briefly describe some generalizations of the systems and results. Because of space limitation, only essential results with brief proofs are given in this paper.

2 Scalar case: Sigma-Delta modulation

In the scalar case, we can limit our consideration to the following system [29, 30]:

$$x^+ = ax - \mathrm{sgn}\,(ax). \tag{3}$$

When $a > 0$, we call this system Type-I, and when $a < 0$, Type-II. Here, it should be noted that the case of $x^+ = ax + \mathrm{sgn}\,(ax)$ leads to divergence when $|a| > 1$.

An orbit (x_i) is periodic with period $p \geqslant 0$ if, for all i,

$$x_{i+p} = x_i.$$

A period p of an orbit (x_i) is prime if it is the smallest of such a period, and in this case, we say that the orbit (x_i) is p-periodic.

The following results are straightforward.

Theorem 1 1) When $-1 < a < 1$, the only global attractor is the following set of two points:

$$\{-1/(1+|a|), 1/(1+|a|)\}. \tag{4}$$

When $0 \leqslant a < 1$, the two points in (4) are 2-periodic; when $-1 \leqslant a < 0$, the two points in (4) are (1-periodic) fixed points.

2) When $a = 1$, any point in the half-open interval $(-1, 1]$ is a 2-periodic point. When $a = -1$, all points but $\pm 1/2$ in the closed interval $(-1, 1]$ are 2-periodic; $\pm 1/2$ are fixed points.

3) When $a \leqslant -2$, (3) has n-periodic points for all $n \geqslant 1$; when $a \geqslant 2$, (3) has n-periodic points for all $n \geqslant 1$.

The last conclusion also follows from the next result, which is preceded by some definitions.

For a given positive integer $k \geqslant 2$, by an ordered set of k "sign" parameters, we mean the set $\{\theta_0, \theta_1, \cdots, \theta_{k-1}\}$, in which $\theta_i \in \{-1, 1\}$, $i = 0, 1, \cdots, k-1$. An ordered set of polynomials with "sign" coefficients $\mathcal{P}_{\{\theta_0, \theta_1, \cdots, \theta_{k-1}\}}$ corresponding to a given ordered set of "sign" parameters $\{\theta_0, \theta_1, \cdots, \theta_{k-1}\}$ is defined as follows:

$$\left\{ P_i(a) = \sum_{j=0}^{k-1} \theta_{i+j} a^{k-j-1}, 0 \leqslant i \leqslant k-1 \right\}, \tag{5}$$

where $\theta_{i+j} = \theta_{(i+j) \mathrm{mod}\ (k)}$.

The ordered set of "sign" parameters $\{\theta_0, \theta_1, \cdots, \theta_{k-1}\}$ is called (strictly) shift-definite at a if

$$\theta_i P_i(a) \geqslant (>)\ 0,$$

for all $P_i(a) \in \mathcal{P}_{\{\theta_0, \theta_1, \cdots, \theta_{k-1}\}}$, $i = 0, 1, \cdots, k-1$. Define a function of a on $(1, \infty)$ by

$$\underline{\boldsymbol{P}}_{\{\theta_0, \cdots, \theta_{k-1}\}}(a) = \min\left\{\theta_i P_i(a) | P_i \in \mathcal{P}_{\{\theta_0, \cdots, \theta_{k-1}\}}\right\}, \tag{6}$$

and call it the minimal value function w.r.t. $\{\theta_0, \theta_1, \cdots, \theta_{k-1}\}$. If there is a $P_i(a) \in \mathcal{P}_{\{\theta_0, \theta_1, \cdots, \theta_{k-1}\}}$ such that

$$\underline{\boldsymbol{P}}_{\{\theta_0, \theta_1, \cdots, \theta_{k-1}\}}(a) = \theta_i P_i(a),$$

and for all $j \neq i$, $P_j(a) \in \mathcal{P}_{\{\theta_0, \theta_1, \cdots, \theta_{k-1}\}}$,

$$\underline{\boldsymbol{P}}_{\{\theta_0, \theta_1, \cdots, \theta_{k-1}\}}(a) < \theta_j P_j(a),$$

then the minimal value function is said to be strictly minimal at a.

The following results are for Type-I systems.

Theorem 2 i) A point $x_0 \in \mathbb{R}$ is a periodic point with period n if and only if there is a set of n "sign" parameters, $\{\theta_0, \theta_1, \cdots, \theta_{n-1}\}, \theta_i \in \{-1, 1\}, i = 0, 1, \cdots, n-1$, which is shift-definite, such that

$$x_0 = \frac{1}{a^n - 1} \sum_{i=0}^{n-1} a^{n-i-1} \theta_i. \tag{7}$$

ii) A point $x_0 \in \mathbb{R}$ is an n-periodic point (with prime period n) if and only if n is the smallest positive integer such that i) holds.

iii) If the minimal value function w.r.t. a shift-definite set of "sign" parameters $\{\theta_0, \theta_1, \cdots, \theta_{n-1}\}$ is strictly minimal at a, then the periodic point given by (7) has a prime period n.

To characterize when periodic points of a certain order exist, we make use of three groups of polynomials whose definitions and properties are summarized in the following three lemmas.

Lemma 1 Consider the system of polynomials defined by $\boldsymbol{P}_2(a) = a - 1$, and for positive integers $m \geqslant 1$,

$$\boldsymbol{P}_{2^{m+1}}(a) = (a^{2^m} - 1)\boldsymbol{P}_{2^m}(a). \tag{8}$$

These polynomials have the following properties:

i) For $m \geqslant 1$, $\boldsymbol{P}_{2^{m+1}}(a) = (a-1)\boldsymbol{P}_{2^m}(a^2)$.

ii) For every $m \geqslant 1$, use $\{\theta_0', \theta_1', \cdots, \theta_{2^m-1}'\}$ to denote the ordered set of parameters of the polynomials $\boldsymbol{P}_{2^m}(a)$, corresponding to the coefficients in decreasing order of powers. This ordered set of parameters are shift-definite at any $a \in [1, \infty)$. The polynomial $\boldsymbol{P}_{2^m}(a)$ itself is the minimal value function on the interval $(1, \infty)$, and it is strictly minimal at all $a \in (1, \infty)$.

Lemma 2 Consider the system of polynomials defined by $\boldsymbol{Q}_3(a) = a^2 - a - 1$, and for $k > 1$,

$$\boldsymbol{Q}_{2k+3}(a) = a^2 \boldsymbol{Q}_{2k+1}(a) + a - 1 \tag{9}$$

and

$$\boldsymbol{Q}_{(2k+1)2^m}(a) = \boldsymbol{Q}_{2k+1}(a^{2^m})\boldsymbol{P}_{2^m}(a).$$

These polynomials have the following properties:

i) $\boldsymbol{Q}_{(2k+1)2^{m+1}}(a) = (a-1)\boldsymbol{Q}_{(2k+1)2^m}(a^2)$, and it has a unique root $a_{(2k+1)2^m}$ in the interval $(1, \infty)$;

ii) $\sqrt{2} < a_{2k+3} < a_{2k+1}$;

iii) $\lim\limits_{k\to+\infty} a_{2k+1} = \sqrt{2}$;

iv) $a_{(2k+1)2^m} = (a_{2k+1})^{1/2^m}$;

v) $\boldsymbol{Q}_{(2k+1)2^n}(a)$ is the strictly minimal value function, w.r.t. the ordered set of parameters consisting of coefficients of $\boldsymbol{Q}_{(2k+1)2^n}(a)$, in decreasing order of powers. When $a \geqslant a_{(2k+1)2^n}$, this ordered set of parameters is shift-definite.

Lemma 3 Consider the system of polynomials defined by $\boldsymbol{H}_2(a) = \boldsymbol{P}_2(a)$, and for $k \geqslant 1$,

$$\boldsymbol{H}_{2k+2}(a) = a^2\boldsymbol{H}_{2k}(a) - \boldsymbol{H}_2(a). \tag{10}$$

These polynomials have the following properties:

i) For all $k > 1$, $\boldsymbol{H}_{2k}(a) = a\boldsymbol{Q}_{2k-1}(a) + 1$.

ii) For every $k > 2$, the polynomial $\boldsymbol{H}_{2k}(a)$ has a unique real root in the interval $(1, \infty)$.

iii) The sequence of real roots of polynomials $\boldsymbol{H}_{2k}(a)$ in the interval $(1, \infty)$, denoted by $\bar{a}_{2k}$, is strictly monotonically increasing when $k \geqslant 3$. In particular, $\bar{a}_6 = a_6$, and $\lim\limits_{k\to+\infty} \bar{a}_{2k} = \sqrt{2}$.

We sketch the proofs for parts ii) and iii) of Lemma 2.

From the definition of $\boldsymbol{Q}_{2k+3}(a)$, we calculate the following:

$$\boldsymbol{Q}_{2k+3}(a) = \frac{a^{2k+3} - 2a^{2k-1} - 1}{a+1}.$$

We need only to prove that the polynomial defined by $\bar{\boldsymbol{Q}}_{2k+3}(a) = a^{2k+3} - 2a^{2k+1} - 1$ has a unique root in $(1, \infty)$. Since

$$\frac{\mathrm{d}\bar{\boldsymbol{Q}}_{2k+3}(\mathrm{a})}{\mathrm{da}} = a^{2k}((2k+3)a^2 - 2(2k+1)),$$

we see that $\dfrac{\mathrm{d}\bar{\boldsymbol{Q}}_{2k+3}(\mathrm{a})}{\mathrm{da}}$ is zero in $(1, \infty)$ only when $a = a^* = \sqrt{\dfrac{2(2k+1)}{2k+3}}$, and

$$\begin{cases} \dfrac{\mathrm{d}\bar{\boldsymbol{Q}}_{2k+3}(\mathrm{a})}{\mathrm{da}} < 0, \text{ when } 1 \leqslant a < a^*, \\ \dfrac{\mathrm{d}\bar{\boldsymbol{Q}}_{2k+3}(\mathrm{a})}{\mathrm{da}} > 0, \text{ when } a > a^*. \end{cases}$$

Therefore, we have, for $a \in (1, a^*]$, $\bar{\boldsymbol{Q}}_{2k+3}(a) < \bar{\boldsymbol{Q}}_{2k+3}(1) = -2$. $\bar{\boldsymbol{Q}}_{2k+3}(a)$ is strictly monotonically increasing in the interval $[a^*, \infty)$. Since $\bar{\boldsymbol{Q}}_{2k+3}(2) = 3 \times 2^{2k+1} - 1 > 0$, we know that $\bar{\boldsymbol{Q}}_{2k+3}(a)$ has a unique root in (a^*, ∞).

From the above, we see that $\bar{\boldsymbol{Q}}_{2k+3}(\sqrt{2}) = -1$, so we conclude that $a_{2k+3} > \sqrt{2}$. To prove $a_{2k+3} < a_{2k+1}$, we note that $\boldsymbol{Q}_{2k+3}(a) = a^2\boldsymbol{Q}_{2k+1}(a) + a - 1$, so that $\boldsymbol{Q}_{2k+3}(a_{2k+1}) = a_{2k+1} - 1 > 0$, therefore, $a_{2k+3} < a_{2k+1}$.

The conclusion of part ii) in Lemma 2 guarantees the existence of a limit of the sequence $\{a_{2k+1}\}$, denoted by a_∞, when k tends to infinity, and $a_\infty \geqslant \sqrt{2}$. Note that a_{2k+1} is also the unique root of $\bar{\boldsymbol{Q}}_{2k+1}(a)$; therefore, we have

$$a_{2k+1}^{2k+1} - 2a_{2k+1}^{2k-1} - 1 = 0,$$

$$a_\infty^2 - 2 - \lim_{k\to\infty} \frac{1}{a_{2k+1}^{2k-1}} = 0.$$

Since $a_\infty \geqslant \sqrt{2}$, the third term on the left-hand side of the last equation is zero; thus, we have $a_\infty^2 - 2 = 0$, that is, $a_\infty = \sqrt{2}$.

We are now ready to present our results for three distinct cases: 2^m-periodic points, odd-order periodic points, and even-order periodic points.

Theorem 3 i) If $a > 1$, then there exists a 2^m-periodic point in $[-1, 1]$ for all $m > 0$.

ii) For every positive integer $k \geqslant 1$, system (3) has a $(2k+1)$-periodic point if and only if $a \geqslant a_{2k+1}$.

iii) System (3) has a periodic point of a prime period $2^n k$, where $k \geqslant 3$ is odd, if and only if $a \geqslant a_{2^n k}$.

Proof We prove i) for illustration. From part ii) in Lemma 1 and part i) in Theorem 2, the point defined by

$$x_{2^m} = \frac{1}{a^{2^m} - 1}\boldsymbol{P}_{2^m}(a),$$

with $m \geqslant 1$, is a periodic point in $[-1, 1]$ with prime period 2^m. Part ii) in Lemma 1 and part iii) in Theorem 2 together imply that x_{2^m} is a 2^m-periodic point.

Properties of polynomials of Q and H are used in the proof of ii) and iii). Results for Type-II systems are similarly obtained.

3 Sigma-Delta modulation with multiple delays

The first extension to the higher-dimensional case is the Sigma-Delta modulation with multiple delays [25]:

$$x_{n+k} = ax_k + u, \quad a \neq 0, \tag{11}$$

where x and u are both real, and

$$u = -\mathrm{sgn}\ (ax_k). \tag{12}$$

In the state-space form (1), this corresponds to a controllable canonical form, in which

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a & 0 & 0 & \cdots & 0 \end{bmatrix}, \ b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \ a \neq 0. \tag{13}$$

When $a > 0$, system (13) is classified to be of Type-I, and when $a < 0$, Type-II. We present results only for $|a| < 1$.

Define the following set of 2^n points:

$$\Omega_{\mathrm{ca}} = \left\{ \frac{1}{1+|a|}\theta, \ \theta = (\theta_1, \theta_2, \cdots, \theta_n)^{\mathrm{T}}, \ \theta_i \in \{-1, 1\}, \ i = 1, 2, \cdots, n \right\}.$$

Theorem 4 Any $x \in \Omega_{\rm ca}$ is a periodic point of the closed-loop system (11) under control of the Δ-modulated feedback (12). For Type-I systems, a positive integer l is a period for some $x \in \Omega_{\rm ca}$ if and only if l is not a divisor of n, but is a divisor of $2n$. For Type-II systems, a positive integer l is a period for some $x \in \Omega_{\rm ca}$ if and only if l is a divisor of n.

Proof For any $x = \dfrac{1}{1+|a|}\theta \in \Omega_{\rm ca}$, denoting $x^{(1)} = f(x)$, one can verify that, for Type-I systems, $x^{(n)} = -x$ and $x^{(2n)} = x$. Therefore, every point in $\Omega_{\rm ca}$ is a periodic point of f, and a positive integer l is the period of a point of $\Omega_{\rm ca}$ only if it is a divisor of $2n$ but not a divisor of n.

On the other hand, if a positive integer l is a divisor of $2n$, but not a divisor of n, then, according to the integer division algorithm, there exists a non-negative integer α and a positive integer β, strictly less than l, such that

$$n = \alpha l + \beta. \tag{14}$$

From the above expression for n, it follows that l divides $2n$ implies l divides 2β, that is, $2\beta = k\,l$ for some positive integer k. Because $0 < \beta < l$, it follows necessarily that $k = 1$, and therefore $l = 2\beta$. From (14), we have

$$n = (2\alpha + 1)\frac{l}{2} = (2\alpha + 1)\beta.$$

Construct the following vector of n numbers:

$$\Bigg(\overbrace{\underbrace{1,\cdots,1}_{\beta},\underbrace{-1,\cdots,-1}_{\beta},\cdots,\underbrace{1,\cdots,1}_{\beta},\underbrace{-1,\cdots,-1}_{\beta},\underbrace{1,\cdots,1}_{\beta}}^{2\alpha+1}\Bigg)^{\rm T}.$$

It is easily seen that the vector obtained via multiplying this vector by $\dfrac{1}{1+a}$ belongs to $\Omega_{\rm ca}$ and has period l.

Similar arguments apply to Type-II systems. Therefore, we have the following characterization of all possible periods.

Theorem 5 When $n > 1$, denote $n = \prod\limits_{i=1}^{q} p_i^{n_i}$, where $n_i > 0$, and let $p_i,\ i = 1, 2, \cdots, q$, be different prime factors of n in increasing order. Then, the following conclusions hold:

i) When $n = 1$, $\Omega_{\rm ca}$ consists of only one forward-orbit of period 2 for systems of Type-I, and two equilibria for systems of Type-II.

ii) When $n > 1$, for systems of Type-II, a positive integer l is the period of a point $x \in \Omega_{\rm ca}$ if and only if it takes the form

$$l = \prod_{i=1}^{q} p_i^{d_i},\ 0 \leqslant d_i \leqslant n_i,\ i = 1, 2, \cdots, q. \tag{15}$$

There are two periodic-1 (fixed) points in $\Omega_{\rm ca}$.

For any $l \neq 1$ of the form (15), let p_m, $1 \leqslant m \leqslant q$, be the smallest prime factor of l, i.e., $d_1 = \cdots = d_{m-1} = 0$, $d_m \neq 0$. Denote $\bar{l} = \dfrac{l}{p_m}$. Then, the number of distinct periodic orbits in $\Omega_{\rm ca}$ with period l equals $2^l - 2^{\bar{l}}$.

iii) For systems of Type-I, factorize n into the following form, for $i = 1, \cdots, q$,

$$n = 2^{n_0} \prod_{i=1}^{q} p_i^{n_i},\ p_i \neq 2,\ n_0 \geqslant 0,\ n_i > 0. \tag{16}$$

Then, a positive integer l is the period of a point $x \in \Omega_{\rm ca}$ if and only if it takes the following form, for $i = 1, \cdots, q$,

$$l = 2^{n_0+1} \prod_{i=1}^{q} p_i^{d_i},\ 0 \leqslant d_i \leqslant n_i. \tag{17}$$

There is one periodic-2 orbit in $\Omega_{\rm ca}$.

For any $l \neq 2$ of the form (17), let p_m, $1 \leqslant m \leqslant q$, be the second smallest prime factor of l, *i.e.*, $d_1 = \cdots = d_{m-1} = 0$, $d_m \neq 0$. Denote $\bar{l} = \dfrac{l}{p_m}$. Then, the number of distinct periodic orbits in $\Omega_{\rm ca}$ with period l equals $2^l - 2^{\bar{l}}$.

4 Modulation along arbitrary direction of stable systems

In this section, we assume that A is a stable matrix, i.e., the eigenvalues of A lie inside the unit circle.

The following result concerning periodic orbits generated by external periodic excitation is well-known.

Theorem 6 i) For a periodic input sequence of period L, there is a periodic orbit of period L for system (1).

ii) This periodic orbit is globally attracting.

Now, we turn to the situation of Δ-modulated control of system (1). In this case, the controller u is a Δ-modulated feedback defined by

$$u = {\rm sgn}\ (c^{\rm T}x), \tag{18}$$

in which $c \in \mathbb{R}^n\backslash\{0\}$ is an arbitrary, but fixed, modulation direction.

Suppose $\{x_0, x_1, \cdots, \}$ is an orbit of the closed-loop system (1) and (18) starting from x_0. The sequence defined by $\{s_0, s_1, \cdots, \}$, where $s_i = {\rm sgn}\ (c^{\rm T}x_i)$, for $i = 0, 1, \cdots$, is a binary sequence of 1's and -1's. We will call it a modulated orbit of the closed-loop system (1) and (18) corresponding to the orbit $\{x_0, x_1, \cdots\}$.

Obviously, the modulated orbit of a periodic orbit of the closed-loop system (1) and (18) is periodic. Therefore, to determine the periodicity of an orbit of a Δ-modulated system, from Theorem 6, it is decisive to see whether the Δ-modulation in (18) introduces a periodic binary sequence. This is addressed by the following theorem [31].

Theorem 7 The Δ-modulated system (1) and (18) has a periodic orbit of period L if and only if there are $\sigma_0, \sigma_1, \cdots, \sigma_{L-1} \in \{-1, 1\}$ such that, for $\sigma_i = 1$,

$$c^{\rm T}(I - A^L)^{-1} \sum_{j=0}^{L-1} A^{L-j-1} b\, \sigma_{i+j} \geqslant 0, \tag{19}$$

and for $\sigma_i = -1$,

$$c^{\mathrm{T}}(I-A^L)^{-1}\sum_{j=0}^{L-1}A^{L-j-1}b\,\sigma_{i+j}<0 \tag{20}$$

for $i=0,1,\cdots,L-1$, in which $\sigma_{i+j}=\sigma_{(i+j)\mathrm{mod}L}$.

Proof (Necessity) If $\{x_0,x_1,\cdots\}$ is a periodic orbit with period L, then denote $\sigma_i = s_i = \mathrm{sgn}\ (c^{\mathrm{T}}x_i)$, for $i=0,1,\cdots,L-1$. Since $\{x_0,x_1,\cdots\}$ is periodic with period L, we have

$$x_i=(I-A^L)^{-1}\sum_{j=0}^{L-1}A^{L-j-1}b\,\sigma_{i+j},$$

for $i=0,1,\cdots,L-1$. Hence,

$$c^{\mathrm{T}}(I-A^L)^{-1}\sum_{j=0}^{L-1}A^{L-j-1}b\,\sigma_{i+j}=c^{\mathrm{T}}x_i,$$

which implies the conditions of the theorem.

(Sufficiency) Denote

$$x^{(i)}=(I-A^L)^{-1}\sum_{j=0}^{L-1}A^{L-j-1}b\,\sigma_{j+i}, \tag{21}$$

and

$$f(x)=Ax+b\ \mathrm{sgn}\ (c^{\mathrm{T}}x).$$

It is straightforward to verify that under the conditions of the theorem, $f^{(i)}(x^{(0)})=x^{(i)}$, for $i=0,1,\cdots,L-1$ and $f^{(L)}(x^{(0)})=x^{(0)}$, i.e., the orbit starting at $x^{(0)}$ has period L.

A Δ-modulated system can have many periodic points. The first interesting result is the following.

Corollary 1 i) If (A,b) is controllable, then there is a $c\in\mathbb{R}^n$ such that the closed-loop system (1) and (18) has n-periodic orbits.

ii) If (c^{T},A) is observable, then there is a $b\in\mathbb{R}^n$ such that the closed-loop system (1) and (18) has n-periodic orbits.

Proof We prove i) only. For $n=1$, choose $c=b$, and for $n=2$, choose

$$c=(I-A^2)(Ab-b).$$

It can be verified that these two choices result in 1-periodic points for $n=1$ and 2-periodic points for $n=2$, respectively.

For $n\geqslant 3$, since the controllability of (A,b) implies the existence of the inverse in the expression, we choose c^{T} as

$$(1,0,\cdots,0)\begin{pmatrix}A^{n-1}b & \cdots & Ab & b\end{pmatrix}^{-1}(I-A^n).$$

Then, for any binary sequence $\{s_0,s_1,\cdots,s_{n-1}\}$,

$$c^{\mathrm{T}}(I-A^n)^{-1}(A^{n-1}bs_0+\cdots+bs_{n-1})=s_0.$$

So, the inequalities in (19) and (20) hold. By Theorem 7, for this choice of c, any n binary sequence gives rise to an orbit of period n.

Choose a sequence $s_0=1$, $s_i=-1$, for $i=1,\cdots,n-1$. According to (21), the periodic orbit generated by it consists of the following n points:

$$x^{(i)}=(I-A^n)^{-1}\sum_{j=0}^{n-1}A^{n-j-1}bs_{j+i}.$$

It can be verified that these n points are different; therefore, this orbit is n-periodic. Actually, it can be proved that $x^{(0)},x^{(1)},\cdots,x^{(n-1)}$ are linearly independent.

The criterion in (19) and (20) is useful in deriving concrete results about the existence of periodic orbits of a certain order.

Proposition 1 i) System (1) under the Δ-modulation of (18) has a fixed (1-periodic) point iff $c^{\mathrm{T}}(I-A)^{-1}b\geqslant 0$.

ii) System (1) under the Δ-modulation of (18) has a 2-periodic orbit iff $c^{\mathrm{T}}(I+A)^{-1}b<0$.

iii) System (1) under the Δ-modulation of (18) has a 3-periodic orbit iff

$$\begin{aligned}&2\max\{c^{\mathrm{T}}(I-A^3)^{-1}b,c^{\mathrm{T}}(I-A^3)^{-1}Ab\}\\&\quad<c^{\mathrm{T}}(I-A)^{-1}b\leqslant 2c^{\mathrm{T}}(I-A^3)^{-1}A^2b.\end{aligned}$$

iv) System (1) under the Δ-modulation of (18) has a 4-periodic orbit iff

$$\begin{aligned}&2\max_{0\leqslant i\leqslant 2}c^{\mathrm{T}}(I-A^4)^{-1}A^ib<c^{\mathrm{T}}(I-A)^{-1}b\\&\leqslant 2c^{\mathrm{T}}(I-A^4)^{-1}A^3b,\end{aligned}$$

or

$$\begin{aligned}c^{\mathrm{T}}(I-A)^{-1}b>2\max\{&c^{\mathrm{T}}(I-A^4)^{-1}(A+I)b,\\&c^{\mathrm{T}}(I-A^4)^{-1}(A^2+A)b\}.\end{aligned}$$

5 Other generalizations

Some generalizations are made in [26∼28].

5.1 Unbalanced Δ-modulated systems

In the first case, the following discrete-time nonlinear system is considered:

$$x_{n+1}=ax_n+u, \tag{22}$$

under the so-called unbalanced Δ-modulated feedback (UDMF)

$$u=\Delta(ax)\overset{\mathrm{def}}{=}\begin{cases}-\Delta_1, & ax\geqslant 0,\\ \Delta_2, & ax<0,\end{cases} \tag{23}$$

where Δ_1 and Δ_2 are given positive real numbers, $\Delta_1\neq\Delta_2$.

It should be noted that Δ-modulated control is a special case of UDMF, e.g., the balanced case with $\Delta_1=\Delta_2$. In the same application area of transmitting power control, it has witnessed the flexibility of unbalanced Δ-modulated feedback in, i.e., [32, 33]. All these motivate a careful study of systems (22) and (23).

Define $\gamma=\dfrac{\Delta_2}{\Delta_1}$. Then, $\gamma\neq 1$.

System (22) is referred to as a system of Type I when $a>0$, and system of Type II when $a<0$, respectively.

We have only considered the case when the parameter $0 <| a |\leqslant 1$.

The results in [26] can be summarized in the following.

A system of Type II has only two fixed points and the set of fixed points is globally attracting. For $0 < a < 1$, systems of Type I have no fixed points, and there is a denumerable set of values for the ratio $\gamma = \dfrac{\Delta_2}{\Delta_1}$, and for each parameter γ of the denumerable set, systems of Type I have no periodic orbits and, in this case, every orbit is dense in the state interval $[-\Delta_1,\ \Delta_2)$. To each of the other rate values of γ, systems of Type I all have a unique periodic orbit. The structural property of the periodic motion is robust; i.e., there exists an interval including this value γ such that all parameters in this interval correspond to those periodic orbits of the same structural property. For the case of $a = 1$, all points in the interval $[-\Delta_1,\ \Delta_2)$ are $n-$periodic with $n \geqslant 3$ when γ is a rational number, and every orbit is dense in the interval $[-\Delta_1,\ \Delta_2)$ when γ is an irrational number. Moreover, every such unique periodic orbit is globally attracting for both types of systems.

5.2 Quantized feedback systems

A first-order discrete-time control system with a two-level quantized feedback is considered in [27]:

$$x^+ = f(x) \overset{\text{def}}{=} ax - q(x), \tag{24}$$

where the scaling factor $a > 0$ is a real number, the quantized feedback $q(x)$ is defined as

$$q(x) = \begin{cases} 1, & x \geqslant 0.5, \\ 0, & -0.5 < x < 0.5, \\ -1, & x \leqslant -0.5. \end{cases}$$

A first simple result is the following.

Proposition 2 When $0 < a \leqslant 1$, there are only three periodic points of the system (1) $\{0, \pm 1/(a+1)\}$, and 0 is 1-periodic (fixed point), and $\pm 1/(a+1)$ are 2-periodic. The set $\{0, \pm 1/(a+1)\}$ are globally attracting.

Then the study also starts with special classes of polynomials.

We define the following four sets of polynomials:

$$\begin{aligned} p_1(a) &= p^1(a) = a - 1, \\ q_1(a) &= q^1(a) = a - 3, \\ p_2(a) &= p^2(a) = a^2 - 3, \\ q_2(a) &= q^2(a) = a^2 - 2a - 1, \end{aligned}$$

and for $n \geqslant 3$,

$$\begin{aligned} p_n(a) &= a^n - 2a^{n-2} - 1, \\ q_n(a) &= a^n - 2a^{n-1} - 1, \\ p^n(a) &= a^n - 2a^{n-2} - 2a^{n-3} - \cdots - 2a - 3, \\ q^n(a) &= a^n - 2a^{n-1} - 2a^{n-3} - \cdots - 2a - 3. \end{aligned}$$

These polynomials have a very special property: the polynomials $p_1(a)$ and $p^1(a)$ have their only root at $a = 1$, and all other polynomials have only one real root in $(1, \infty)$.

Lemma 4 i) For $n \geqslant 2$, each of the polynomials $p_n(a), q_n(a), p^n(a)$ and $q^n(a)$ has only one real root in $(1, \infty)$.

ii) Denote $\underline{p}_1, \underline{q}_1, \bar{p}_1$ and $\bar{q}_1$ the root of $p_1(a)$, $q_1(a)$, $p^1(a)$ and $q^1(a)$, respectively, and for $n \geqslant 2$, denote $\underline{p}_n, \underline{q}_n, \bar{p}_n$ and $\bar{q}_n$ the only root of $p_n(a), q_n(a), p^n(a)$ and $q^n(a)$ in $(1, \infty)$, respectively. Then for $n \geqslant 3$,

ii.1) $\underline{p}_{n+1} < \underline{p}_n$ and $\lim\limits_{n\to\infty} \underline{p}_n = \sqrt{2}$;

ii.2) $\underline{q}_{n+1} < \underline{q}_n$ and $\lim\limits_{n\to\infty} \underline{q}_n = 2$;

ii.3) $\bar{p}_{n+1} > \bar{p}_n$ and $\lim\limits_{n\to\infty} \bar{p}_n = 2$;

ii.4) $\bar{q}_{n+1} > \bar{q}_n$ and

$$\begin{aligned} \lim_{n\to\infty} \bar{q}_n &= 1 + \frac{\sqrt[3]{27 + 11\sqrt{6}}}{3} + \frac{1}{\sqrt[3]{27 + 11\sqrt{6}}} \\ &\doteq 2.5241. \end{aligned}$$

Then, results on the periodicity of system (24) can be partially characterized in the following theorem.

Theorem 8 i) For any $n = 1, 2, \cdots$, system (24) has non-zero n-periodic points if $\underline{p}_n < a \leqslant \bar{q}_n$.

ii) $a \leqslant \bar{q}_n$ is necessary for having an n-periodic point.

5.3 Uplink delayed systems

In the last generalizaation, we performed a complete spectral analysis of system

$$x_{n+1} = x_n - \mathrm{sgn}\,(x_{n-k}), \tag{25}$$

in which $x_i \in \mathbb{R}$ is a scalar signal, and the fixed integer $k \geqslant 0$ represents the loop delay. A special feature of the Δ-modulated feedback system (25) is the existence of a delay in the feedback loop. This kind of systems with loop delays appear in, among others, transmitting power control of a mobile unit in a Direct Sequence Code Division Multiple Access (DS-CDMA) cellular network [11] and in the study of nonlinear dynamics of digital bang-bang Phase-Locked-Loops (PLLs) [34].

It will be seen that the spectral properties of system (25) is completely different from that of the "up-link" delayed system. If we realize system (1) by a higher-order system, then it is only critically stable and the results of [31] cannot be directly applied.

The results on the existence of periodic orbits of system (25) can be elegantly stated and rigorously proved: every orbit of system (25) is eventually periodic with a prime period of the form $2(2j+1)$ for some integer j, $0 \leqslant j \leqslant k$, such that $2(2j+1)$ divides $k - j$. A byproduct of these results is: in case of multiple periods, all the smaller periods divide the maximal period. In other words, in case of multiple frequencies, it consists of the primary frequency and some of its harmonics (multiples of the primary frequency).

"Overtune" will never happen to the system.

Periods for delays up to 39 are given in the table.

Table 1 Periods for delays up to 39.

delay k	period	delay k	period
0	2	20	2, 82
1	6	21	86
2	2, 10	22	2, 10, 18, 90
3	14	23	94
4	2, 18	24	2, 98
5	22	25	6, 102
6	2, 26	26	2, 106
7	6, 30	27	22,110
8	2, 34	28	2, 114
9	38	29	118
10	2, 42	30	2, 122
11	46	31	6, 14, 126
12	2,10,50	32	2, 10, 26, 130
13	6, 54	33	134
14	2, 58	34	2, 138
15	62	35	142
16	2, 66	36	2, 146
17	14,70	37	6, 30, 150
18	2, 74	38	2, 154
19	6, 78	39	158

6 Conclusions

The existence of periodic points of all orders of Sigma-Delta modulation with "leaky" integration has been completely characterized by making use of some interesting groups of polynomials with "sign" coefficients. The results have also been extended to Sigma-Delta modulations with multiple delays in a natural way. Further extensions have been made to the existence of periodic points arising from Δ-modulated feedback control of a stable linear system in an arbitrary direction, for which some necessary and sufficient conditions have been derived. Thus, this investigation is self-contained and relatively complete. Finally, the paper has also described some possible generalizations of the results.

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