Robust Synchronization of a Class of Nonlinear Systems: Applications to Chaotic Coupled Electromechanical Systems

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Abstract: This article treats the robust synchronization problem of a class of nonlinear systems from a control theoretical point of view. Because of the tremendous complexity of nonlinear systems, the problem is restricted to chaotic electromechanical devices. The results are discussed in the context of complete synchronization. A new dynamic output feedback is applied to perform synchronization in spite of master/slave mismatches. The main idea is to construct an augmented dynamical system from the synchronization error system, which is itself uncertain. The advantage of this method over the existing results is that the synchronization time is explicitly computed. Numerical simulations are provided to verify the operation of the proposed algorithm.

Keywords: Nonlinear systems, chaotic systems, electromechanical devices, synchronization, observers.

NOMENCLATURE

$\mathbb{R}$	denotes the set of real numbers
$\mathbb{R}^n$	is the n-dimensional real linear space
x, y	represents a column vector of R^n, and x^T a vector row
$\dot{x}$	is the first derivative of real-valued C^1 function $x(t) = (x_1(t), ..., xn(t))^T$
$\mathbf{A}, \mathbf{B}, \mathbf{N}, \mathbf{X}$	are matrices
$\mathbf{A}^T$	is the transpose of the matrix $\mathbf{A}$
$\mathbf{A}_{ij}$ and $\mathbf{A}_{-ij}$	are elements of $\mathbf{A}$ and $\mathbf{A}^{-1}$, respectively
s and t	are real scalars

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Figures 2–4 appears in color online: http://jvc.sagepub.com

1. INTRODUCTION

The study of the synchronization problem for chaotic oscillators has been very important from the point of view of nonlinear science, with applications in biology, medicine, cryptography, secure data transmission and elsewhere (Pecora and Caroll, 1990; Lakshamanam and Murali, 1996; Chen and Dong, 1998; Bocaletti et al., 2002; Andrievskii and Fradkov, 2003, 2004). The idea of synchronization is to use the output of the master system to control the slave system, so that the output of the slave system follows the output of the master system asymptotically. In general, synchronization research has been focused on two areas, related to either *state observers* or *control laws*. The main applications of state observers lie in the synchronization of nonlinear oscillators with the same model structure and order, but different initial conditions and/or parameters (Morgul and Solak, 1996; Grassi and Mascolo, 1997; Nijmeijer and Mareels, 1997, Feki, 2003; Liao and Tsai, 2000; Bowong et al., 2004). The use of *control laws*, on the other hand, allows us to achieve synchronization between nonlinear oscillators, with different structures and order, with the state variables of the slave system being forced to follow the trajectories of the master system. This approach can be seen as a tracking problem (Gonzalez et al., 1999; Femat et al., 1999; Muraly, 2000; Bowong, 2004; Moukam Kakmeni et al., 2004); some authors design the controller based on the dynamic of the synchronization error, because this approach allows the transformation of the tracking problem to a regulation problem with the origin (zero) as the corresponding set point (Femat and Solis-Peralez, 1999).

Several other control approaches have been tried, using neural-network, fuzzy, adaptive, and sliding and other techniques (Bowong, 2004). Other traditional control methods (Ott et al., 1990) consider the introduction of an additive feedback controller, to force the system to reach the desired reference (set point), i.e., $\left\| x(t) - x_{ps}(t) \right\| \to 0 || x(t)$ as $t \to \infty$. These methodologies are based on the cancellation of the nonlinear terms of the chaotic systems, in order to impose a desired behavior. Under this philosophy, nonlinear differential geometric control techniques have been successfully employed (Isidori, 1989; Nijmeijer and Van der Shaft, 1990). They correspond to systems that can be fully or partially linearized by a change of coordinates and/or state feedback following differential-geometric concepts (Gonzalez et al., 1999). This type of nonlinear systems can be linearized by a state feedback control, which, assuming perfect knowledge of the mathematical model, cancels all the nonlinearities, producing global asymptotic stability (Femat et al., 1999). A drawback of exact linearization techniques and other model based controllers is that they rely on exact cancellation of nonlinearities.

In practice, exact knowledge of the system dynamics is not possible. A more realistic situation is to know some nominal functions of the corresponding nonlinearities, which are employed in the control design. However, the use of nominal model nonlinearities can lead to performance degradation and even closed-loop instability. In fact, when the systems possess strong nonlinearities, the standard linearizing, generic model, and active controllers cannot cancel completely such nonlinearities, and instabilities can be induced. The worst case occurs if knowledge of the nonlinearities is very poor or completely absent, such that conventional linearizing techniques are inadequate. To avoid these problems, the geometric approach for the design of nonlinear controllers based on uncertainty observers has been employed, and these kinds of techniques show satisfactory capabilities for a wide range of systems (Aguilar et al., 2002; Aguilar-Lopez and Alvarez-Ramirez, 2002). The use of proportional observers coupled with linearizing controllers has been very successful, but the proportional observers have several problems. Notably, they are very sensitive to noisy measurements, and robustness issues have not been completely addressed. For these reasons, more sophisticated observers have been designed, in order to generate better open-loop and closed-loop performances. PI, sliding-mode, and numeric observers have been developed, along with other types (Martinez-Guerra et al., 2004; Aguilar et al., 2001).

However, despite the large amount of theoretical and experimental results already obtained, a great deal of effort is still required to determine the optimal parameters, in order to shorten the synchronization time (Woafo and Kreankel, 2002; Chembo and Woafo, 2002), define the synchronization threshold parameters (Pyragas, 1998), and avoid loss of synchronization and instability during the synchronization process (Osipov et al., 1997). This problem is important in all fields where synchronization has or may gain practical interest. For instance, if we consider the application of synchronization in secure communications, the range of synchronization time during which the chaotic oscillators are not synchronized is the period during which the encoded message cannot be recovered or sent. Not merely an irreversible loss of information, this is a catastrophe in digital communications, since the first bits of standardized bit strings always contain signalization data or the identity card of the message. Hence, it is clear that the synchronization time must be minimized, so that the chaotic transmitted oscillators synchronize as fast as possible. In this context, it is fair to say that there a need for study of this problem.

The purpose of this work is to make a novel contribution to robust synchronization of a class of continuous-time systems. Because of the tremendous complexity of nonlinear systems, the problem is restricted to chaotic electromechanical devices, which are widely encountered in electromechanical engineering; for instance (in its linear version) electrodynamic loudspeakers. Previously, Woafo et al. (2005) investigated the problem of the chaotic behavior of a class of electromechanical systems. They found that chaos can arise following a period-doubling bifurcation cascade or an invariant torus bifurcation series.

In this article, based on a rigorous mathematical analysis and use of the Lyapunov Direct Method, an adaptive feedback controller is designed such that two coupled chaotic electromechanical systems with uncertainties can be synchronized. The source of such uncertainties could be modelling errors, parametric mismatching, or external disturbances. The proposed strategy is an input-output control scheme comprising an uncertainty estimator and a linearizer-like feedback. The robust controller is designed by means of the following procedure:

i) the uncertainties are lumped in a nonlinear function
ii) the lumped nonlinear function is interpreted as an augmented state, in such a way that the extended system is dynamically equivalent to the original system
iii) in order to obtain an estimate of the augmented state, a state estimator is designed for the extended system
iv) the estimated value of the uncertainties is provided for the control law (via the estimated value of the augmented state).

Our stability analysis provides an easy and explicit procedure for computing the synchronization time, which depends on the initial conditions and two suitable positive parameters.

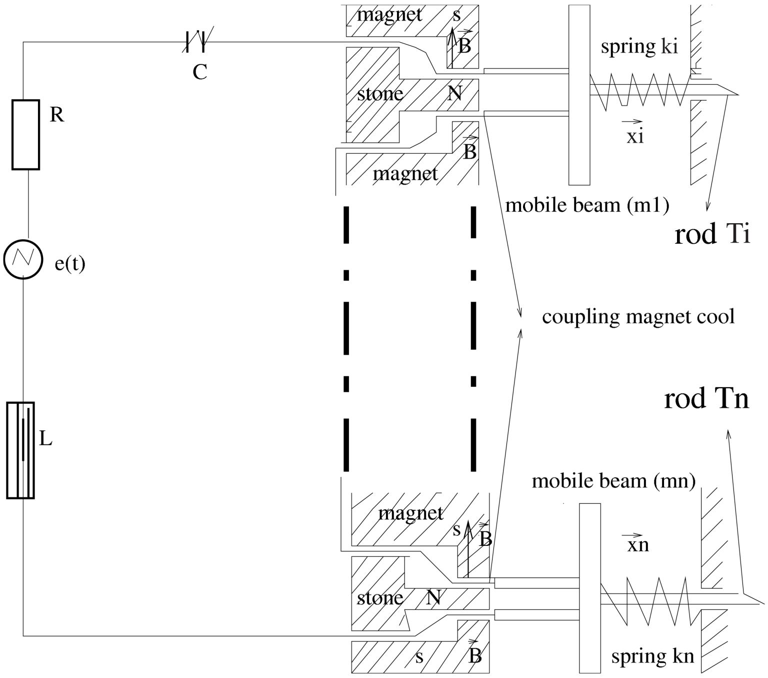


Figure 1. Schematic of the multi-function electromechanical system.

More importantly, we show how the parameters of the feedback control law affect the convergence rate of the synchronization error. The robustness of the feedback control law against model uncertainties in the actuator is shown using numerical simulations.

We hope that the methodology developed for this specific chaotic system will be applicable to other types of chaotic systems such as the Rossler system, Chua's circuit, Lorenz systems and many other types of chaotic system. We think that the technique developed in this work provides a strong tool for control theory and is full of promise, as it could be applied to a great range of problems: Stabilization, implementation, observers, adaptive synchronization, etc.

2. PRELIMINARIES

2.1. System Description

The electromechanical system, as described by Woafo et al. (2005), is schematically represented in Figure 1. It consists of an electrical part coupled magnetically to a mechanical part governed by n linear mechanical oscillators. The coupling between the two parts is realized by the electromechanical force of a permanent magnet. The electrical part of the system consists of a resistor R, an inductor L, a capacitor C with nonlinear characteristics (Woafo et al., 2005) and a sinusoidal voltage source, all connected in series, while the mechanical part is composed of n mobile beams which can move respectively along the $\vec{x}_i$ $(i = 1, ..., n)$ axis on both sides. The rods Ti are bound to mobile beams with springs of constants k_i. The motion of the entire electromechanical system is governed by $n + 1$ nondimensional coupled nonlinear differential equations of the form

$$\begin{aligned} \ddot{x} + \gamma \dot{x} + x + \beta x^3 + \sum_{i=1}^{n} \lambda_i \dot{x}_i &= E_0 \cos \omega t, \\ \ddot{x}_i + \gamma_i \dot{x}_i + \omega_i^2 x_i + \lambda_{i1} \dot{x} &= 0, \quad i = 1, ..., n, \end{aligned} \tag{1}$$

where the overdot denotes differentiation with respect to time. The electrical part (Dung electrical oscillator) is represented by the variable x, while x_i denote the mechanical parts (the n linear mechanical oscillators). x denotes the instantaneous electrical charge of the capacitor, and x_i the displacements of the i^{th} mobile beam. γ and γ_i are, respectively, the damping coefficients of the Dung electrical part and of the i^{th} linear mechanical part. The quantities λ_i and λ_{i1} are the coupling coefficients, β is the nonlinear coefficient, and ω_i is the natural frequency of the i^{th} oscillator. E_0 and ω are the amplitude and frequency, respectively, of the external excitation (sinusoidal voltage source).

The model shown in Figure 1 is widely encountered in electromechanical engineering. In particular, in its linear version, it describes the electrodynamic loudspeaker (Olson, 1967). In this case, the sinusoidal signal $e(t)$ represents an incoming pure message. Because of recent advances in the theory of nonlinear phenomena, it is interesting to consider such an electrodynamic system containing one or more nonlinear components, or in the state where one or several of its components react nonlinearly. One such state occurs in the electrodynamic loudspeaker, due to the nonlinear character of the diaphragm suspension system, resulting in signal distortion and subharmonic generation (Olson, 1967). Moreover, the model can serve as a servo-command mechanism, which can be used for a range of purposes. Here, one would like to take advantage of the nonlinear responses of the model in manufacturing processes.

2.2. Problem Formulation

In order to observe the synchronization behavior in the electromechanical system, we assume that the master system is given by system (1) and the slave is described by

$$\begin{aligned} \ddot{y} + \gamma \dot{y} + y + \beta y^3 + \sum_{i=1}^{n} \lambda_i \dot{y}_i &= E_0 \cos \omega t + u, \\ \ddot{y}_i + \gamma_i \dot{y}_i + \omega_i^2 y_i + \lambda_{i1} \dot{y} &= 0, \quad i = 1, 2, ..., n, \end{aligned} \tag{2}$$

where we have introduced the control input $u \in R$. The control input u is to be determined, for the purpose of synchronizing two identical coupled electromechanical systems with the same parameters in spite of the difference in initial conditions. Without loss of generality, we will assume, in what follows, that all parameters of both the master and slave electromechanical systems are positive.

For further analysis of stability and synchronization, we define the state error vector between the master and slave electromechanical systems as $e = y - x$, $\dot{e} = \dot{y} - \dot{x}$ and $\dot{e}_i = \dot{y}_i - \dot{x}_i$ $(i = 1, ..., n)$. Using equations (1) and (2), the error dynamics can be described by

$$\begin{aligned} \ddot{e} + \gamma \dot{e} + e + \beta e^3 + 3\beta e x(e + x) + \sum_{i=1}^{n} \lambda_i \dot{e}_i &= u \\ \ddot{e}_i + \gamma_i \dot{e}_i + \omega_i^2 e_i + \lambda_{i1} \dot{e} &= 0, \quad i = 1, 2, ..., n. \end{aligned} \tag{3}$$

Remark 1. System (3) is minimum phase with respect to $e = 0$; that is, the second equation of system (3) is uniformly exponentially stable about $\dot{e} = 0$. In fact, when $\dot{e} = 0$, the zero dynamics are given by

$$\ddot{e}_i + \gamma_i \dot{e}_i + \omega_i^2 e_i = 0 \quad i = 1, ..., n. \tag{4}$$

In order to show the convergence to zero of the above system, we can define the function

$$V(\dot{e}_i, e_i) = \frac{1}{2} \sum_{i=1}^{n} \left(\dot{e}_i^2 + \omega_i^2 e_i^2 \right) \tag{5}$$

as a candidate Lyapunov function. Its time derivative along (4) satisfies

$$V(\dot{e}_i, e_i) = -\sum_{i=1}^{n} \gamma_i \dot{e}_i^2 \leq 0$$

which implies that $V(\dot{e}_i, e_i)$ is semi-definite negative because $\gamma_i > 0$. Furthermore, it follows from the LaSalle invariance principle (LaSalle and Letschetz, 1961) that the largest invariant set contained in $E = \{(e_i, \dot{e}_i) \in \mathbb{R}^{2n}, \dot{V}(\dot{e}_i, e_i) = 0\}$ is the manifold $\dot{e}_i = 0$. According to system (4), $\dot{e}_i = 0$ implies that $e_i = 0$. Therefore, the origin is the largest invariant set contained in E. Since $e_i = \dot{e}_i = 0$, one can conclude that the synchronization error states $(e_i(t), \dot{e}_i(t))$ remain at zero for all $t \geq t_0 \geq 0$, since the manifold $(e_i, \dot{e}_i) = 0$ is the largest invariant set of $\mathbb{R}^{2n}$. This implies that when $\dot{e}_i = 0$, $\lim_{t\to\infty} e_i(t) = 0$ and $\lim_{t\to\infty} \dot{e}_i(t) = 0$.

Thus, when we have taken action to achieve $\lim_{t\to\infty} e_i(t) \to 0$, the second equation of system (3) converges asymptotically to zero for the so-called minimum-phase character. Therefore, we only consider the first equation of system (3) in what follows.

Now, assume that the outputs of the master and slave systems are, respectively, $y_m = x$ and $y_s = y$. By substituting $z_1 = e$ and $z_2 = \dot{e}$, the first equation of system (3) can be rewritten as

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= \Theta(z, x, \dot{e}_i) + u \\ y_z &= z_1 \end{aligned} \tag{6}$$

where $z = (z_1 z_2)^T$, y_z is the error system output and $\Theta(z, x, \dot{e}_i) = -\gamma z_2 - z_1 - \beta z_1^3 - 3\beta z_1 x(z_1 + x) - \sum_{i=1}^{n} \lambda_i \dot{e}_i$.

The synchronization problem can be stated as follows: given the transmitted signal y_m and least prior information about the structure of the nonlinear filter, system (1), to design a receiver signal u such that

$$z(t) = 0 \quad \text{for all} \quad T_s \geq 0 \tag{7}$$

where Ts is the synchronization time.

To describe the new design and analysis, two hypotheses are needed.

H$_1$: Only the system output $y_z = z_1$ is available for feedback.
H$_2$: The function $\Theta(z, x, e_i)$ is uncertain.

Some comments regarding these are in order. Assumption 1 is realistic. For instance, in the case of secure communications, only the transmitted signal and receiver signal are available for feedback from measurements. Another example of the problem can be found in neuron synchronization, where master neuron transmits a scalar signal. The slave neuron tracks the signal of the master neuron. Assumption 2 refers to a general and practical situation, as the term $\Theta(z, x, e_i)$ involves the uncertainties in the system. Hence, the nonlinear function $\Theta(z, x, e_i)$ is unknown and cannot be used directly in linearizing-type feedback. These kinds of uncertainties have previously been studied in the context of chaos control and synchronization by various authors (Gonzalez et al., 1999; Femat et al., 1999; Bowong, 2004; Moukam Kakmeni et al., 2004).

Our proposal for dealing with the uncertain term $\Theta(z, x, e_i)$ in equation (6) is to lump it into a new state. Thus, let $\eta = \Theta(z, x, e_i)$. In this way, system (6) can be rewritten (Gonzalez et al., 1999; Femat et al., 1999; Bowong, 2004; Moukam Kakmeni et al., 2004) as the dynamically equivalent extended system

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= \eta + u \\ \dot{\eta} &= \Xi(z, \dot{x}, \eta, e_i, \dot{e}_i, u) \\ y_z &= z_1 \end{aligned} \tag{8}$$

where

$$\begin{aligned} \Xi(z, \dot{x}, \eta, e_i, \dot{e}_i, u) &= z_2 \partial\Theta(z, x, \dot{e}_i)/\partial z_1 + (\eta + u)\partial\Theta(z, x, \dot{e}_i)/\partial z_2 + \dot{x}\partial\Theta(z, x, \dot{e}_i)/\partial x \\ &- \sum_{i=1}^{n} (\gamma_i \dot{e}_i + \omega_i^2 e_i + \lambda_{i1} z_2)\partial\Theta(z, x, \dot{e}_1)/\partial \dot{e}_i. \end{aligned}$$

With respect to systems (6) and (8), we can say:

Proposition 1. System (8) is dynamically equivalent to system (6), that is, system (8) has the same solution as the system (6) module $\Pi(z, \eta) = z$.

Proof. It is straightforward to prove that $\Psi(z, x, \eta, \dot{e}_i) = \eta - \Theta(z, x, \dot{e}_i)$ is a first integral of system (8). In order to prove this property, it suffices to show that along the trajectories of system (8), $d\Psi(z, x, \eta, \dot{e}_i)/dt = 0$ for all $t \geq 0$, or, equivalently $z_2\partial\Psi(z, x, \eta, \dot{e}_i)/\partial z_1 + (\eta + u)\partial\Psi(z, x, \eta, \dot{e}_i)/\partial z_2 + \dot{\eta}\partial\Psi(z, x, \eta, \dot{e}_i)/\partial\eta + \dot{x}\partial\Psi(z, x, \eta, \dot{e}_i)/dx - (\gamma_i\dot{e}_i + \omega_i^2 e_i + \lambda_{i1}z_2)\partial\Psi(z, x, \eta, \dot{e}_i)/\partial\dot{e}_i = 0$. This is automatically satisfied, because $\partial\Psi(z, x, \eta, \dot{e}_i)/\partial\eta = 1$ and $\dot{\eta} = z_2\partial\Psi(z, x, \eta, \dot{e}_i)/\partial z_1 + (\eta+u)\partial\Psi(z, x, \eta, \dot{e}_i)/\partial z_2 + \dot{x}\partial\Psi(z, x, \eta, \dot{e}_i)/\partial x - (\gamma_i\dot{e}_i + \omega_i^2\dot{e}_i + \lambda_{i1}z_2)\partial\Psi(z, x, \eta, \dot{e}_i)/\partial\dot{e}_i$. Hence, system (8) is dynamically equivalent to system (6). This implies that the augmented state η provides the dynamics of the uncertain function $\Theta(z, x, \dot{e}_i)$.

For the sake of compactness, we introduce the following afilternative description for system (8)

$$\begin{aligned} \dot{z} &= Az + B(\eta + u) \\ \dot{\eta} &= \Xi(z, \dot{x}, \eta, e_i, \dot{e}_i, u) \end{aligned} \tag{9}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

3. A FEEDBACK CONTROL LAW WITH ESTIMATION OF UNCERTAINTIES

In this section, we propose a physically realizable feedback controller u to attain the synchronization objective stated above, i.e., the stabilization at the origin and in a finite time of uncertain system (9). The approach developed in this work is able to guarantee robust stability (in fact, robust synchronization) from incomplete state measurements, and requires no detailed model of the system. Our approach includes a state/uncertainty observer and gives a robust feedback control scheme. The expression of the synchronization time is explicitly computed. Let $\mathbf{M}(\alpha, \beta)$ be the matrix

$$M(\alpha, \beta) = \begin{pmatrix} \dfrac{2\left(\frac{\alpha}{\beta}\right)^3}{(\alpha+1)(\alpha+2)(\alpha+3)} & \dfrac{-\left(\frac{\alpha}{\beta}\right)}{(\alpha+1)(\alpha+2)} \\ \dfrac{-\left(\frac{\alpha}{\beta}\right)}{(\alpha+1)(\alpha+2)} & \dfrac{\left(\frac{\alpha}{\beta}\right)}{\alpha+1} \end{pmatrix} \tag{10}$$

where α and β are positive constants. Also, let $N(\theta)$ be the matrix

$$N(\theta) = \frac{\alpha}{\beta}\theta^{\frac{1}{\alpha}} \int_0^1 (1-t)^{\alpha} e^{-\frac{\alpha}{\beta}A^T\theta^{\frac{1}{\alpha}}t} BB^T e^{-\frac{\alpha}{\beta}A\theta^{\frac{1}{\alpha}}t} dt. \tag{11}$$

More precisely, we have:

Lemma 1. $N(\theta)$ is symmetric, positive and definite, and is the solution of the differential matrix equation

$$\frac{dX}{d\theta} = \frac{1}{\beta}\theta^{\frac{1}{\alpha}-1}\left[-A^T X - XA - \beta\theta^{\frac{1}{\alpha}} X + BB^T\right]. \tag{12}$$

This Lemma is proved in Appendix A. Note that $N(\theta)$ is determined by A and B. Equation (12) will be useful in proving that the dynamics of the closed-loop system are asymptotically stable. This is related to the fact that the proposed control scheme is based on the use of bounded positive functions that are nonincreasing along the solutions of the closed-loop system. Moreover, we have

$$N(\theta) = \begin{pmatrix} \dfrac{2\left(\frac{\alpha}{\beta}\theta^{\frac{1}{\alpha}}\right)^3}{(\alpha+1)(\alpha+2)(\alpha+3)} & \dfrac{-\left(\frac{\alpha}{\beta}\theta^{\frac{1}{\alpha}}\right)}{(\alpha+1)(\alpha+2)} \\ \dfrac{-\left(\frac{\alpha}{\beta}\theta^{\frac{1}{\alpha}}\right)}{(\alpha+1)(\alpha+2)} & \dfrac{\left(\frac{\alpha}{\beta}\theta^{\frac{1}{\alpha}}\right)}{\alpha+1} \end{pmatrix}. \tag{13}$$

However, in Appendix B we prove

Lemma 2. $\theta = \theta(z)$ is of class C^1 and is the unique positive solution of

$$\theta^{1+\frac{3}{\alpha}} = \sum_{i,j=1}^{2} \tilde{M}_{ij}(\alpha,\beta)\theta^{\frac{1}{\alpha}(i+j-2)} z_i z_j, \tag{14}$$

where $\tilde{M}_{ij}(\alpha,\beta)$ are elements of the inverse matrix of **M,** which is given by

$$M^{-1}(\alpha,\beta) = \begin{pmatrix} \dfrac{(\alpha+2)^2(\alpha+3)}{\left(\frac{\alpha}{\beta}\right)^3} & \dfrac{(\alpha+2)(\alpha+3)}{\left(\frac{\alpha}{\beta}\right)^2} \\ \dfrac{(\alpha+2)(\alpha+3)}{\left(\frac{\alpha}{\beta}\right)^2} & \dfrac{2(\alpha+2)}{\left(\frac{\alpha}{\beta}\right)} \end{pmatrix}.$$

If we now consider the following linearizing-like control law

$$u = -\eta - \frac{1}{2}B^T N^{-1}(\theta)z \tag{15}$$

where $N^{-1}(\theta)$ is the inverse matrix of $N(\theta)$ which is given by

$$N^{-1}(\theta) = \begin{pmatrix} \dfrac{(\alpha+2)^2(\alpha+3)}{\left(\frac{\alpha}{\beta}\theta^{\frac{1}{\alpha}}\right)^3} & \dfrac{(\alpha+2)(\alpha+3)}{\left(\frac{\alpha}{\beta}\theta^{\frac{1}{\alpha}}\right)^2} \\ \dfrac{(\alpha+2)(\alpha+3)}{\left(\frac{\alpha}{\beta}\theta^{\frac{1}{\alpha}}\right)^2} & \dfrac{2(\alpha+2)}{\left(\frac{\alpha}{\beta}\theta^{\frac{1}{\alpha}}\right)} \end{pmatrix}.$$

Remark 2. A simple computation can prove that equation (14) can be rewritten as

$$\begin{aligned} \theta^{1+\frac{3}{\alpha}}(z) &= \tilde{M}_{11}(\alpha,\beta)z_1^2 + (\tilde{M}_{12}(\alpha,\beta) + \tilde{M}_{21}(\alpha,\beta)\theta^{\frac{1}{\alpha}}z_1z_2 + \tilde{M}_{22}(\alpha,\beta)\theta^{\frac{2}{\alpha}}z_2^2 \\ &= \frac{(\alpha+2)^2(\alpha+3)z_1^2}{\left(\frac{\alpha}{\beta}\right)^3} + \frac{2(\alpha+2)(\alpha+3)z_1z_2}{\left(\frac{\alpha}{\beta}\right)^2} + \frac{2(\alpha+2)z_2^2}{\left(\frac{\alpha}{\beta}\right)}. \end{aligned}$$

With this in mind, we can see that

$$\theta(z) = \frac{(\alpha+2)^2(\alpha+3)z_1^2}{\left(\frac{\alpha}{\beta}\theta^{\frac{1}{\alpha}}\right)^3} + \frac{2(\alpha+2)(\alpha+3)z_1z_2}{\left(\frac{\alpha}{\beta}\theta^{\frac{1}{\alpha}}\right)^2} + \frac{2(\alpha+2)z_2^2}{\left(\frac{\alpha}{\beta}\theta^{\frac{1}{\alpha}}\right)} = z^T N^{-1}(\theta)z.$$

Substitution of the linearizing-like controller (15) into (9) gives

$$\begin{aligned} \dot{z} &= \left(A - \frac{1}{2}BB^T N^{-1}(\theta)\right)z \\ \dot{\eta} &= \Xi(z, \dot{x}, \eta, e_i, \dot{e}_i, u). \end{aligned} \tag{16}$$

Now, we can establish the following result.

Theorem 1. *Let $z_0 = z(0)$ be the initial condition of $z(t)$. If $z_0 \neq 0$, $\alpha \geq 1$ and $\beta > 0$, then the synchronization error $z(t)$ converges asymptotically to zero within a finite time*

$$T_8 = \frac{\alpha}{\beta}\theta^{\frac{1}{\alpha}}(z_0) \tag{17}$$

that is, $z(t) \to \infty$ for all $t \geq T_8 > 0$.

Without loss of generality, we assume that $z(t)$ is defined in the interval $[0, T_8]$ so that $\theta \neq 0$ and the matrices $N(\theta)$ and $N^{-1}(\theta)$ exist. We define the Lyapunov candidate function as

$$\theta(z) = z^T N^{-1}(\theta) z. \tag{18}$$

Now, consider the function $F(z, \theta) = \theta(z) - z^T N^{-1}(\theta) z$. From equation (18), we know that $dF = \frac{\partial F}{\partial \theta} d\theta + \frac{\partial F}{\partial z} dz = 0$, which implies that $F'_\theta \frac{\partial \theta}{\partial z} + F'_z = 0$, where $F'_z = \frac{\partial F}{\partial z} = -2N^{-1}(\theta) z$ and $F'_\theta = \frac{\partial F}{\partial \theta} = z^T \left[\frac{1}{\theta} N^{-1}(\theta) - \frac{d}{d\theta} N^{-1}(\theta)\right] z$. From this, we can deduce that $\frac{\partial \theta}{\partial z} = -\frac{F'_z}{F'_\theta}$. Thus, the time derivative of (18) along the trajectories of (16) satisfies

$$\begin{aligned}
\dot{\theta}(z) &= \left\langle \frac{\partial \theta}{\partial z}, \left(A - \frac{1}{2} N^{-1}(\theta) B B^T\right) z \right\rangle \\
&= -\left\langle \frac{F'_z}{F'_\theta}, \left(A - \frac{1}{2} N^{-1}(\theta) B B^T\right) z \right\rangle \\
&= \frac{1}{F'_\theta} z^T \left\langle A^T N^{-1}(\theta) + N^{-1}(\theta) A^T - N^{-1}(\theta) B B^T N^{-1}(\theta) \right\rangle z
\end{aligned} \tag{19}$$

where $\langle ., . \rangle$ is the inner product. Using equation (12), one may easily prove that

$$A^T N^{-1}(\theta) + N^{-1}(\theta) A^T - N^{-1}(\theta) B B^T N^{-1}(\theta) = \beta \theta^{\frac{1}{\alpha} - 1} \left[\frac{d}{d\theta} N^{-1}(\theta) - \frac{1}{\theta} N^{-1}(\theta)\right]. \tag{20}$$

Finally, we get

$$\dot{\theta}(z) = -\beta \theta^{1 - \frac{1}{\alpha}}(z), \tag{21}$$

which is negative definite if $\alpha \geq 1$ and $\beta > 0$. This means that if $\alpha \geq 1$ and $\beta > 0$, the synchronization error $z(t)$ converges asymptotically to zero. Since θ is a continuous function, one can easily prove (by applying the LaSalle invariance principle) that the origin is the largest invariant set contained in $\varepsilon = \{z \in \mathbb{R}^2, \dot{\theta}(z) = 0\}$. Thus, the synchronization error remains at zero for all $t \geq t_0 \geq 0$.

The convergence of $\eta(t)$ to zero follows from the fact that the closed-loop system is in cascade form. Since $\Theta(z, x, \dot{e}_i)$ is smooth, the control dynamics is given by

$$\dot{u} = -\Xi(z, \dot{x}, \eta, e_i, \dot{e}_i, u) - \frac{1}{2} B^T \dot{\theta} \frac{\partial}{\partial \theta} N^{-1}(\theta) - \frac{1}{2} B^T N^{-1}(\theta) \dot{z}.$$

Since $\Xi(z, \dot{x}, \eta, e_i, \dot{e}_i, u)$ is a smooth function, $\dot{u}$ is also a smooth function. Consequently, $\Psi(z, x, \eta, \dot{e}_i)$ is a first-integral of the closed-loop system. Then, from equation (15), the augmented state becomes $\eta = -u - \frac{1}{2} B^T N^{-1}(\theta) z$. Hence, the augmented state η is bounded, and its dynamics are also bounded. Finally, since $z(t)$ asymptotically converges to zero, $\eta(t)$ also asymptotically converges to zero. This guarantees the convergence to the origin of closed-loop system (16).

To compute the synchronization time, we have to follow the time trajectory of closed-loop system (16). In this case, synchronization is achieved when $z(t)$ is zero for all $t \geq T_s \geq 0$. If we integrate equation (21), we get $\theta^{\frac{1}{\alpha}} = \frac{1}{\alpha}(-\beta t + c)$ where c is an integration constant. Since $z_0 = 0$, $\theta(z_0) = 0$, and one can deduce that $c = \alpha\theta^{\frac{1}{\alpha}}(z_0)$. With this in mind, and using the fact that $\theta(z) = 0$ at $t = T_s$, one may easily prove that the synchronization time is defined as in equation (17). This completes the proof.

Remark 3. Given the feedback parameters á and β, it is not immediately apparent how one chooses the function θ so that synchronization objective (7) is satisfied. Furthermore, it is not easy to determine the solutions of equation (14) analytically. Fortunately, this equation can be solved numerically. Since error system (3) is minimum phase, the state e_i, $i = 1, ..., n$, will converge to the origin, although in an arbitrary time. However, it is clear that its convergence rate will also depend on the feedback parameters α and β. Determining the quantitative relation between the convergence rate of the states z and e_i, $i = 1, ..., n$, requires further work, however.

Nevertheless, the linearizing-like feedback (15) is not physically realizable, as it requires perfect knowledge of the nonlinear term $\Theta(z, x, \dot{e}_i)$. Because of the two assumptions made, this feedback must be modified in such a way as to include consideration of modelling errors and parameter perturbations. We therefore use the estimation of $\Theta(z, x, \dot{e}_i)$ in such a way that the main characteristics of the linearizing-like feedback (15) are retained. As was established by Jiang and Praly (1998), the problem of estimating $(z, \ \eta)$ can be addressed by using a high-gain observer. Thus, we are interested in a dynamical output feedback of the form

$$\begin{aligned} \dot{\hat{z}}_1 &= \hat{z}_2 + \rho k_1(z_1 - \hat{z}_1) \\ \dot{\hat{z}}_2 &= \hat{\eta} + \rho^2 k_2(z_1 - \hat{z}_1) + u \\ \dot{\hat{\eta}} &= \rho^3 k_3(z_1 - \hat{z}_1) \end{aligned} \tag{22}$$

$$u(\hat{z}) = Sat\left\{ -\hat{\eta} - \frac{(\alpha+2)(\alpha+3)\hat{z}_1}{2\left(\frac{\alpha}{\beta}\beta\theta^{\frac{1}{\alpha}}\right)^2} - \frac{(\alpha+2)\hat{z}_2}{\frac{\alpha}{\beta}\theta^{\frac{1}{\alpha}}} \right\} \tag{23}$$

where $\hat{z}_1$, $\hat{z}_2$ and $\hat{\eta}$ are the estimated values of z_1, z_2 and η, respectively, and $\rho > 0$ is the high-gain parameter, which can be interpreted as the uncertainty estimation rate, and can often be chosen as a constant (Femat et al., 1999), and

$$Sat\,\{.\} = \begin{cases} = u_{\max} & \text{if} \quad u > u_{\max} \\ = -\hat{\eta} - \frac{1}{2}B^T N^{-1}(\theta)\hat{z} & \text{if} \quad -u_{\max} \leq u \leq u_{\max} \\ = -u_{\max} & \text{if} \quad u < -u_{\max} \end{cases}$$

θ is the unique positive solution of the equation

$$\theta^{\frac{\alpha+3}{\alpha}} = \frac{2\beta}{\alpha}(\alpha+2)\hat{z}_2^2\theta^{\frac{2}{\alpha}} + \frac{2\beta^2}{\alpha^2}(\alpha+2)(\alpha+1)\hat{z}_1\hat{z}_2\theta^{\frac{1}{\alpha}} + \frac{\beta^3}{\alpha^3}(\alpha+2)^2(\alpha+3)\hat{z}_1^2. \quad (24)$$

The estimation constants k_i, $i = 1, 2, 3$, are chosen in such a way that the polynomial $s^3 + k_3 s^2 + k_2 s + k_1 = 0$ has all its roots located in the open left-hand complex plane. Now, we can demonstrate the following

Theorem 2. *Let $\hat{z}_0 = \hat{z}(0)$ be the initial condition of $\hat{z}(t)$. If $\hat{z}(0) \neq 0$, $\alpha \geq 1$ and $\beta > 0$, then the synchronization error z(t) converges asymptotically to zero at a finite time*

$$T_8 = \frac{\alpha}{\beta}\theta^{\frac{1}{\alpha}}(\hat{z}_0) \quad (25)$$

Proof. Let $\tilde{e} \in \mathbb{R}^3$ be an estimation error vector whose components are defined as follows: $\tilde{e}_i = \rho^{2-i}(z_i \tilde{z}_i)$, for $i = 1, 2$, and $\tilde{e}_3 = \eta - \hat{\eta}$. Substitution of robust feedback controller (23) and the dynamics of the above defined estimation error into (9) yields

$$\begin{aligned} \dot{z} &= \Lambda(z, \tilde{e}, \eta, u) \\ \dot{\eta} &= \Xi(z, \tilde{e}, \dot{x}, \eta, e_i, \dot{e}_i, u) \\ \dot{\tilde{e}} &= \rho D\tilde{e} + B'\Xi(z, \tilde{e}, \dot{x}, \eta, e_i, \dot{e}_i, u) \end{aligned} \quad (26)$$

where $\Lambda(z, \tilde{e}, \eta, u) = Az + B(\eta + u)$ with $u = u(z_i - \rho^{i-2}\tilde{e}_i, \eta - \tilde{e}_3)$,

$$D = \begin{bmatrix} -k_1 & 1 & 0 \\ -k_2 & 0 & 1 \\ -k_3 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B' = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Since the saturation function is a bounded function, there exists a continuous function $\gamma(\|\tilde{e}\|)$ such that $\|\Lambda(z, \tilde{e}, \eta, u)\| \leq \gamma(\|\tilde{e}\|)$. In addition, since $\eta = \Theta(z, x, \dot{e}_i)$ and $u = Sat\{-\hat{\eta} -\frac{1}{2}B^T N^{-1}(\theta)\hat{z}\}$, one can obtain the contraction $\eta = Z(z, \tilde{e}, \dot{x}, \eta, e_i, \dot{e}_i, u)$ (which can be computed from the first integral of the second equation of system (26), i.e., $\eta = \int \Xi(z, \tilde{e}, \dot{x}, \eta, e_i, \dot{e}_i, u)d\tau$. Then, according to the Contraction Mapping Theorem, the state η can be expressed globally and uniquely as a function of the coordinates $(z, \tilde{e})$.

On the other hand, since the matrix D is of Hurwitz construction, the nominal system $\dot{\tilde{e}} = \rho D\tilde{e}$ is quadratically stable. This implies that the Lyapunov equation $D^T P + PD = -I_3$ (where I_3 is the identity matrix of dimension 3) has a positive definite solution P. Since the nonlinear function $\Xi(z, \tilde{e}, \dot{x}, \eta, e_i, \dot{e}_i, u)$ is bounded, the last equation of (26) is quadratically stable.

From this, and the boundedness of $\Lambda(z, \tilde{e}, \eta, u)$, one can conclude that, given a compact set of initial conditions $\Omega_0 \subset \mathbb{R}^3$ containing the origin, there exists an upper bound $u_{\max}$, with $|Sat\{.\}| \leq u_{\max}$ and a high-gain estimation ρ such that Ω_0 is contained in the attraction basin $\Omega_S \times \Omega_{S_0}$ of system (26). Hence, system (26) is semi-globally practically stable, i.e.,

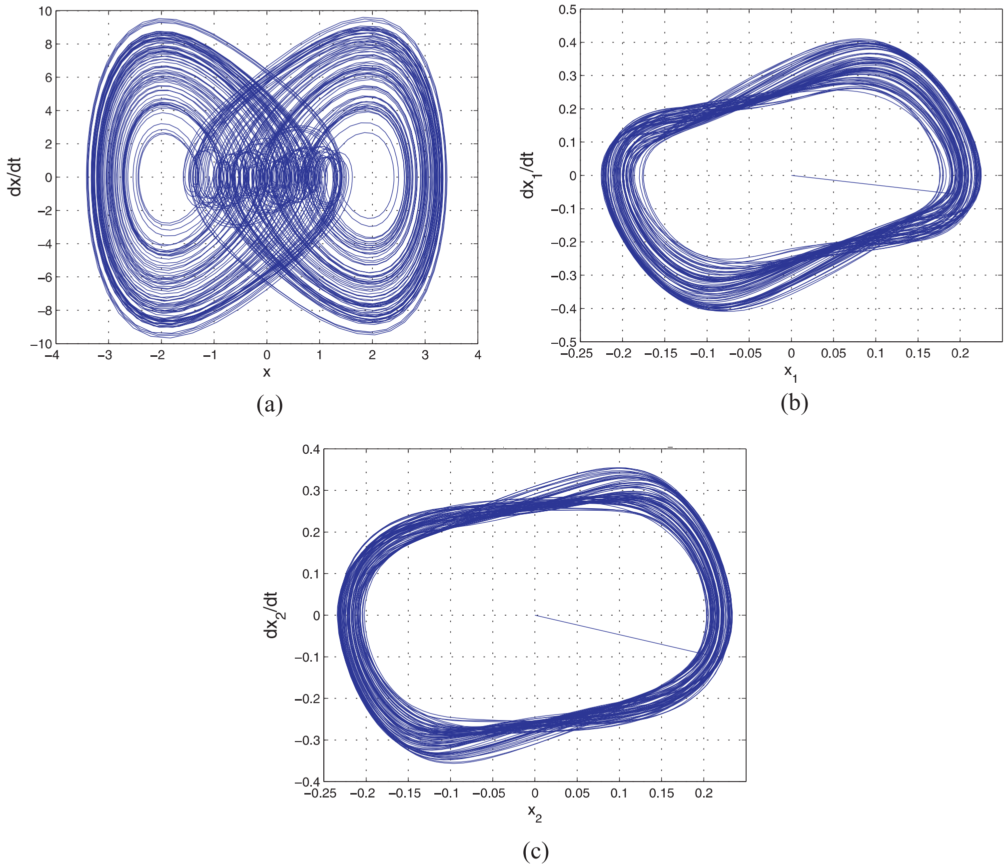


Figure 2. Phase portraits of the electromechanical system showing the chaotic orbit.

$(z, \eta) \rightarrow (0, 0)$. Then, since the solution of (8) is the projection of system (6), one can conclude that $z(t) \rightarrow 0$, via module $\Pi(z, \eta)$, where (z, η) is the projection of system (8) into system (6) for all $t \geq 0$. Therefore, $z(t) \rightarrow 0$ as $t \rightarrow T_s$.

4. NUMERICAL SIMULATIONS

In order to validate the performance of the proposed control law, we will show a series of numerical experiments to demonstrate its effectiveness.

Without loss of generality, we consider the case where $n = 2$. For some parameters such as $\gamma = 0.1$, $\gamma_1 = 0.3$, $\gamma_2 = 0.23$, $\lambda_1 = 0.01$, $\lambda_2 = 0.02$, $\lambda_{11} = 0.06$, $\lambda_{21=}0.04$, $\omega_1 = 1.2$, $\omega_2 = 1.3$, $\omega = 1.4$, $\beta = 2.32$ and $E_0 = 20$, the nonlinear electromechanical system has a chaotic solution (Woafo et al., 2005) as shown in Figure 2. The initial states are at the origin in each part.

The initial conditions for $(\hat{z}_1, \hat{z}_2, \hat{\eta})$ were $(1, 0, -2.5)$, so that

$$\theta(\hat{z}_0) = \left[\frac{\beta^3}{\alpha^3}(\alpha+2)^2(\alpha+3)\right]^{\frac{\alpha}{(\alpha+3)}}.$$

Hence, the synchronization time (25) can be expressed as:

$$T_8 = \frac{1}{\beta^{\frac{\alpha}{(\alpha+3)}}}\left[\alpha^\alpha(\alpha+2)^2(\alpha+3)\right]^{\frac{1}{(\alpha+3)}}. \tag{27}$$

Remark 4. The α, β-parameterization of feedback coupling (23) provides a simple tuning procedure. From the above equation, one can see that for fixed β, if α increases, then $\theta^{\frac{1}{\alpha}}(z_0)$ decreases such that $\alpha\theta^{\frac{1}{\alpha}}(z_0)$ increases. This means that the synchronization time T_s increases with α. Similarly, if β increases while α is fixed, synchronization time T_s decreases. It is thus easy to see how the synchronization time can be minimized. This is of great practical interest, since the synchronization process can be affected as fast as desired, depending on the feedback parameters α and β.

It should be pointed out that the value of θ is obtained via numerical simulations. The high gain parameter is set to $\rho = 30$ and $u_{\max} = 20$. The estimation constants $[k_1, k_2, k_3]$ are $[3, 3, 1]$, so that the eigenvalues of matrix D are located at -1.

Figure 3 shows the synchronization of the master and slave electromechanical systems performed with $\alpha = \beta = 1$. Note that in this case, the analytical value of the synchronization is about 2.445 s. From this figure, it is clear that the synchronization error is stabilized at the origin by output-feedback controller (23) in spite of the fact that the master and slave systems have different initial conditions. From Figures 3a and 3b, one can see that a fairly good convergence of $z \in R^2$ is obtained in about 2.4 s, which corresponds well with the analytical value of the synchronization time. One important feature is that although the control input is acting only on the states $z \in R^2$, $(e_i, \dot{e}_i) \in \mathbb{R}^4$, $i = 1, 2$, is also stabilized at the origin.

Figure 4 shows the performance of output feedback controller (23). Figure 4a shows the current term $\eta(t)$ (solid line) and the estimated term $\hat{\eta}(t)$ (dashed line). After a short transient, i evolves very similarly to η. Such behavior is attained because of the fact the control signal converges exactly to zero. This can be seen in Figure 4b.

Providing further evidence of the effectiveness and efficiency of the proposed synchronization algorithm, we have checked the validity of the analytical results by comparing the values given by equation (27) to those obtained from the numerical simulation. We consider that the synchronization is achieved when the synchronization error is less than a precision or tolerance. Using the equation

$$\|z(t)\| \leq h \tag{28}$$

where $h = 10^{-4}$ is the synchronization precision or tolerance for computing numerical values of the synchronization time. Figures 5a and 5b show, respectively, the synchronization time as a function of β when $\alpha = 1$ and as a function of α when $\beta = 1$. The agreement between

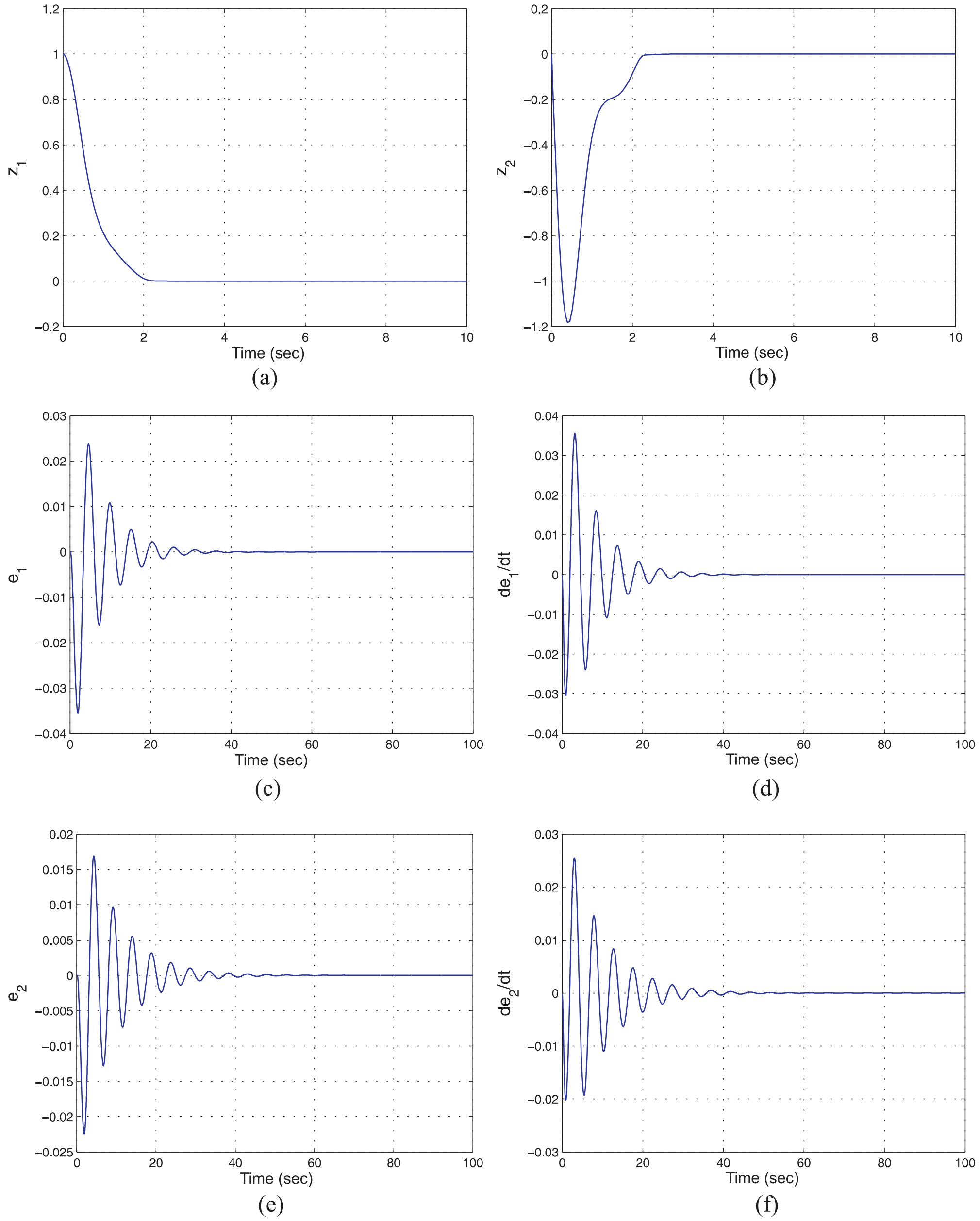
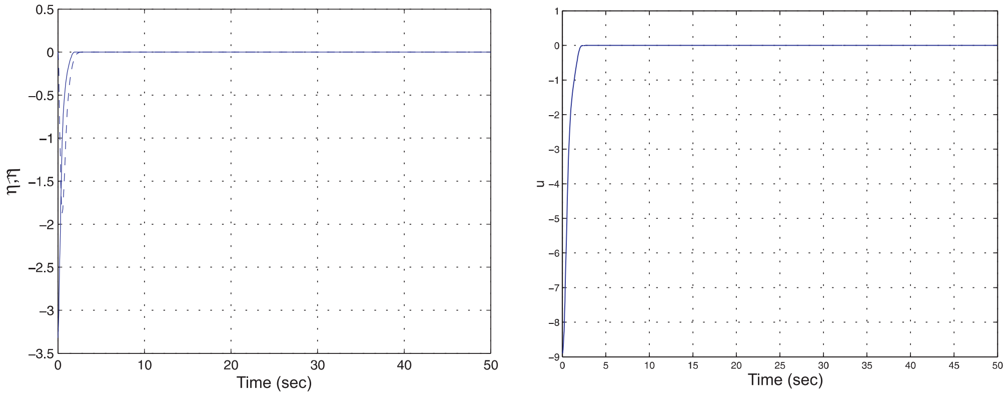


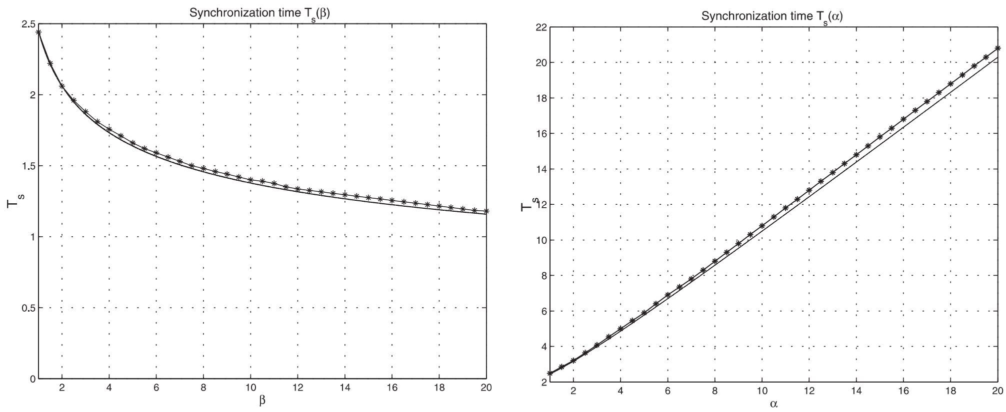
Figure 3. Time evolution of the synchronization error. (a) $z_1(t) = x(t) - y(t)$, (b) $z_2(t) = \dot{x}(t) - \dot{y}(t)$, (c) $e_1(t) = x_1(t) - y_1(t)$, (d) $\dot{e}_1(t) = \dot{x}_1(t) - \dot{y}_1(t)$, (e) $e_2(t) = x_2(t) - y_2(t)$ and (f) $\dot{e}_2(t) = \dot{x}_2(t) - \dot{y}_2(t)$.

the analytical (lines) and numerical (lines with stars) results is good. As predicted by the analysis of Remark 2, one can see that the synchronization time decreases exponentially when β increases (Figure 5a), while the synchronization time increases with the feedback parameter α (Figure 5b).



(a) (b)

Figure 4. (a) Evolution of the current value of η (solid line) and its estimated value $\hat{\eta}$ (dotted line) and (b) time evolution of the control signal.



(a) (b)

Figure 5. Synchronization time T_s: numerical results (lines with stars) and analytical results (lines). (a) As a function of α when $\beta = 1$. (b) As a function of β when $\alpha = 1$.

5. CONCLUSIONS

In this article, a robust synchronization scheme of a class of nonlinear systems was presented from the control theoretical point of view. The problem has been restricted to chaotic electromechanical devices. The synchronization problem was addressed as one of stabilization at the origin of the synchronization error. The mathematical stability conditions are derived from the Lyapunov stability theory. A robust adaptive feedback system is developed, such that two electromechanical chaotic systems can be synchronized. A state observer is used to estimate the systems' uncertainties, and the unmeasurable states, based on the measurable synchronization error. The feedback controller then becomes physically realizable, based on the states of the observer, and can be used to synchronize two electromechanical chaotic systems. An explicit expression of the synchronization time was given in terms of two parameters from which an arbitrary convergence rate of the synchronization error can be prescribed. Simulation results demonstrate that the proposed strategy is able to achieve the synchronization of two electromechanical chaotic systems.

APPENDIX A: PROOF LEMMA 1

In this appendix, we prove that the matrix $N(\theta)$ is symmetric and positive definite. Using $\tau = \theta(1-t)^{\alpha}$, matrix (11) can be expressed as

$$N(\theta) = \frac{1}{\theta} e^{-\frac{\alpha}{\beta} A \theta^{\frac{1}{\alpha}}} \int_0^{\theta} \frac{\tau}{\beta} e^{\frac{\alpha}{\beta} A \tau^{\frac{1}{\alpha}}} B B^T e^{\frac{\alpha}{\beta} A^T \tau^{\frac{1}{\alpha}}} d\tau e^{-\frac{\alpha}{\beta} A^T \theta^{\frac{1}{\alpha}}} \tag{29}$$

which is a symmetric matrix. Now, suppose that for $\theta > 0$, there exists a vector $X_0 \neq 0$ such that $\langle N(\theta) X_0, X_0 \rangle = 0$. Then, from equation (29), one can show that

$$\int_0^{\theta} \left\| \tau^{\frac{1}{2\alpha}} B^T e^{-\frac{\alpha}{\beta} A (\theta^{\frac{1}{\alpha}} - \tau^{\frac{1}{\alpha}})} X_0 \right\|^2 d\tau = 0, \quad \text{for all} \quad \tau \in [0, \theta] \tag{30}$$

which implies that

$$B^T e^{-\frac{\alpha}{\beta} A^T (\theta^{\frac{1}{\alpha}} - \tau^{\frac{1}{\alpha}})} X_0 = 0, \quad \text{for all} \quad \tau \in [0, \theta]. \tag{31}$$

Computing the $n-1$ derivatives of the above equation with respect to τ and setting $\tau = 0$, we get

$$B^T X_0, B^T A^T X_0, B^T (A^T)^2 X_0, ..., B^T (A^T)^{n-1} B X_0 = 0. \tag{32}$$

This leads to a contradiction, since the pair (A, B) is controllable, and one can conclude that $N(\theta)$ is positive definite.

APPENDIX B: PROOF LEMMA 2

Here, we prove that $\theta(z)$ is a C^1 function, and is the unique solution of equation (14).

First, we establish that $\theta(z)$ is the unique solution of equation (11). Consider the function

$$F(z, \theta) = \theta - \left\langle N^{-1}(\theta) z, z \right\rangle. \tag{33}$$

Its derivative with respect to θ satisfies

$$\frac{\partial F(z,\theta)}{\partial \theta} = 1 + \left\langle N^{-1}\frac{dN(\theta)}{d\theta}N^{-1}(\theta)z, z\right\rangle. \tag{34}$$

On the other hand, from equation (11), one can deduce inductively that

$$\frac{dN(\theta)}{d\theta} = \frac{1}{\frac{\alpha}{\beta}\theta^{1+\frac{1}{\alpha}}}\int_0^{\frac{\alpha}{\beta}\theta^{\frac{1}{\alpha}}}\left(1-\frac{s}{\frac{\alpha}{\beta}\theta^{\frac{1}{\alpha}}}\right)^{\alpha-1} se^{-As}BB^Te^{A^Ts}ds > 0. \tag{35}$$

From this observation one can easily prove, as shown in Appendix A, that $\frac{dN(\theta)}{d\theta}$ is positive definite. We now know that $(\partial F(z,\theta)/\partial\theta) > 1$. As a consequence, for large enough θ, $F(z,\ \theta) > 0$ and $F(z,\ \theta)$ must be a strict increasing function with respect to θ. On the other hand, since $N(\theta)$ is symmetric and positive definite, $N_1(\theta)$ is also symmetric and positive definite. With this in mind, one has

$$\left\langle N^{-1}(\theta)z, z\right\rangle \geq \frac{\langle z, z\rangle}{\|N(\theta)\|}. \tag{36}$$

Using equation (11), it is now clear that $N(\theta) \to 0$ when $\theta \to 0$. Thus, for sufficiently small θ, one can deduce that

$$F(z,\theta) \leq \theta - \frac{\langle z, z\rangle}{\|N(\theta)\|} < 0. \tag{37}$$

Thus, since $\theta(z)$ is a continuous function, one can conclude that the equation $\theta = \left\langle N^{-1}(\theta)z, z\right\rangle$ admits a unique positive solution $\theta(z)$.

Now, we will prove that $\theta(z)$ is a C_1 function. Since $(\partial F(z,\theta)/\partial\theta) > 1$ for all $z \neq 0$, one can, by applying the implicit function theorem, deduce that $\theta(z)$ is the solution of $F(z,\ \theta) = 0$ and of class C_1 in the neighborhood of $z \neq 0$. Now, we prove that $\theta(z) < \varepsilon$ for all $z \in \Omega_{s_0}$ with

$$\varepsilon > \Re(\varepsilon)\rho_0^2, \quad \rho_0 > 0 \tag{38}$$

and $\Omega_{s_0} = \{\in \mathbb{R}^2, \|z\| = \rho_0\}$

Suppose that $\theta(z) \geq \varepsilon$ for $z \in \Omega_{s0}$. On the one hand, since for $z \neq 0$, the function $\left\langle N^{-1}(\theta)z, z\right\rangle$ is always decreasing with respect to θ, the function $\left\|N^{-1}(\theta)\right\|$ is also a decreasing function. Thus, for all $\varepsilon > 0$ we have

$$\left\|N^{-1}(\theta)\right\| \leq \Re(\varepsilon) < \infty, \quad \text{for all} \quad \theta \geq \varepsilon.$$

With this inequality in mind, we have

$$\left\langle N^{-1}(\theta)z, z\right\rangle \leq \left\|N^{-1}(\theta)\right\| \|z\|^2 \leq \Re(\varepsilon)\rho_0^2.$$

On the other hand, combining equations (14) and (38), we immediately arrive at the following inequality

$$\theta(z) \geq \varepsilon \geq \Re(\varepsilon)\rho_0^2 \geq \theta(z).$$

This leads to a contradiction and we can thus conclude that $\theta(z) = \varepsilon$ for $z \in \Phi_{s0}$. The results follow.

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